The Gudhi Library:

Simplicial Complexes and Persistent Homology

Clément Maria, Jean-Daniel Boissonnat, Marc Glisse, Mariette Yvinec

1
Introduction

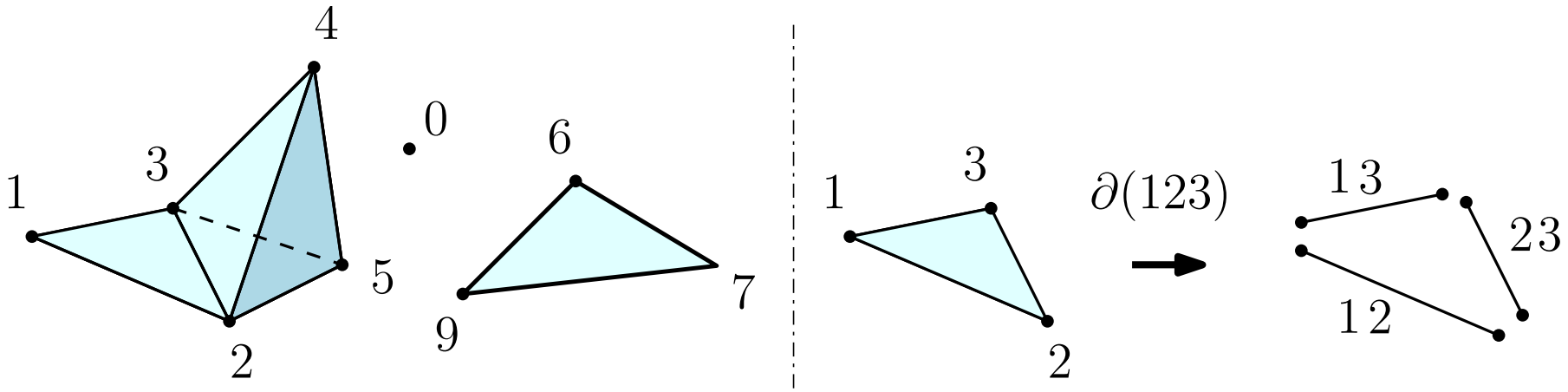
Simplicial Complex

Given a set $V = \{1 \cdots n\}$ of vertices, an **abstract simplicial complex** $\mathcal{K}$ on V is a family of subsets of vertices s.t.:

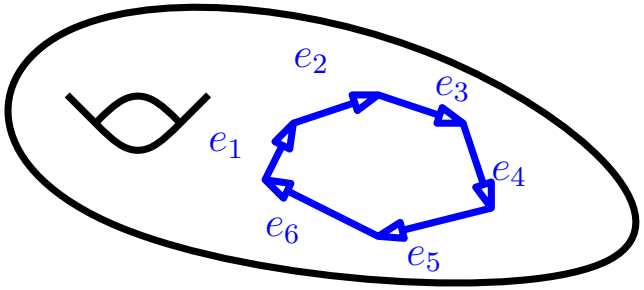
$$\forall \sigma \in \mathcal{K} : \tau \subseteq \sigma \Rightarrow \tau \in \mathcal{K}$$

Such σ is called a <u>simplex</u>

<u>Boundary of a simplex</u> σ: $\partial\sigma = \{\tau \subset \sigma | \mathrm{dim}(\sigma) = \mathrm{dim}(\tau) + 1\}$

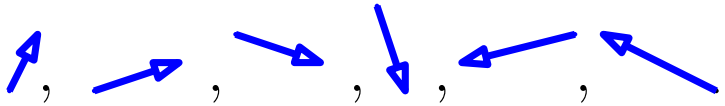


Introduction to Homology



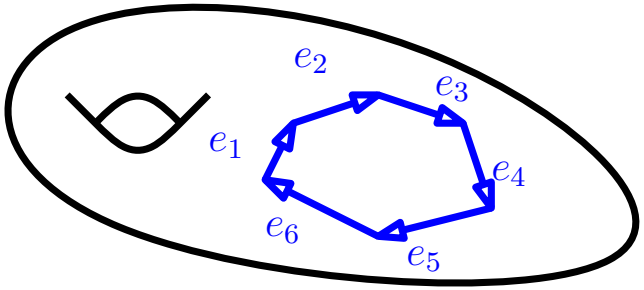
$\{e_1, \; e_2 \; , \; e_3 \; , e_4 , \; e_5 \; , \; e_6 \; , \cdots \}$

$\mathbf{K}^p$: set of p-simplices of $\mathbf{K}$



Introduction to Homology

$e_1 + e_2 + e_3 + e_4 + e_5 + e_6$



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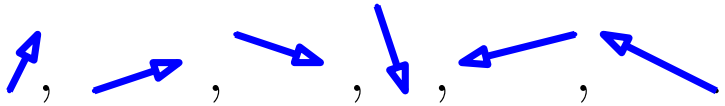

$\mathbf{C}_p$: Abelian group of formal sums of p-simplices with $\mathbb{Z}$-coefficients

$$\mathbf{C}_1 = (\{\textstyle\sum_i k_i e_i\}, +)$$

Introduction to Homology

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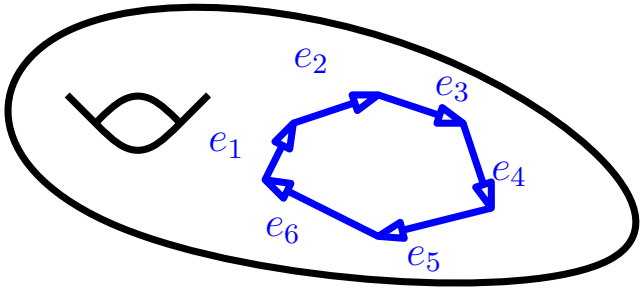

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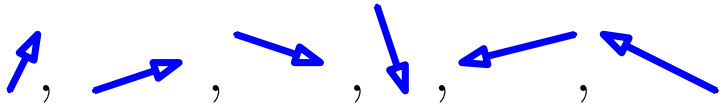

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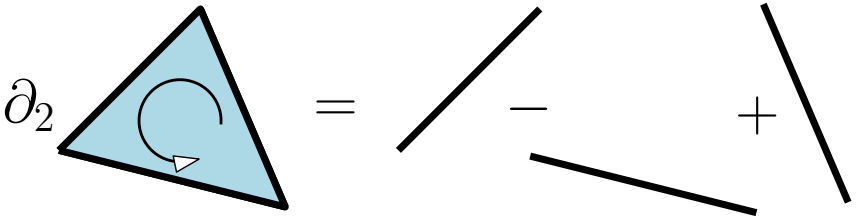

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Introduction to Homology

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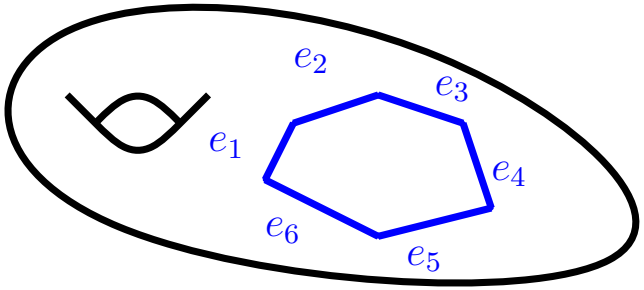

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Field $\mathbb{Z}_p$ or $\mathbb{Q}$

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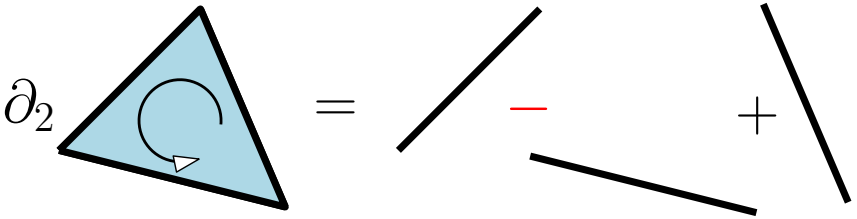

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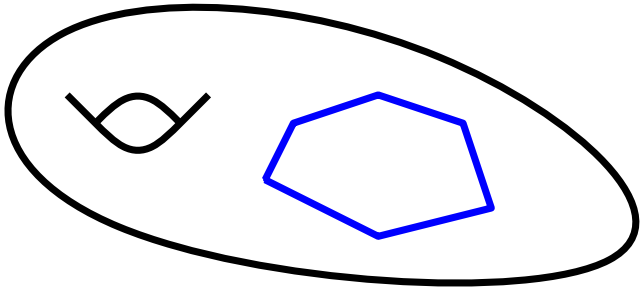

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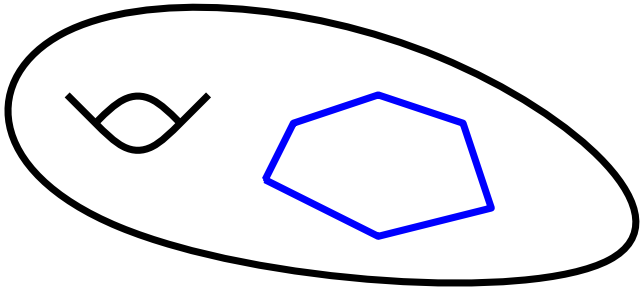

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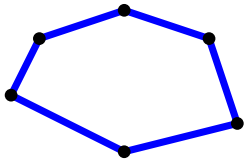
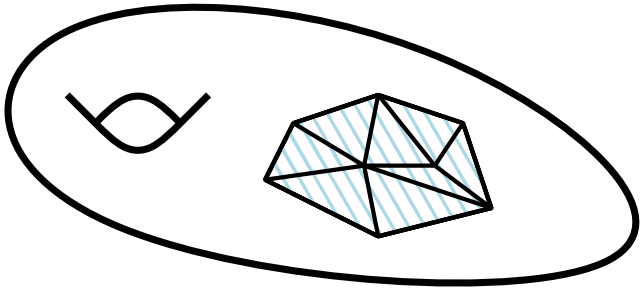
$\mathbf{Z}_p = \ker \partial_p$: the p-cycles ∂_1  $= \bullet\times 2 + \bullet\times 2 + \bullet\times 2 + \bullet\times 2 + \bullet\times 2 + \bullet\times 2 = 0$

Introduction to Homology



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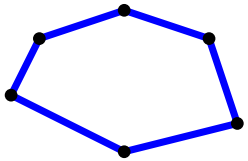
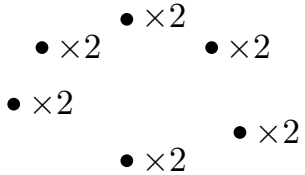
$\mathbf{Z}_p = \ker \partial_p$: the p-cycles ∂_1  $=$ 

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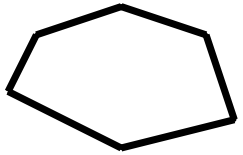
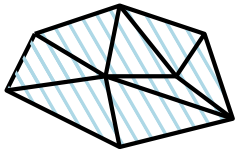
$\mathbf{B}_p = \operatorname{im} \partial_{p+1}$: the p-boundaries  $= \partial_2$ 

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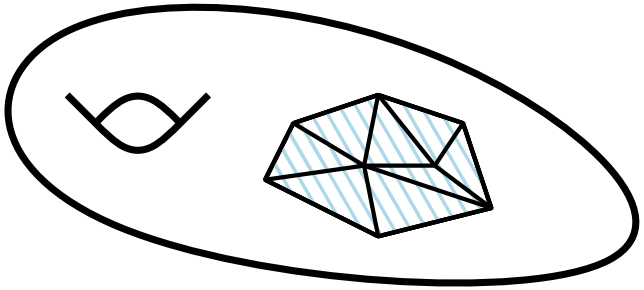

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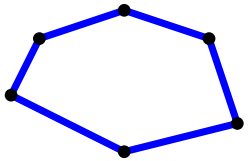
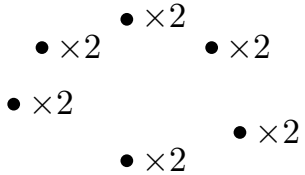
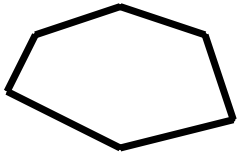
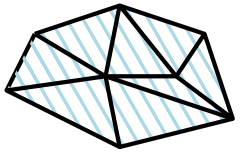
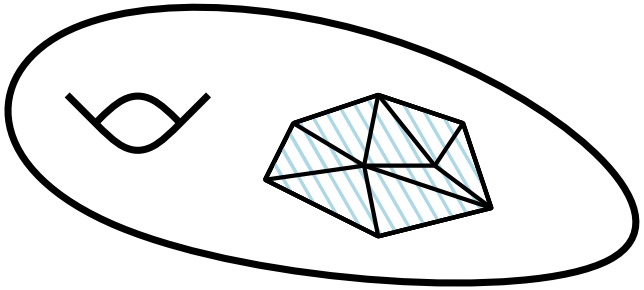
$\mathbf{Z}_p = \ker \partial_p$: the p-cycles ∂_1  $=$  $= \ 0$

$\mathbf{B}_p = \mathrm{im}\ \partial_{p+1}$: the p-boundaries  $=$ ∂_2 

Fundamental property: $\partial_p \circ \partial_{p+1} = 0$

Introduction to Homology



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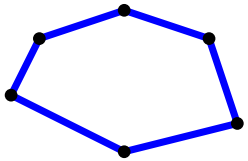
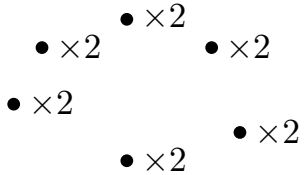
$\mathbf{Z}_p = \ker \partial_p$: the p-cycles ∂_1  $=$ 

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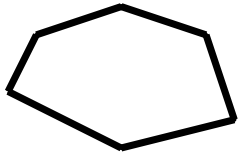
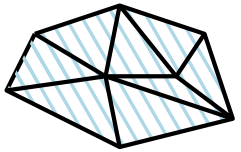
$\mathbf{B}_p = \operatorname{im} \partial_{p+1}$: the p-boundaries  $= \partial_2$ 

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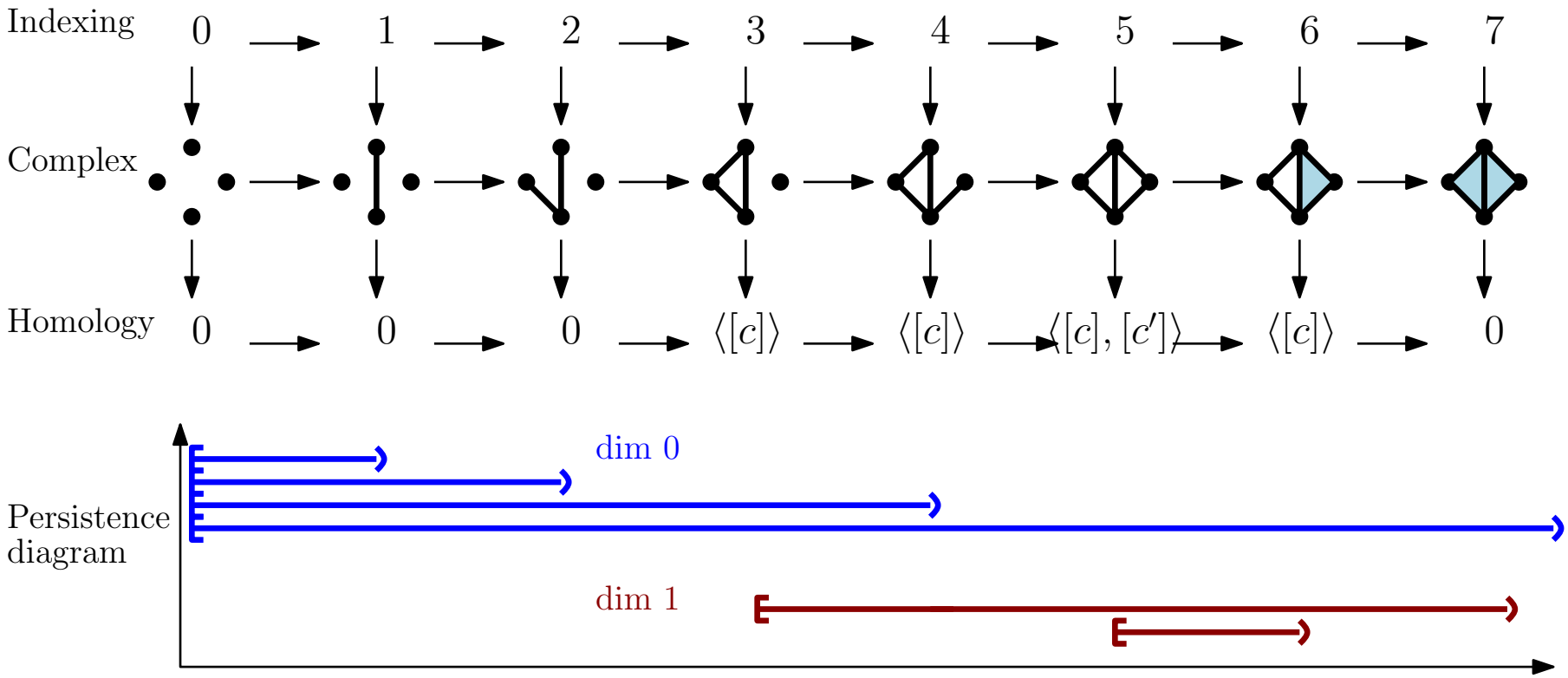
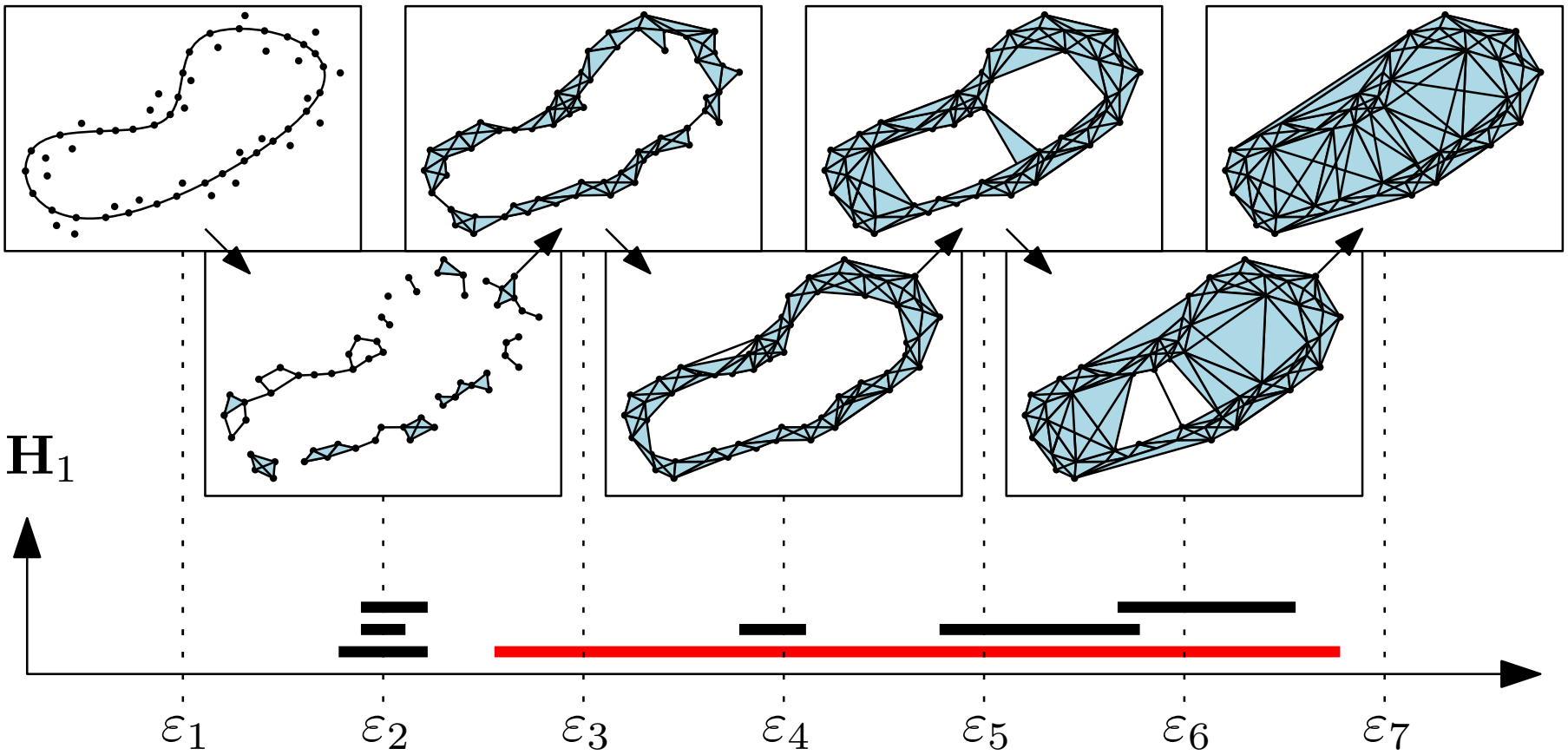
$\mathbf{H}_p = \mathbf{Z}_p / \mathbf{B}_p$: p^{th} homology group

Persistent Homology

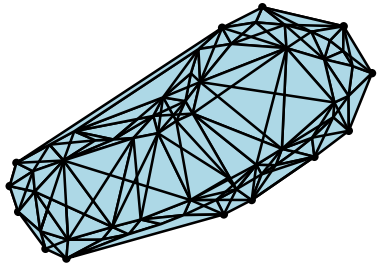


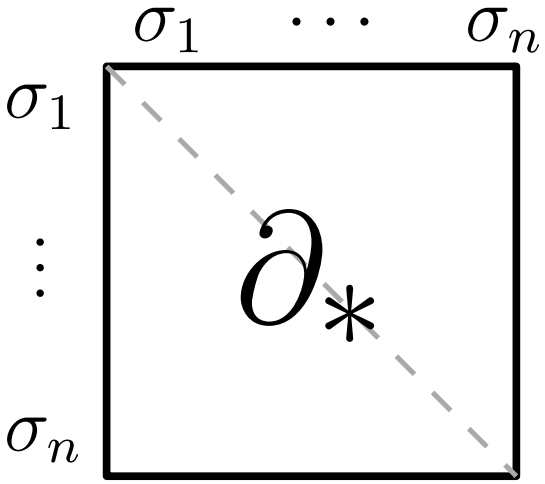
Persistent Homology in Practice



Persistent Homology: Standard Algorithm

- Store the matrix of the boundary operator of the final complex
- Reduce it from left-to-right with column operations



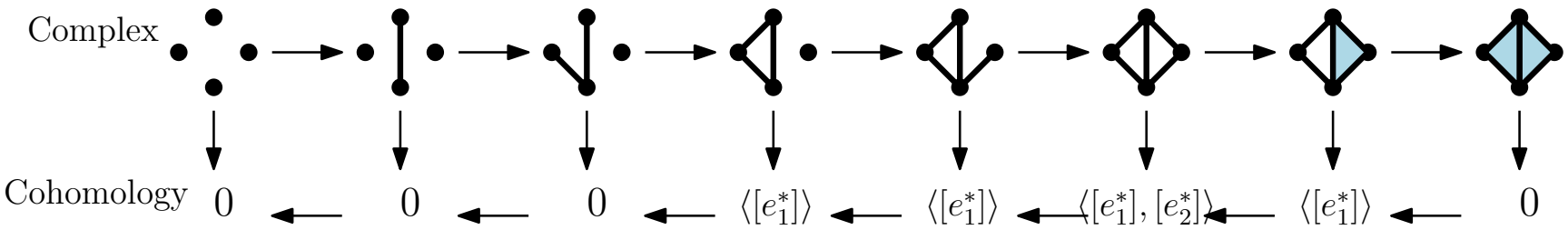


2
Challenge

Practical Performance

[Boissonnat, Dey, M. '13]

On a machine with 3.00 GHz processor and 32 GB RAM,
on a variety of datasets, with different coefficients for cohomology:



Maximal complex size [1]:	$\approx 500 \cdot 10^6$ simplices
Timings [2]:	$2.7 \cdot 10^{-7} \leq \cdot \leq 9.1 \cdot 10^{-7}$ s. per simplex
Size cohomology groups:	negligeable compared to complex

Running time: less than 5 minutes

[1] with a *Simplex Tree*

[2] with a *Compressed Annotation Matrix for persistent cohomology*

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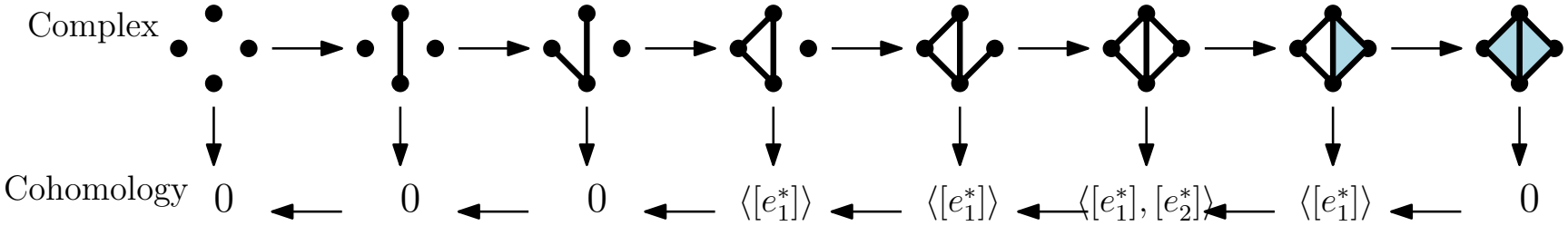

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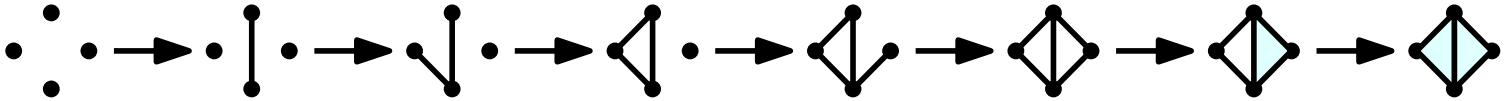
(Recent) Solutions

Standard Persistence: Simplicial Complexes connected by Inclusions



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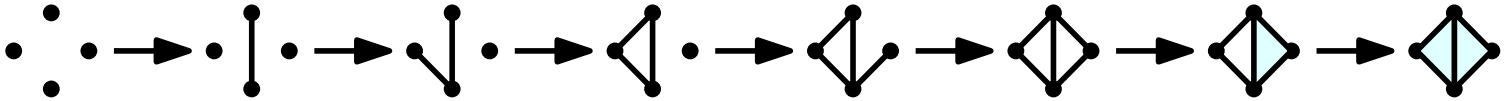
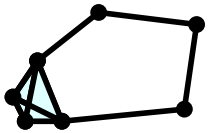


General Cell Complexes  $\cong$  [Mischaikow, Nanda '13]

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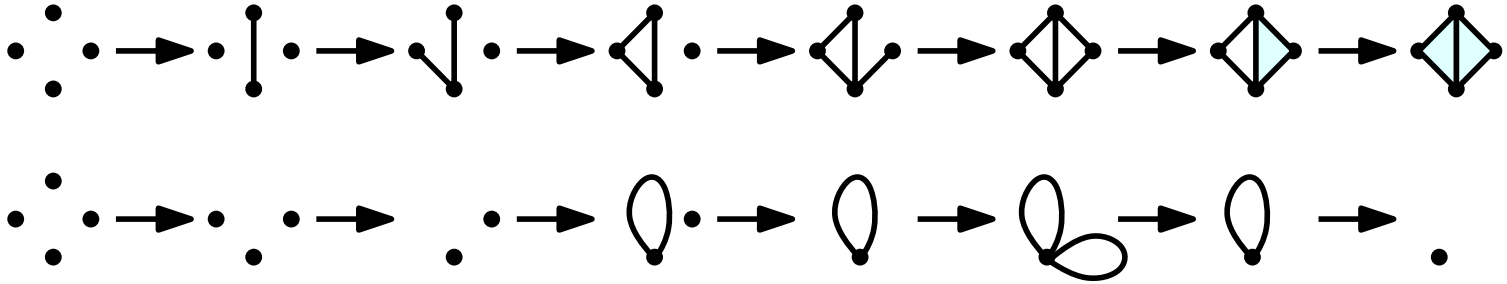
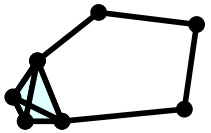


General Cell Complexes  $\cong$  [Mischaikow, Nanda '13]

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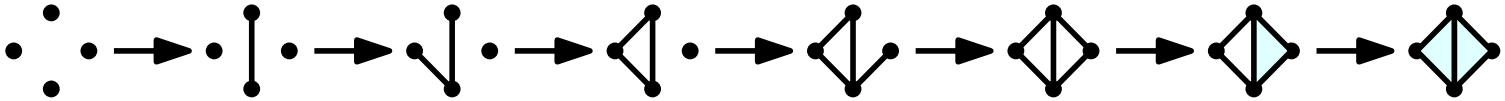
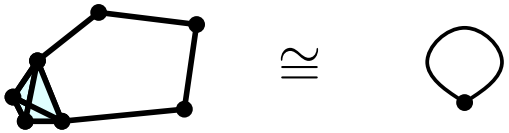
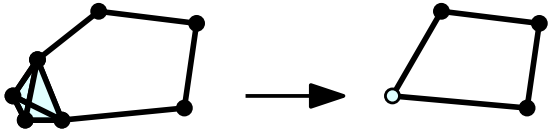


General Cell Complexes  [Mischaikow, Nanda '13]

General Simplicial Maps  [Sheehy '13] [Dey, Fan, Wang '13]

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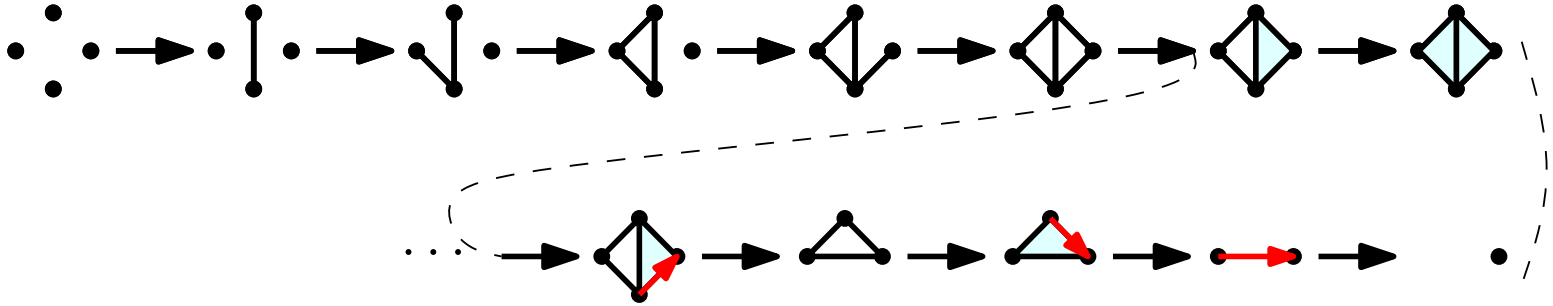

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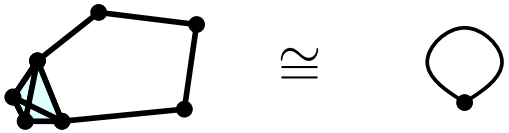

[Mischaikow, Nanda '13]

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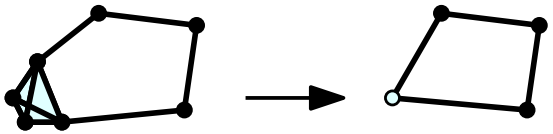

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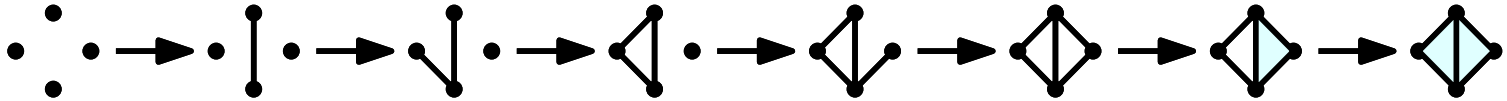
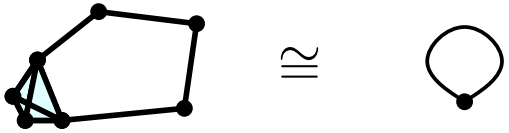
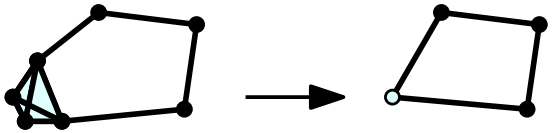


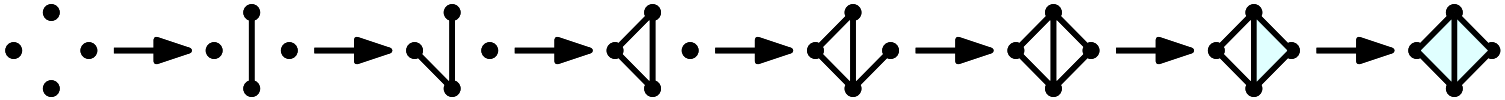
General Cell Complexes  [Mischaikow, Nanda '13]

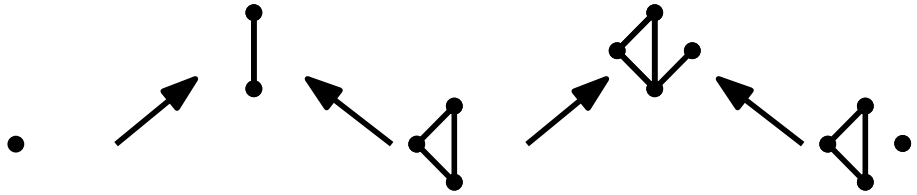
General Simplicial Maps  [Sheehy '13] [Dey, Fan, Wang '13]

Zigzag Persistence  [Carlsson, de Silva, Morozov '09] [M, Oudot '15]

(Recent) Solutions

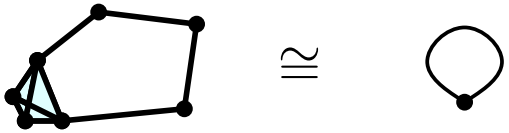
Standard Persistence: Simplicial Complexes connected by Inclusions





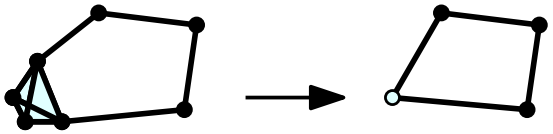
[Oudot, Sheehy '13]

General Cell Complexes



[Mischaikow, Nanda '13]

General Simplicial Maps



[Sheehy '13]
[Dey, Fan, Wang '13]

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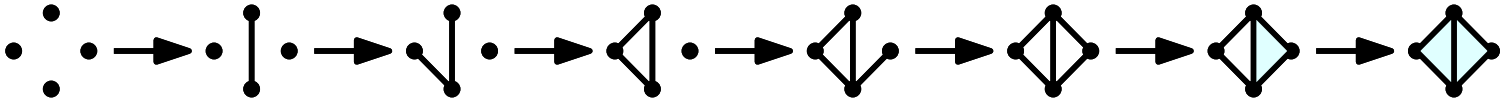
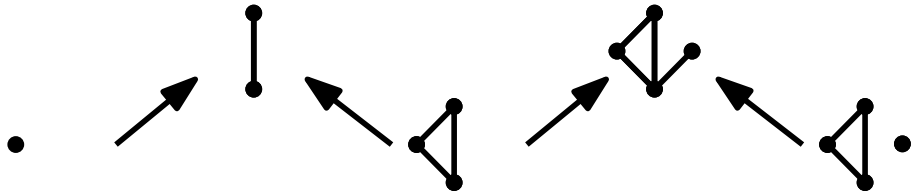
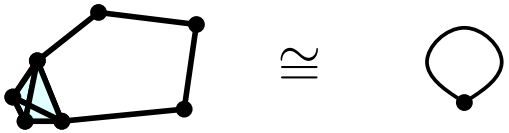
Standard Persistence: Simplicial Complexes connected by Inclusions

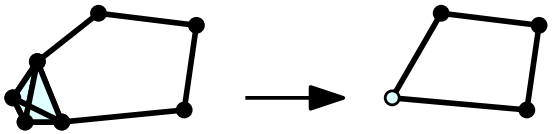




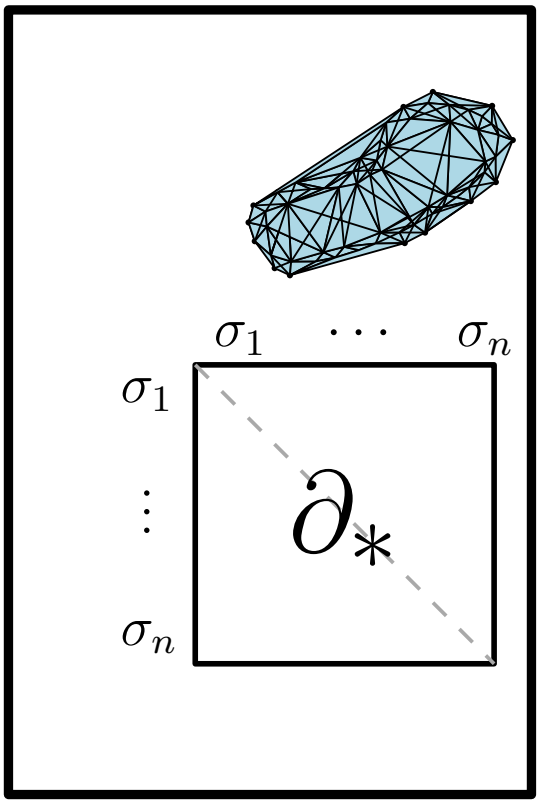
General Cell Complexes



General Simplicial Maps

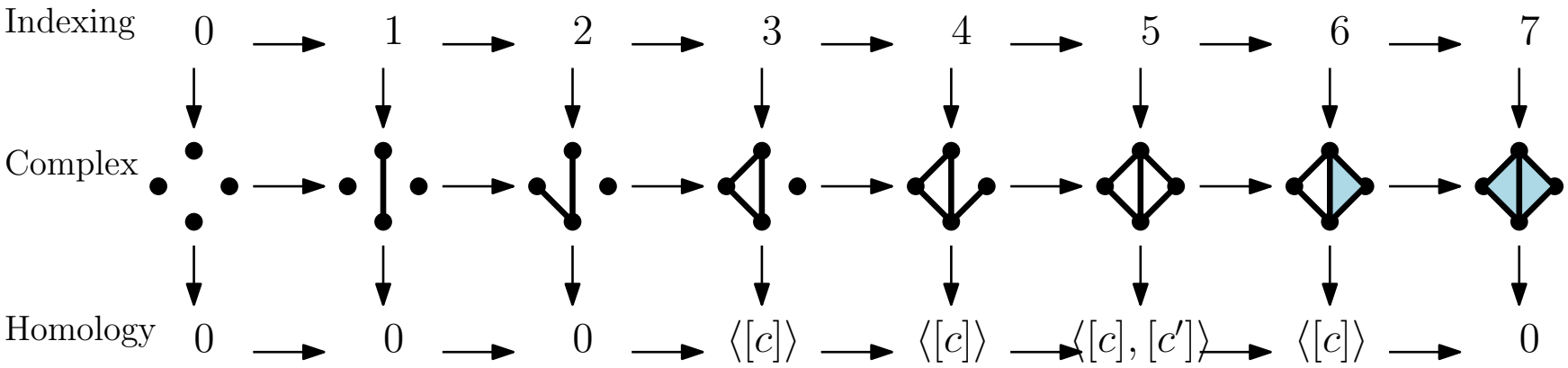


Zigzag Persistence



3
Software Design

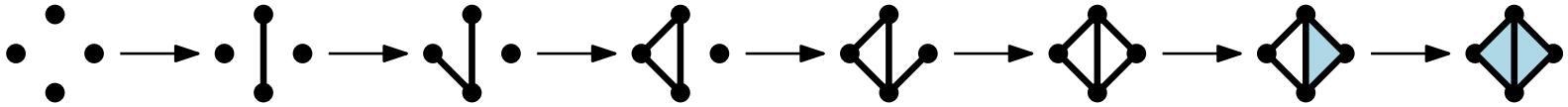
Distinguish Indexing - Complex - Homology



Distinguish Indexing - Complex - Homology

Indexing $0 \longleftrightarrow 1 \longleftrightarrow 2 \longleftrightarrow 3 \longleftrightarrow 4 \longleftrightarrow 5 \longleftrightarrow 6 \longleftrightarrow 7$

Complex



Homology $0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \langle [c] \rangle \longrightarrow \langle [c] \rangle \longrightarrow \langle [c], [c'] \rangle \longrightarrow \langle [c] \rangle \longrightarrow 0$

Distinguish Indexing - Complex - Homology

Indexing $0 \longleftrightarrow 1 \longleftrightarrow 2 \longleftrightarrow 3 \longleftrightarrow 4 \longleftrightarrow 5 \longleftrightarrow 6 \longleftrightarrow 7$

Creation of a concept *IndexingTag*

- `linear-indexing-tag` **for standard persistence**
- `zigzag-indexing-tag` **for zigzag persistence**

Complex

Homology $0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \langle [c] \rangle \longrightarrow \langle [c] \rangle \longrightarrow \langle [c], [c'] \rangle \longrightarrow \langle [c] \rangle \longrightarrow 0$

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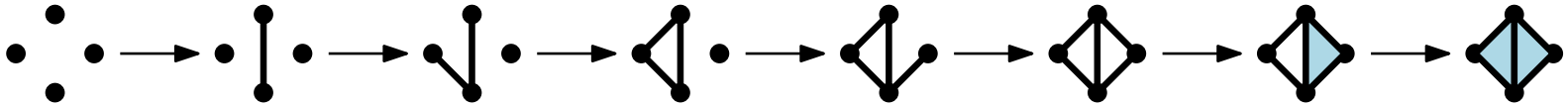

Creation of a concept *FilteredComplex*

- **type** `Simplex-handle`
- **function** `int dimension(Simplex-handle)`
- **function** `Boundary-simplex-range boundary-simplex-range(Simplex-handle)`
- ...

Homology $0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \langle [c] \rangle \longrightarrow \langle [c] \rangle \longrightarrow \langle [c], [c'] \rangle \longrightarrow \langle [c] \rangle \longrightarrow 0$

Distinguish Indexing - Complex - Homology

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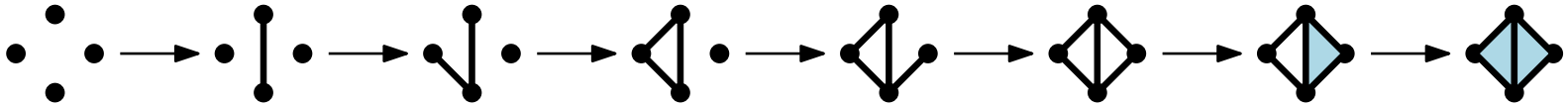

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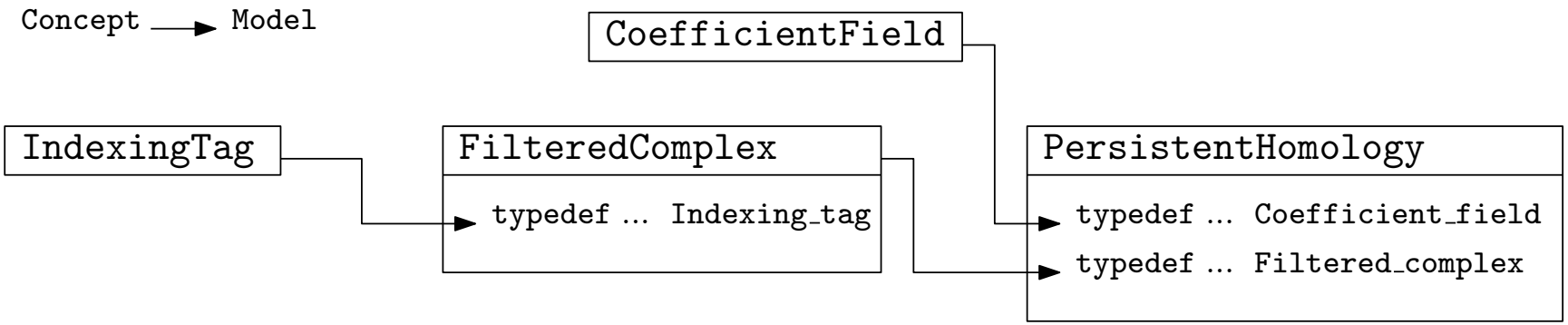
Homology $0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \langle [c] \rangle \longrightarrow \langle [c] \rangle \longrightarrow \langle [c], [c'] \rangle \longrightarrow \langle [c] \rangle \longrightarrow 0$

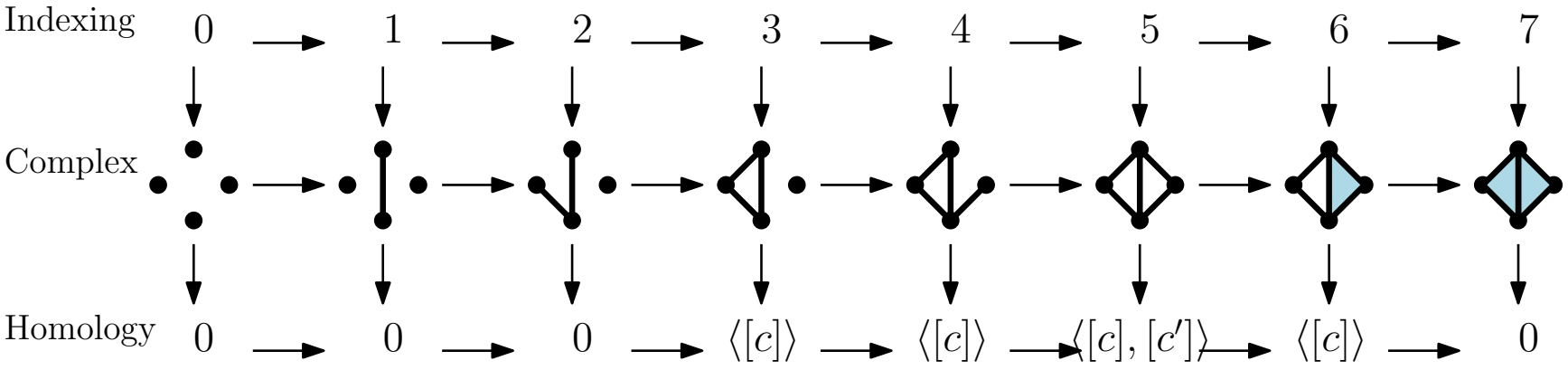
Creation of a concept *PersistentHomology*

- **type** `Coefficient-field`
- ...

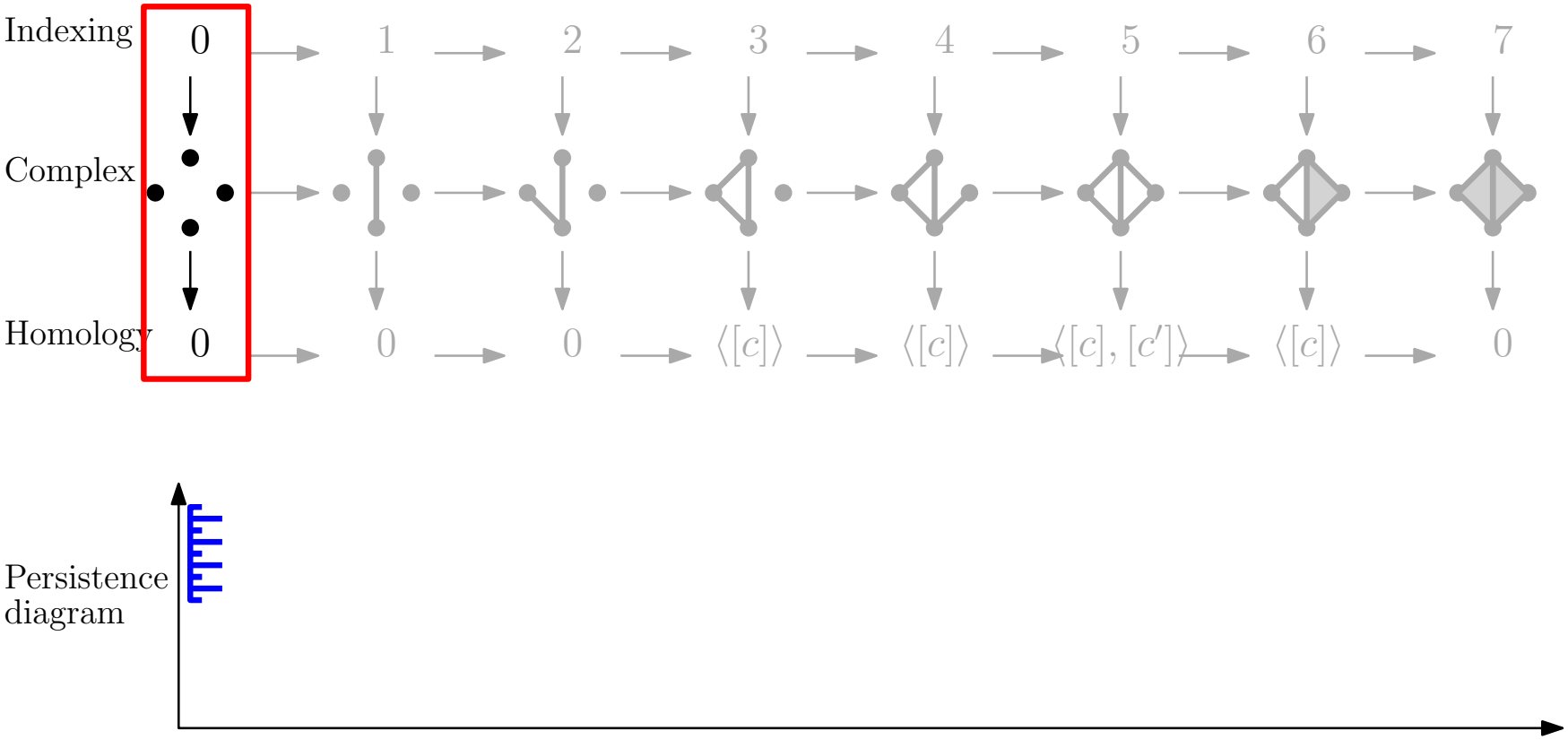
Concepts and Relations



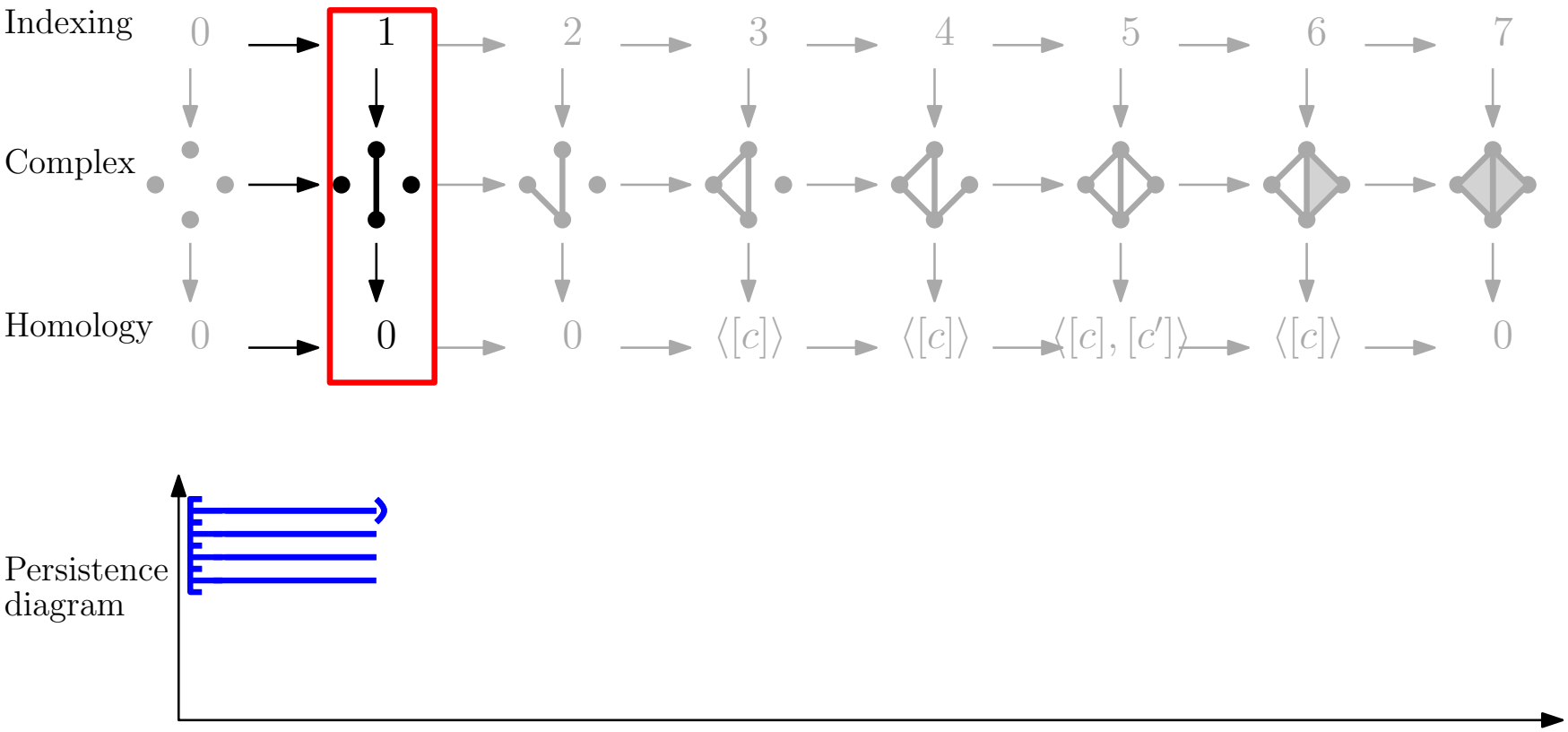
Live in the Present



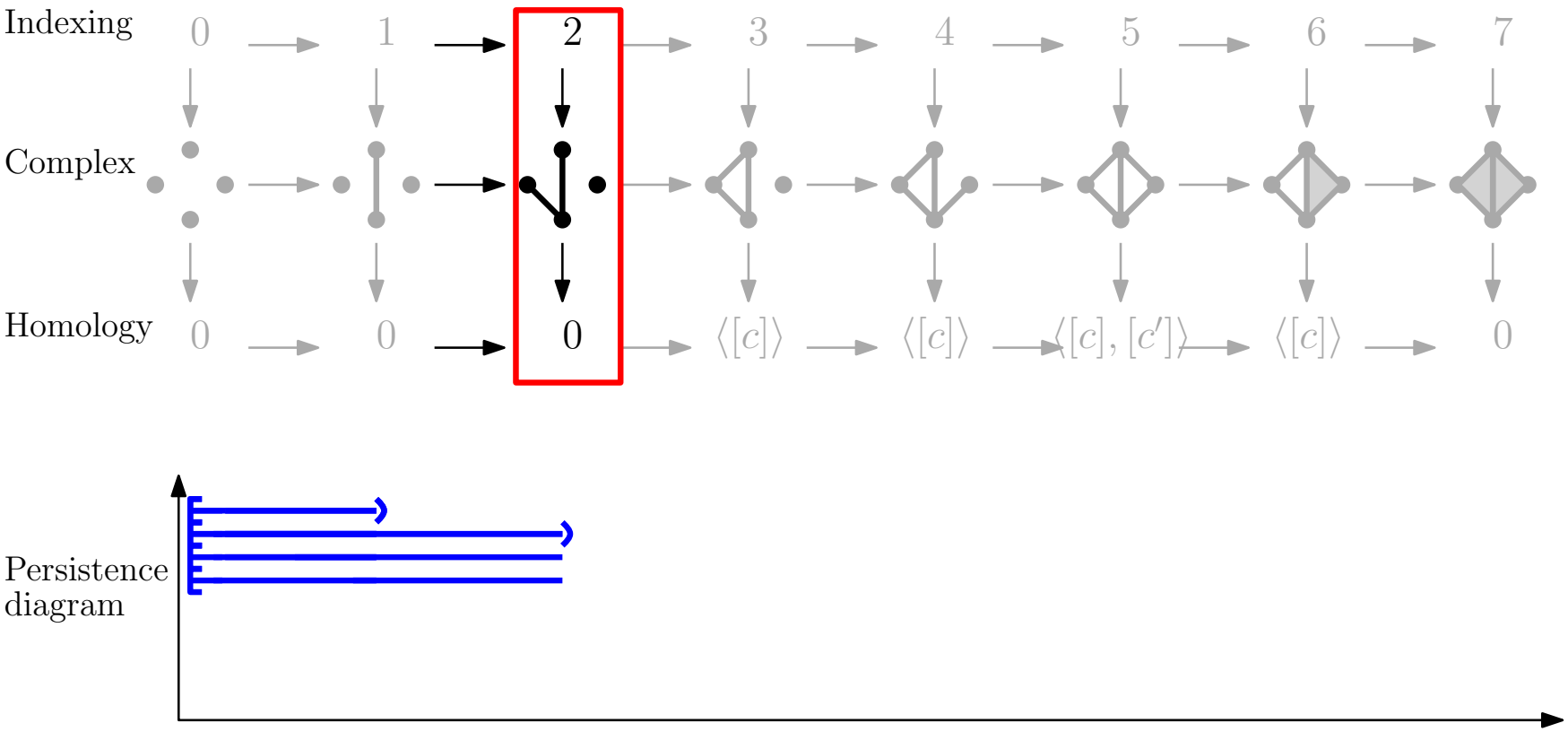
Live in the Present



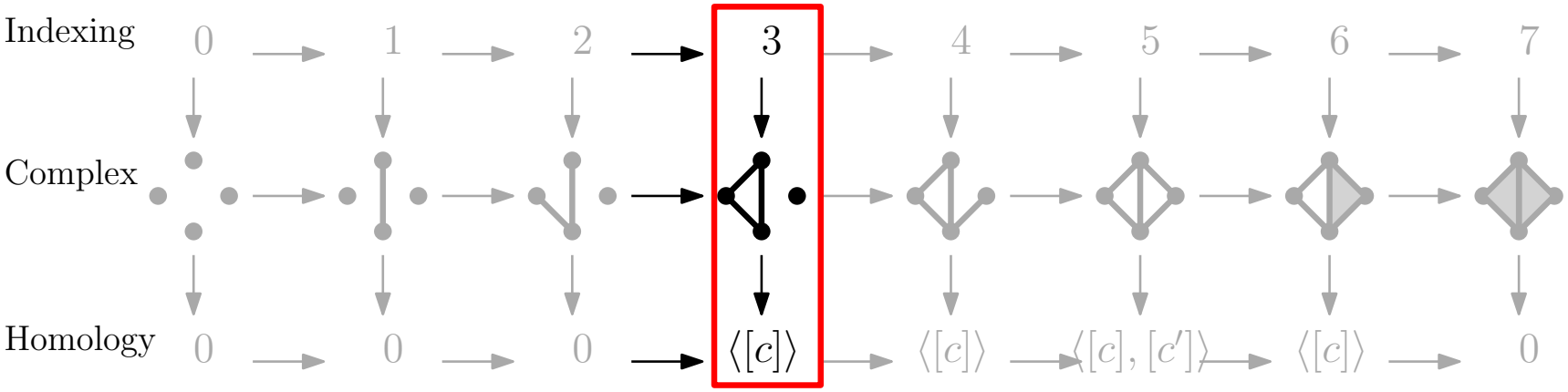
Live in the Present

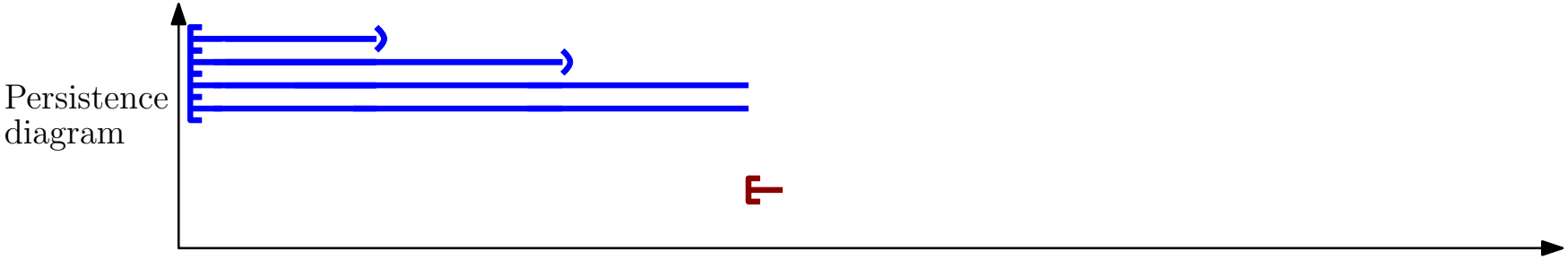


Live in the Present

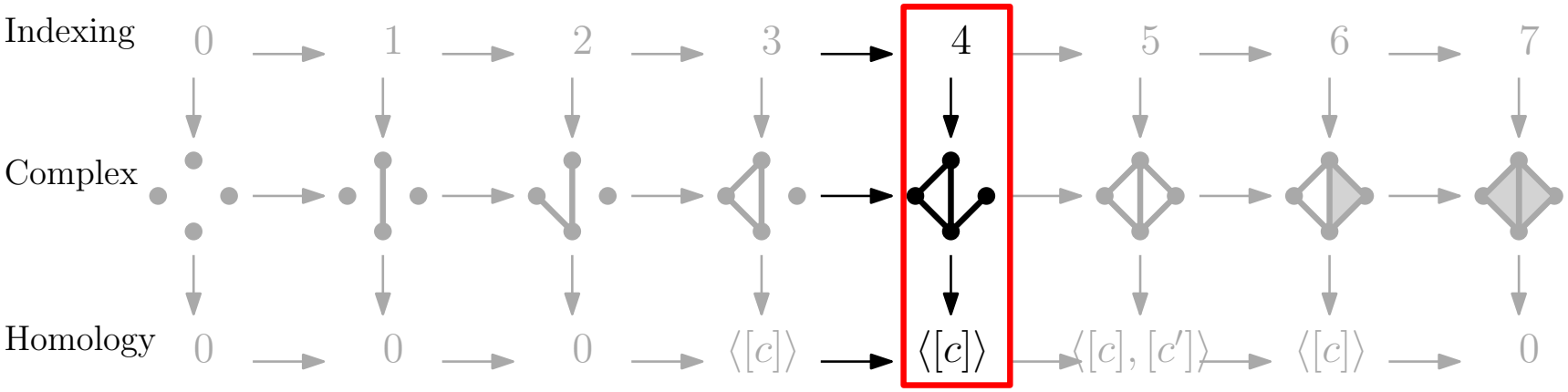


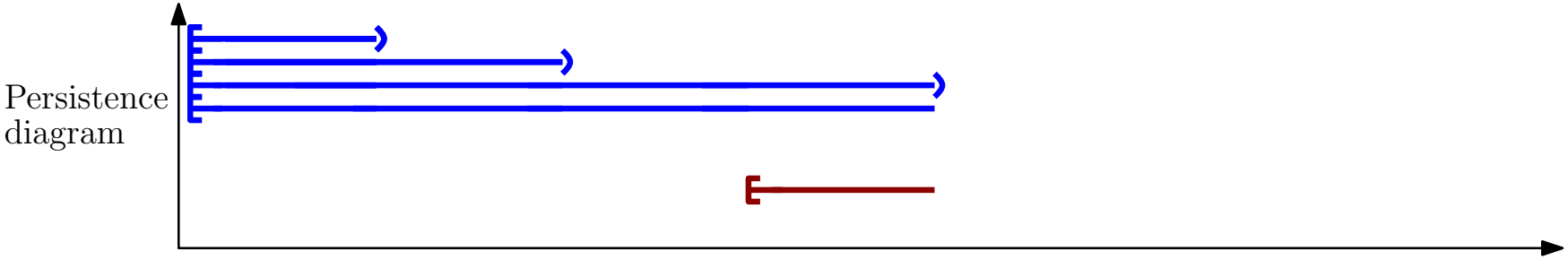
Live in the Present



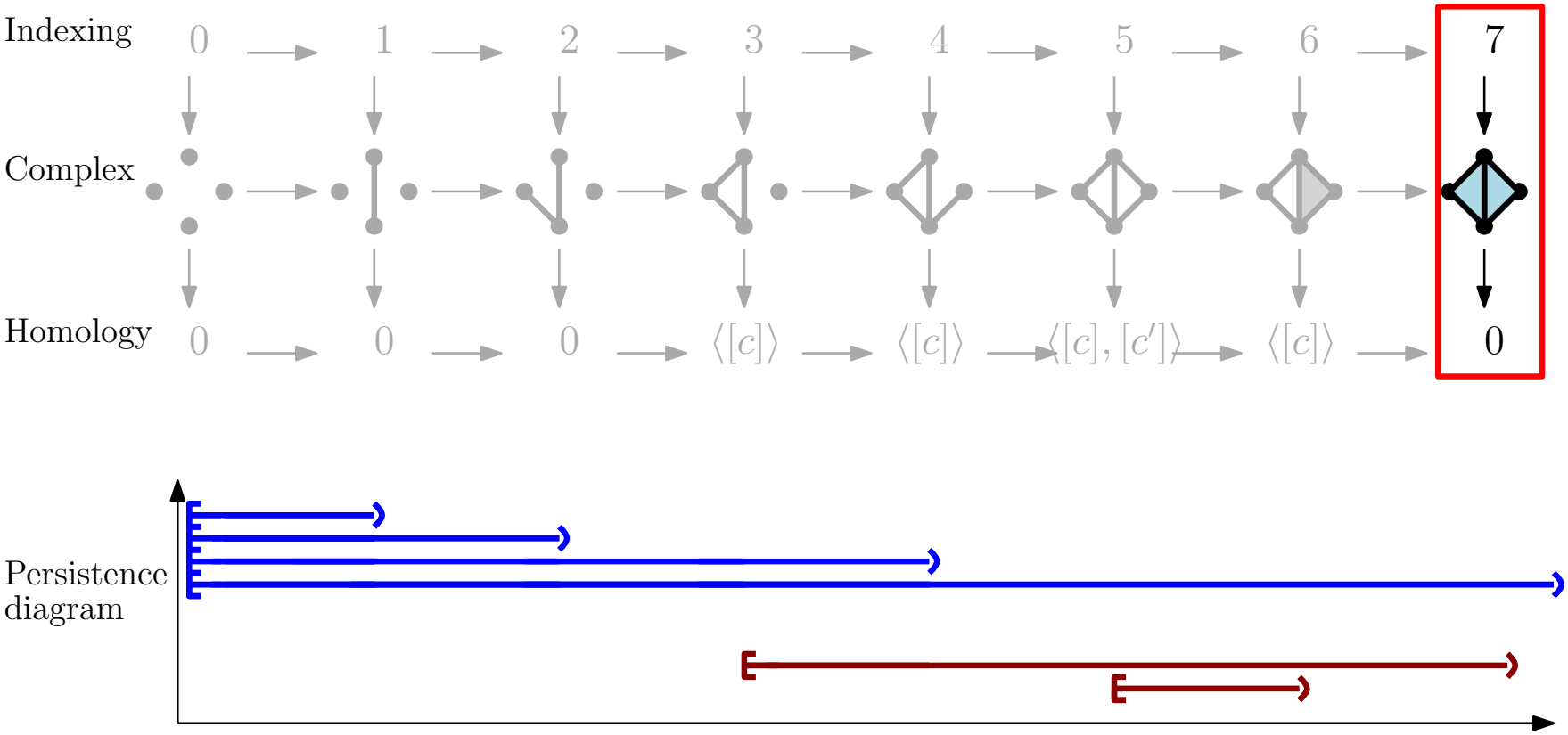


Live in the Present





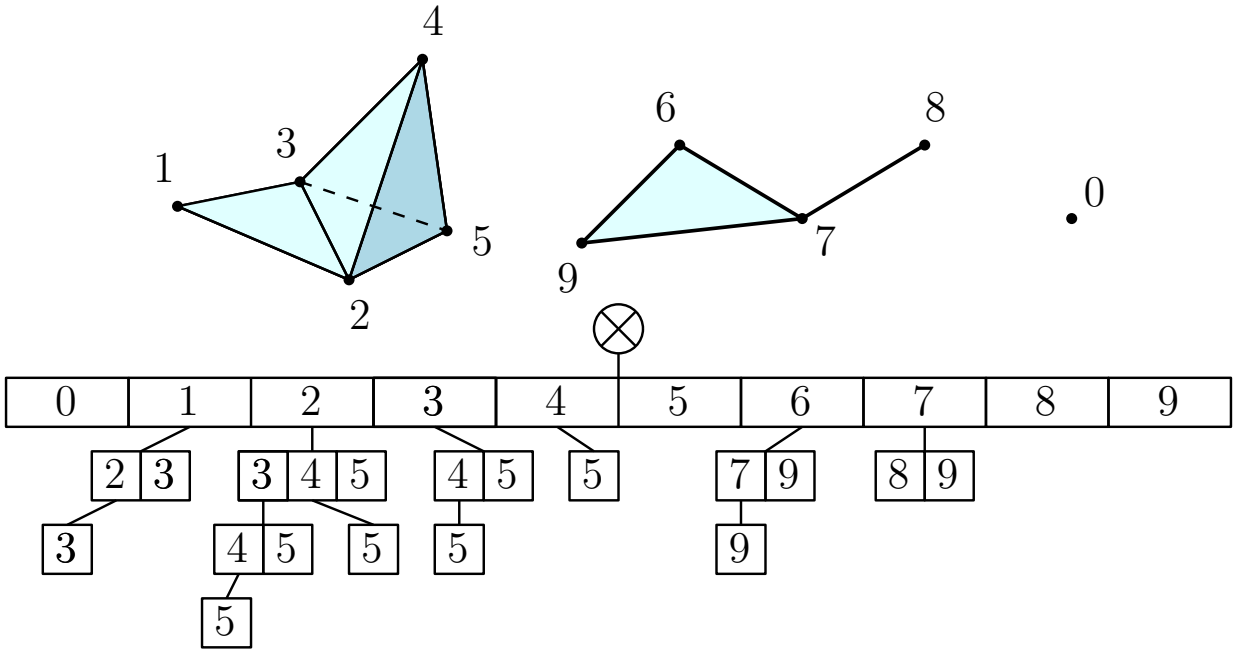
Live in the Present



4
Implementation

Model of *FilteredComplex*: `Simplex_tree`

Simplex Tree for Simplicial Complexes: [Boissonnat, M. '12]

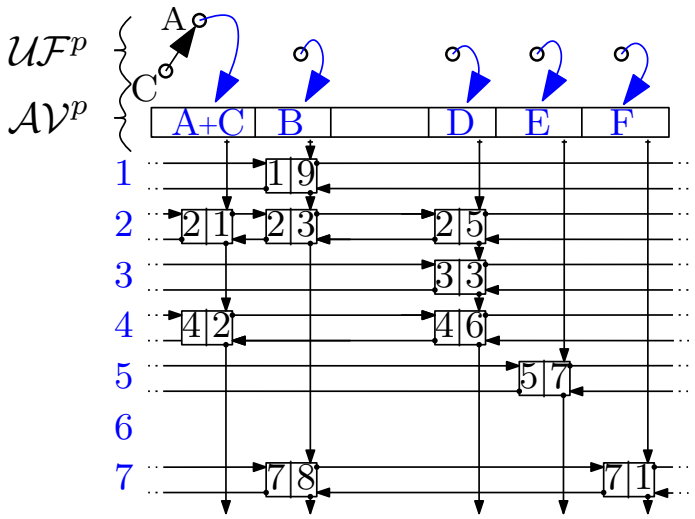


- $O(1)$ memory word per simplex (optimal)
- fast boundary computation
- fast construction of geometric complexes (Rips complex, etc)
- very dynamic: addition/removal simplices, edge contraction ...

Model of *PersistentHomology*: `Persistent_cohomology`

Compressed Annotation Matrix: [Boissonnat, Dey, M. '13]

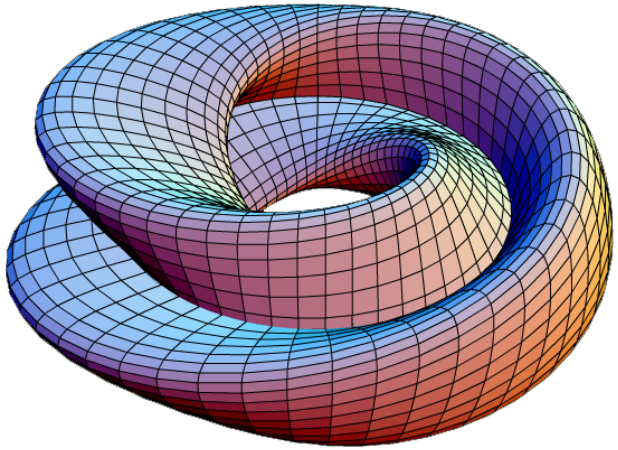
$$
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7
\end{array}
\begin{array}{c}
\begin{array}{cccccc} A & B & C & D & E & F \end{array} \\
\begin{pmatrix}
0 & 9 & 0 & 0 & 0 & 0 \\
1 & 3 & 1 & 5 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 0 \\
2 & 0 & 2 & 6 & 0 & 0 \\
0 & 0 & 0 & 0 & 7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 8 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}
$$

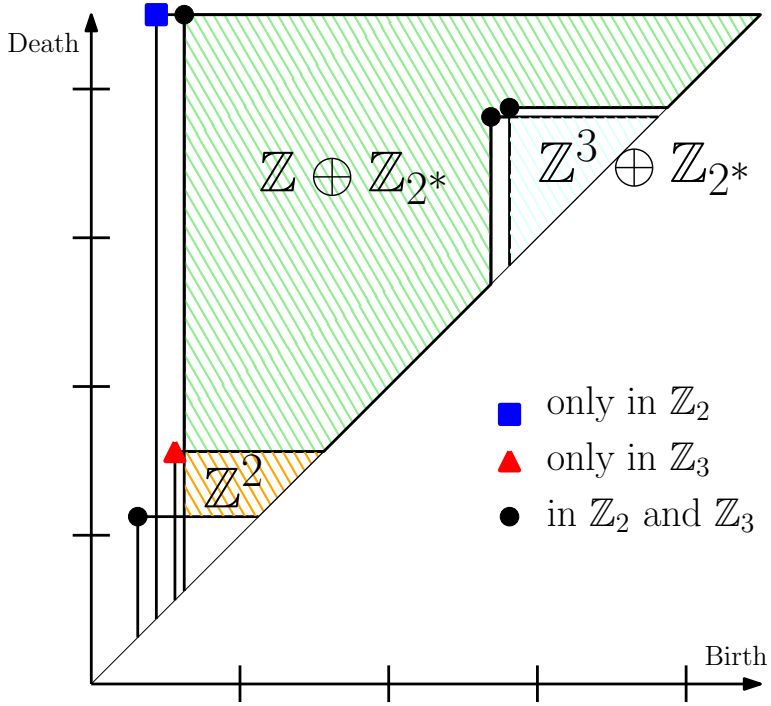


- begin with an empty structure (represent the cohomology groups)
- remain very small all along the computation
- very dynamic: fast update under simplex addition, edge contraction

Model of *CoefficientField*: `Field_Zp` and `Multi_field`

Multi-Field Persistent Homology: [Boissonnat, M. '14]





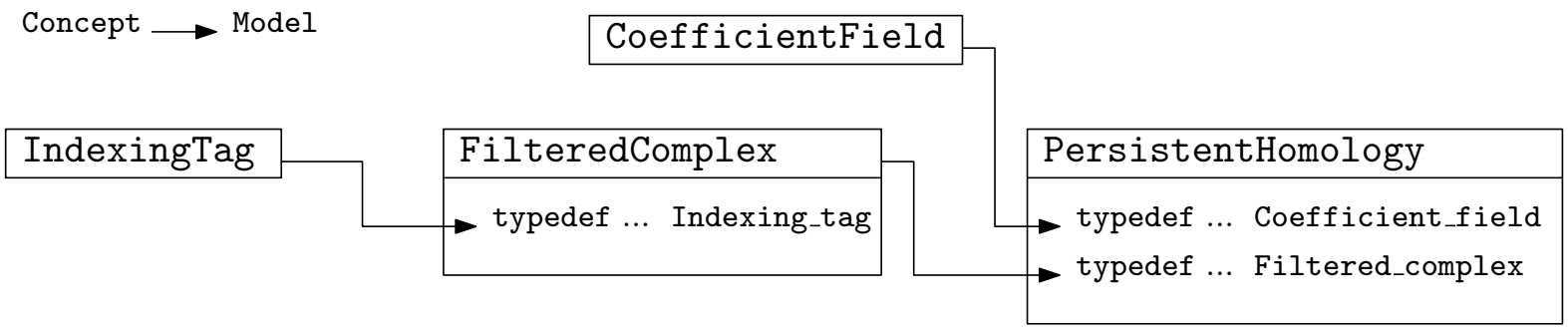
Field:		$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{p\geq 3}$
$\mathbf{H}_0(\mathbb{Z})$	$=$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_p$
$\mathbf{H}_1(\mathbb{Z})$	$=$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_p$
$\mathbf{H}_2(\mathbb{Z})$	$=$	0	$\mathbb{Z}_2$	0

Some Experiments

Data	$\lvert\mathcal{P}\rvert$	D	d	r	$\lvert\mathcal{K}\rvert$	T_{st}	$T^{\mathsf{ph}}_{\mathbb{Z}_2}$	$T^{\mathsf{ph}}_{\mathbb{Z}_{1223}}$	$T^{\mathsf{ph}}_{\mathbb{Z}^2_{1223}}$
Bud	49,990	3	2	0.09	$127 \cdot 10^6$	5.7	161	161	252
Bro	15,000	25	?	0.04	$142 \cdot 10^6$	5.8	252	252	380
Cy8	6,040	24	2	0.8	$193 \cdot 10^6$	8.4	249	249	325
Kl	90,000	5	2	0.25	$114 \cdot 10^6$	8.3	228	227	401
S3	50,000	4	3	0.65	$134 \cdot 10^6$	7.2	176	176	310

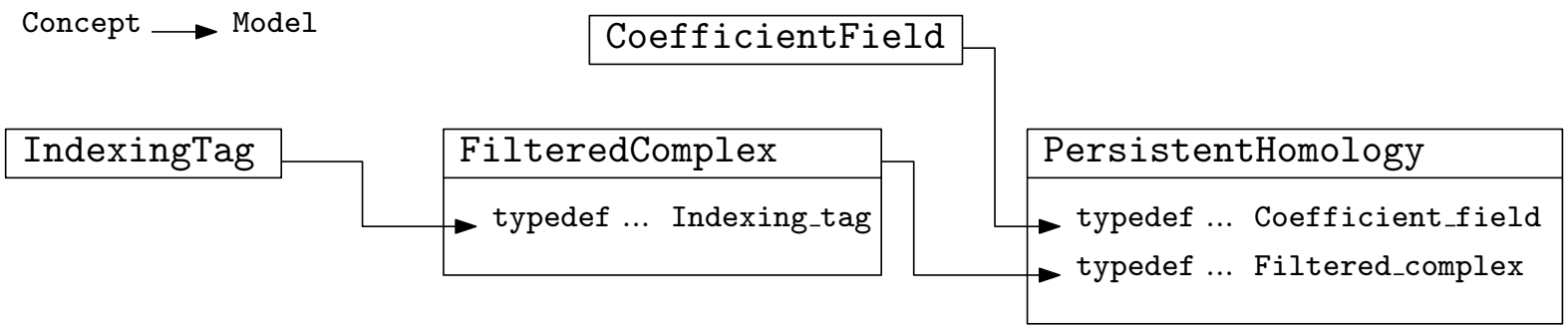
Figure : Timings in seconds for the various algorithms.

Future Directions



- Prototype for zigzag persistence [M., Oudot '15]
- more complexes: witness complex, alpha complex, ... + general cell complexes
- edge contractions and persistence for simplicial maps

Future Directions



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Thank you! Any question?