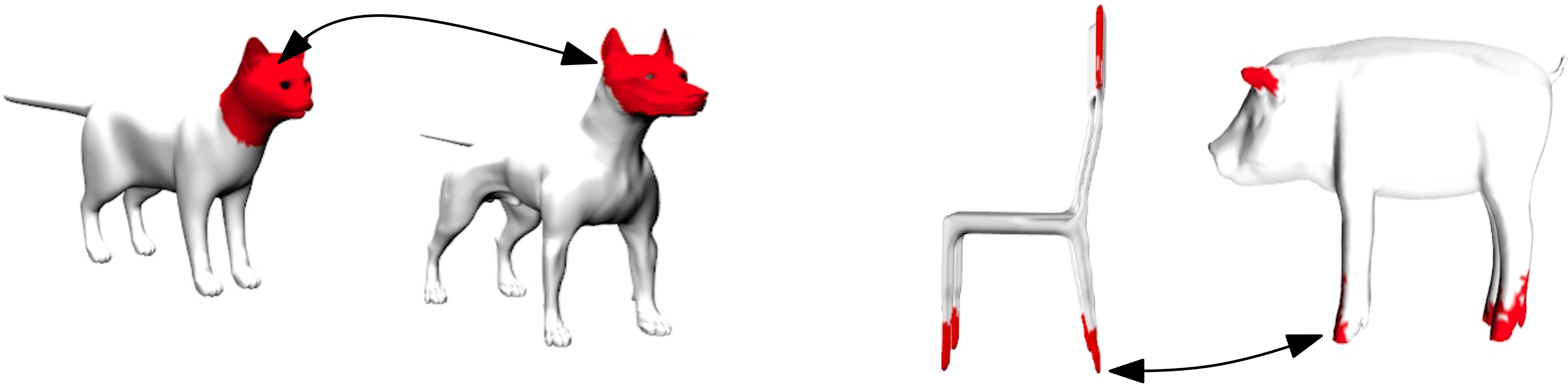
Stable Region Correspondences between non-isometric shapes

Boris Thibert

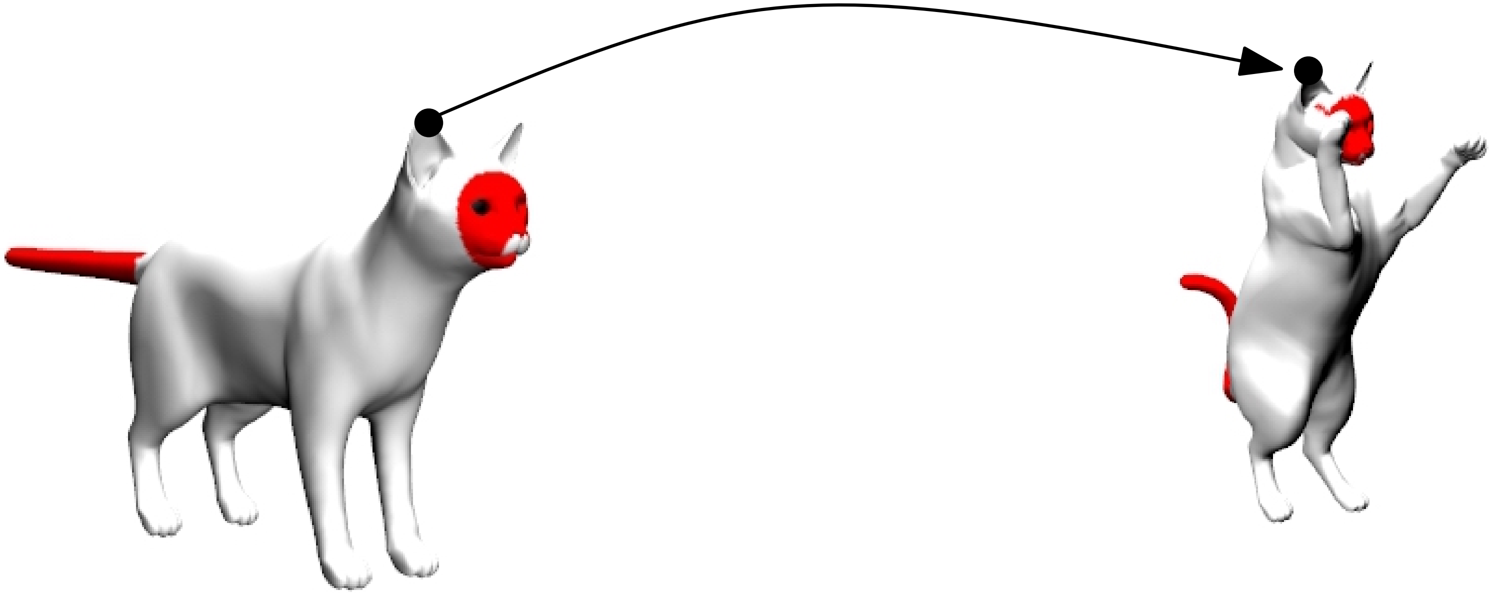
with V.Ganapathi-Subramanian, M. Ovsjanikov, L. Guibas



TopData and Gudhi workshop, October 17,2016

Motivation

Find a map $T : S_1 \rightarrow S_2$??



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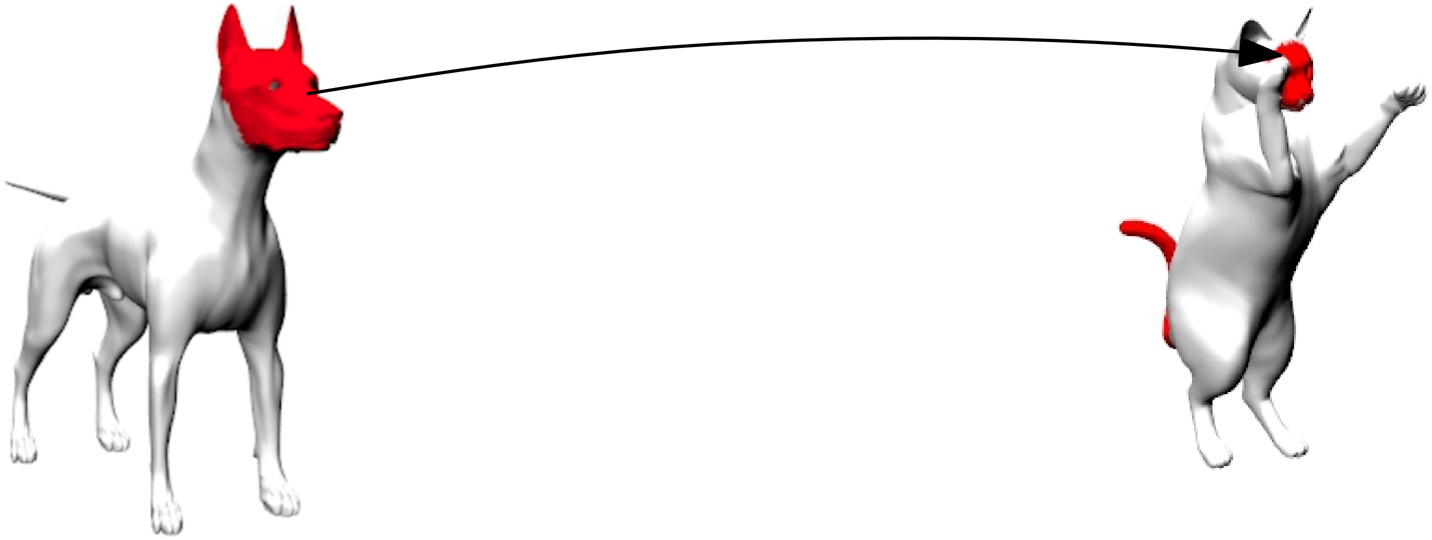

⇒ Find correspondences between regions

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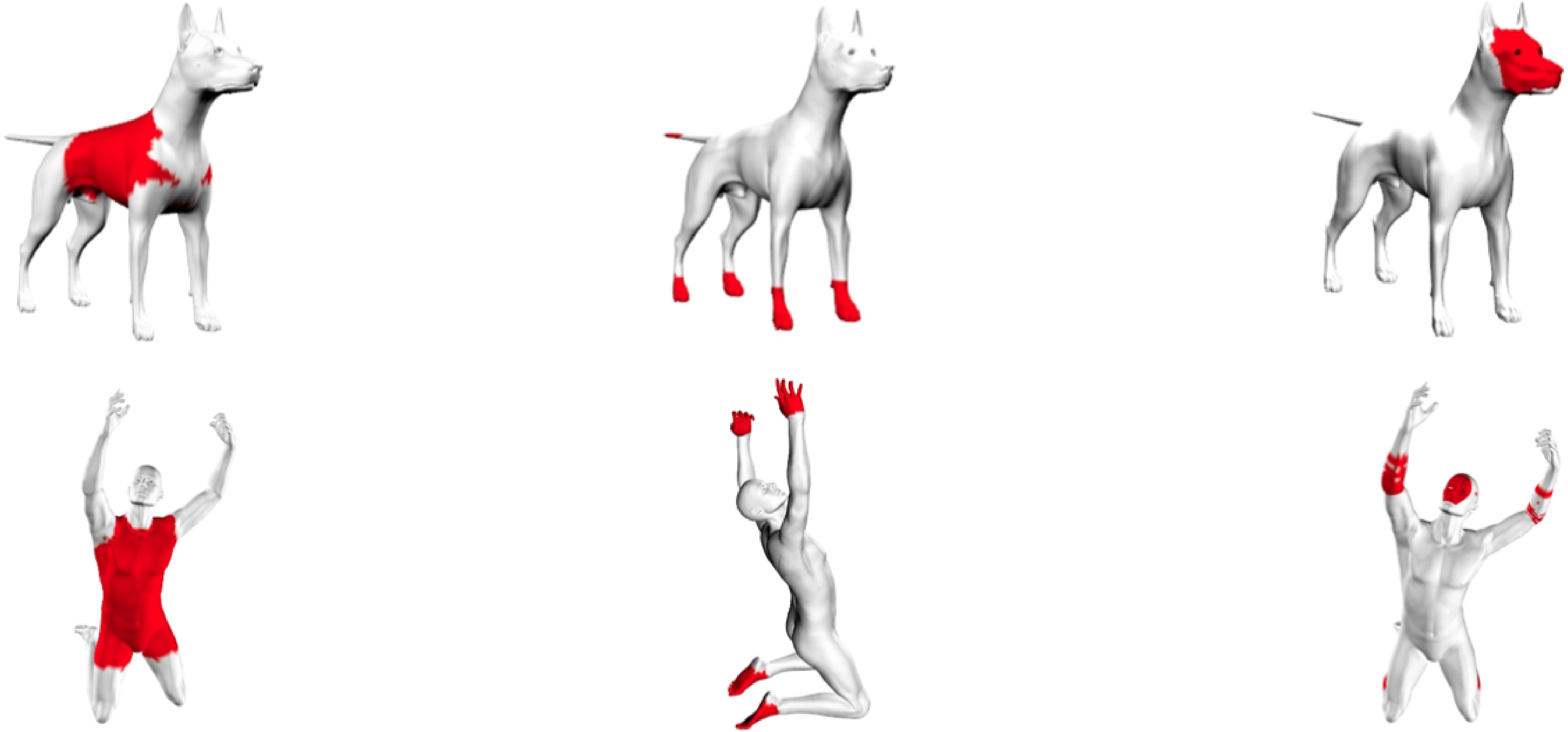

⇒ Find correspondences between regions

An observation

Let $f : S_1 \to \mathbb{R}$ be “geometric” corresponding functions
$d : S_2 \to \mathbb{R}$

Functions such as : WKS, HKS, multiscale mean curvature...

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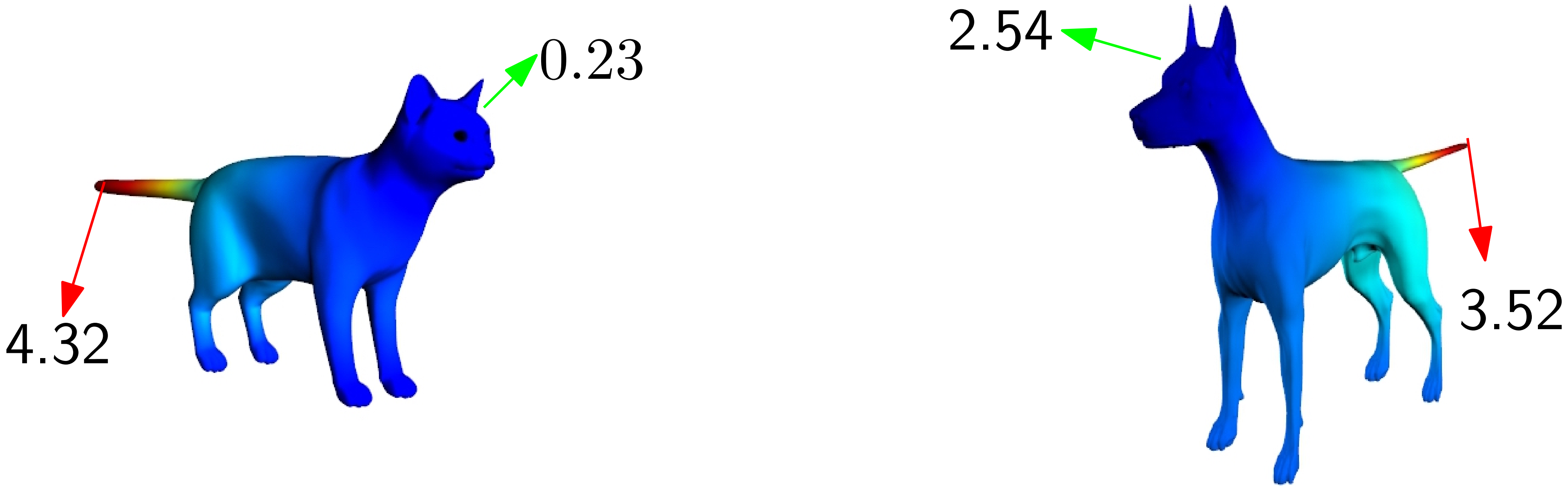

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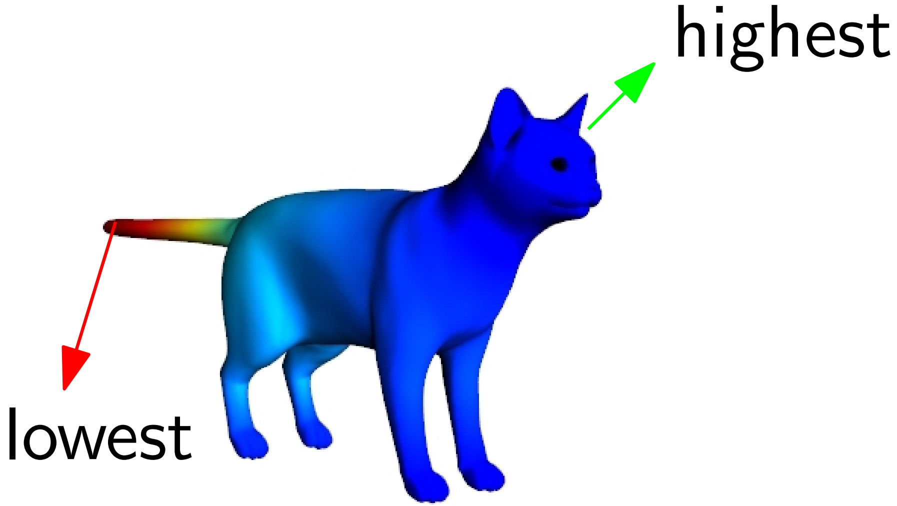

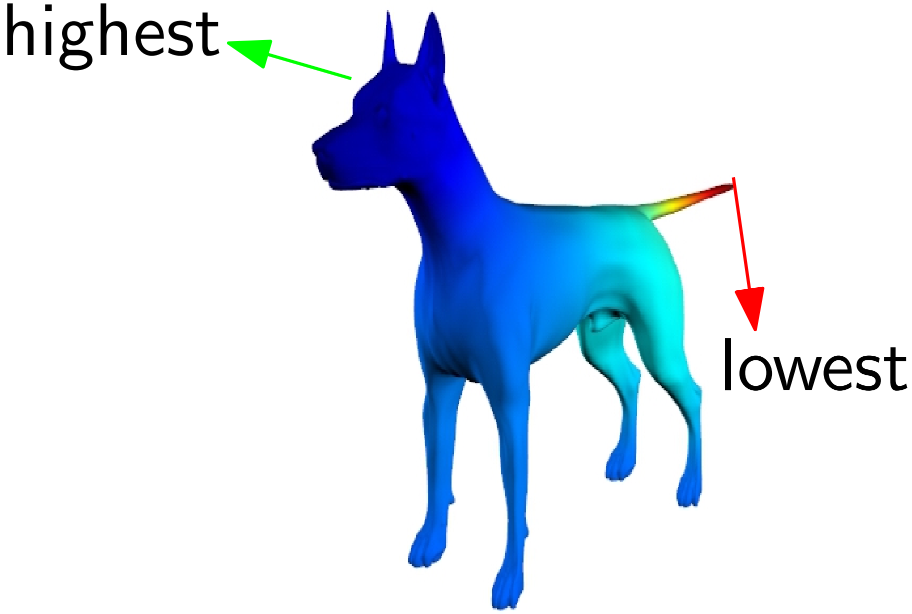

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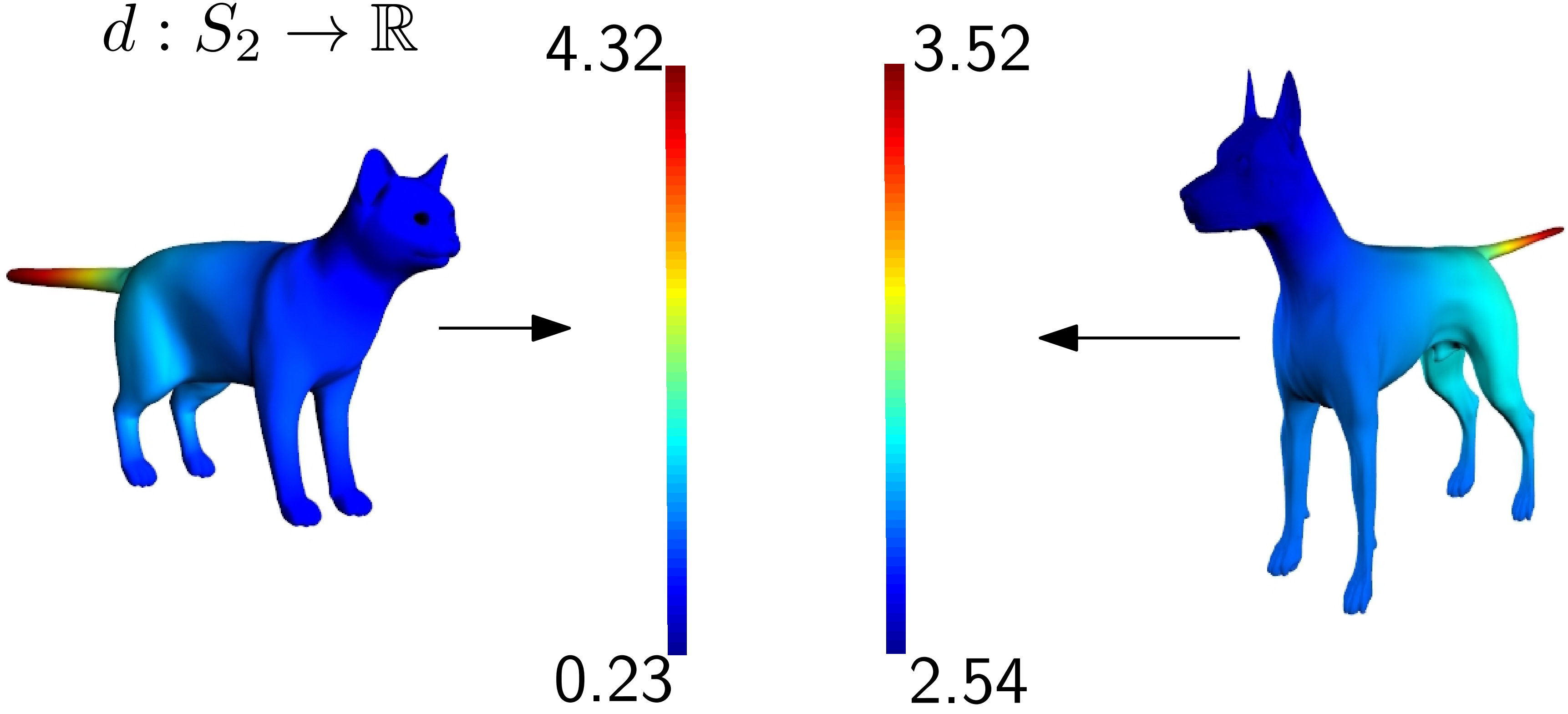

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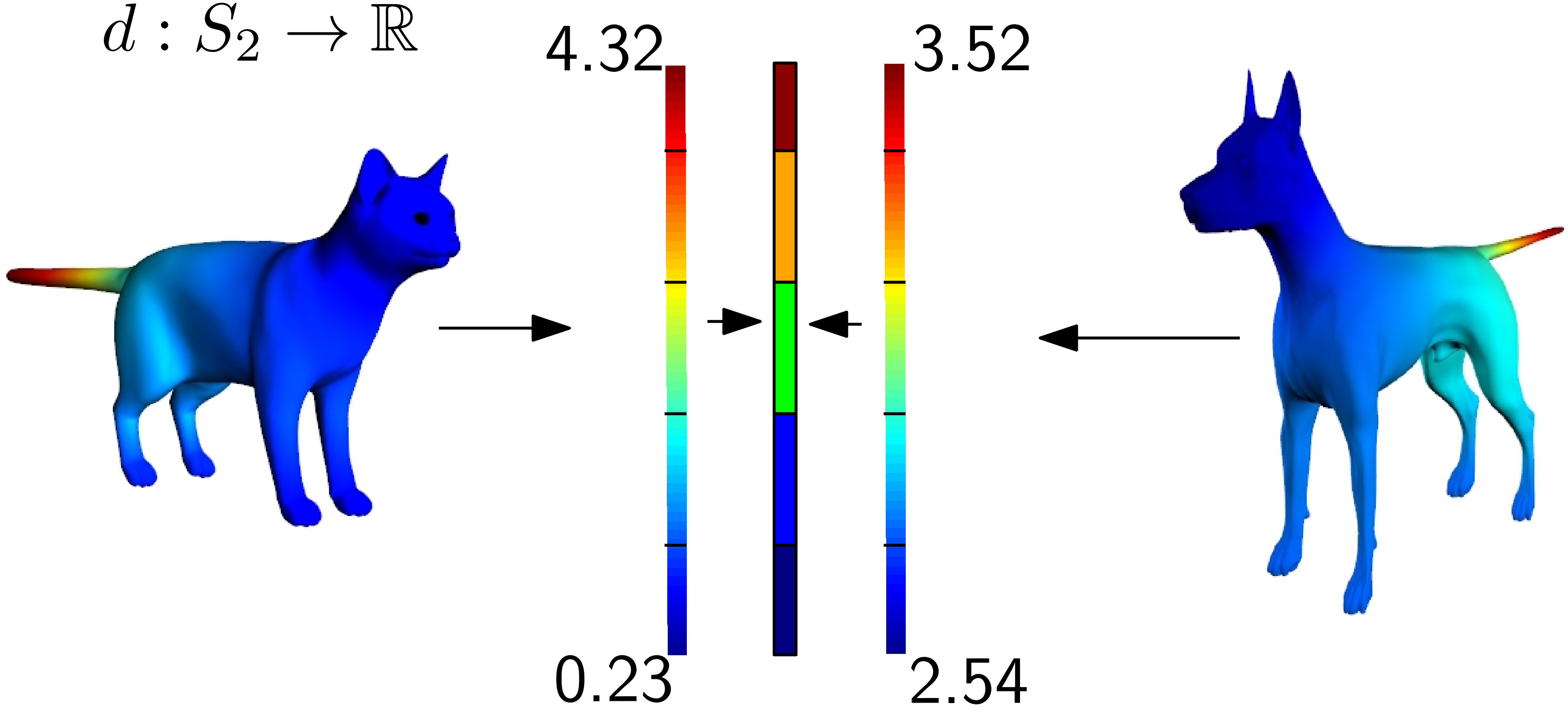

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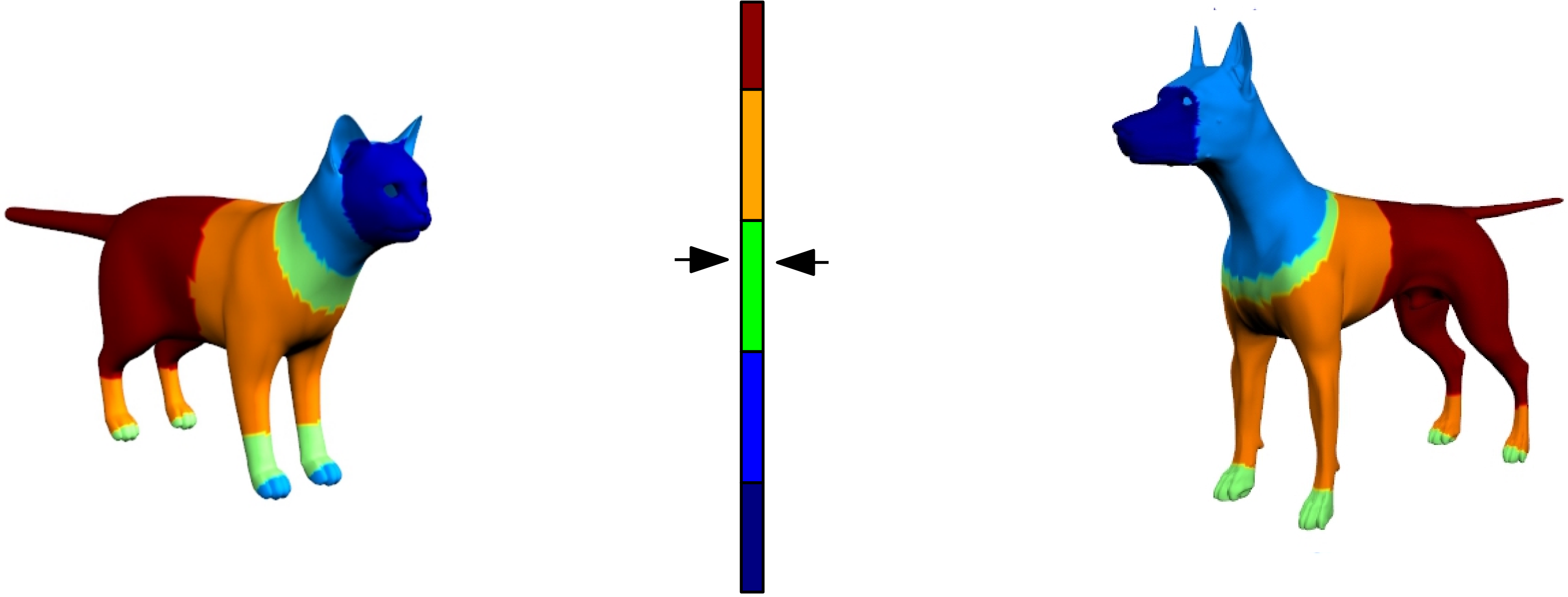
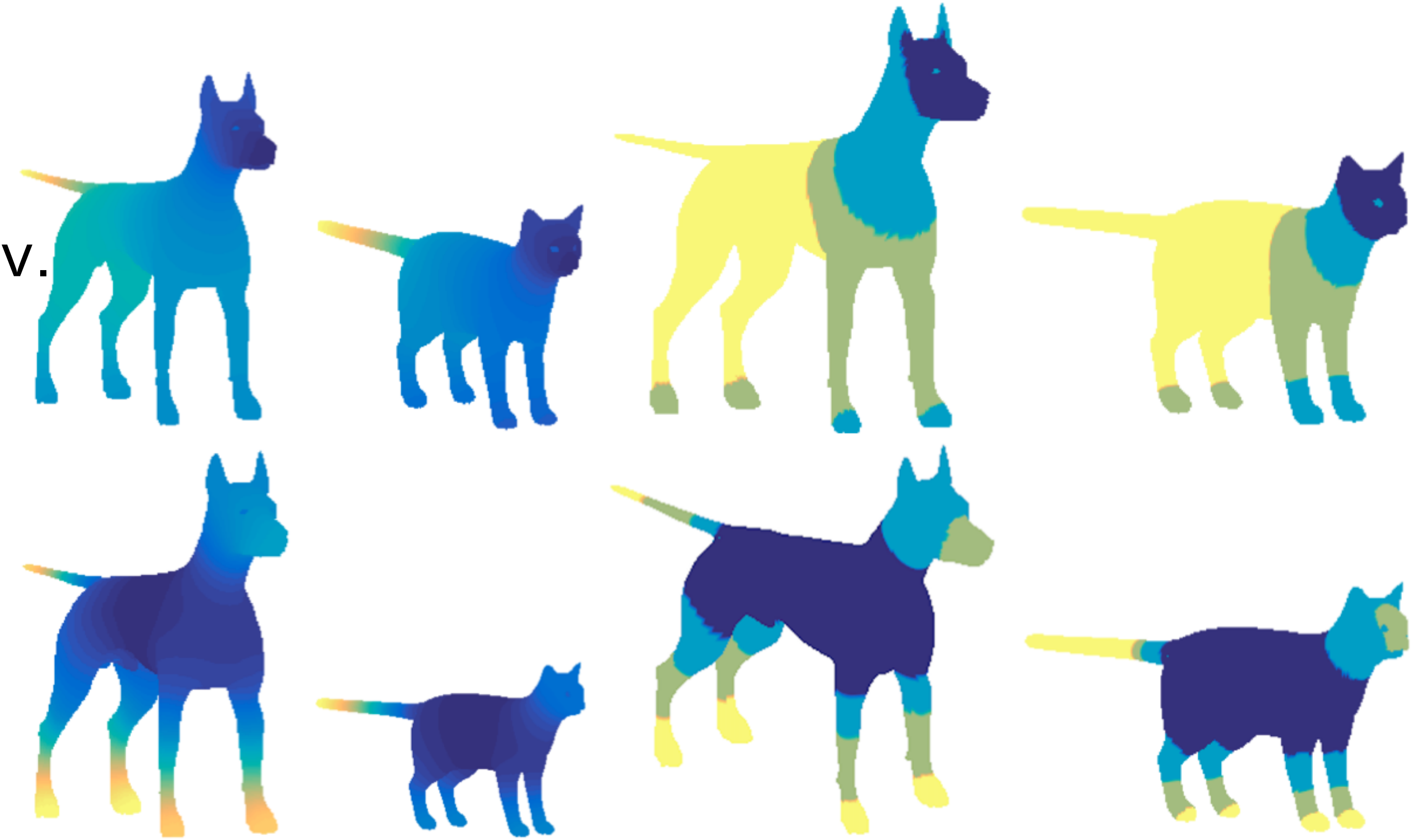


Functions such as : WKS, HKS, multiscale mean curvature...
$\Rightarrow$ high confidence for points with same rank

An observation



⇒ provide consistent information.

Goal

To obtain corresponding regions on both shapes, using a bag of attributes on the two shapes that are likely to correspond.

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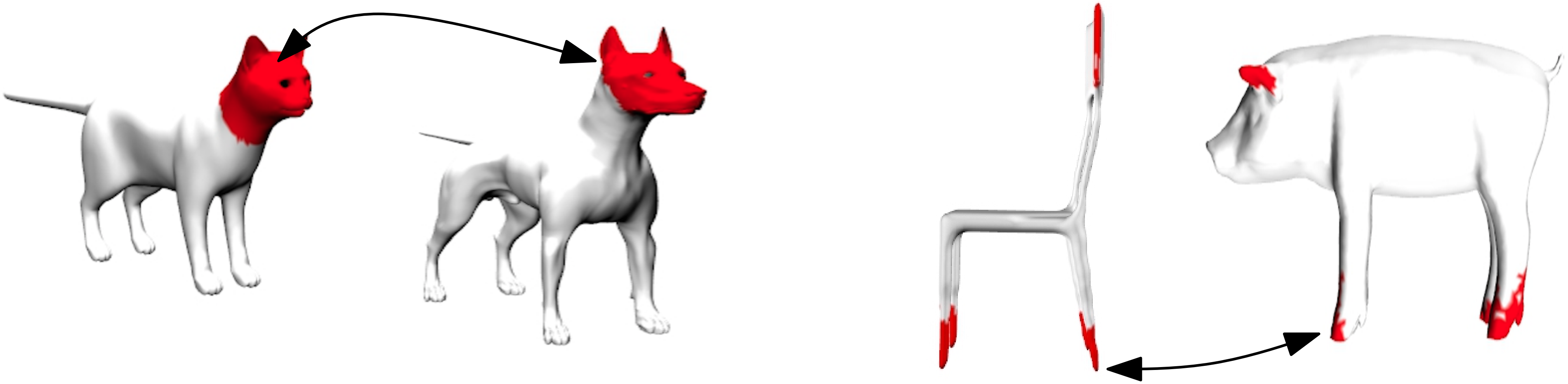
Method Overview

1. To obtain an affinity matrix that measures how well attributes defined on a pair of shapes correspond.
2. To learn regions on the shapes that obey transport consistency.



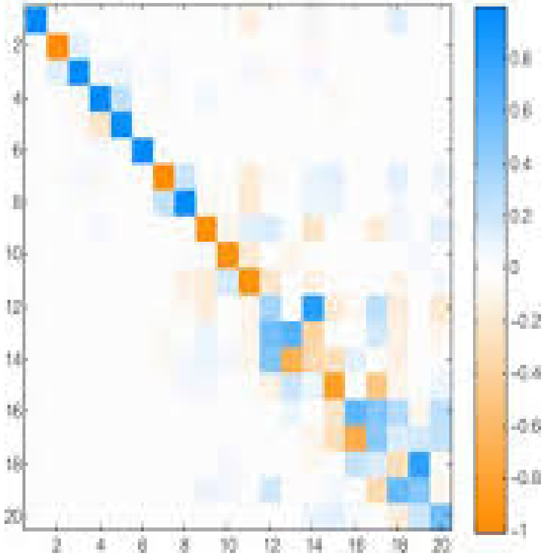
Related work

Functional Maps - Ovsjanikov et al., 2012

Model to compute maps between function spaces defined on shapes

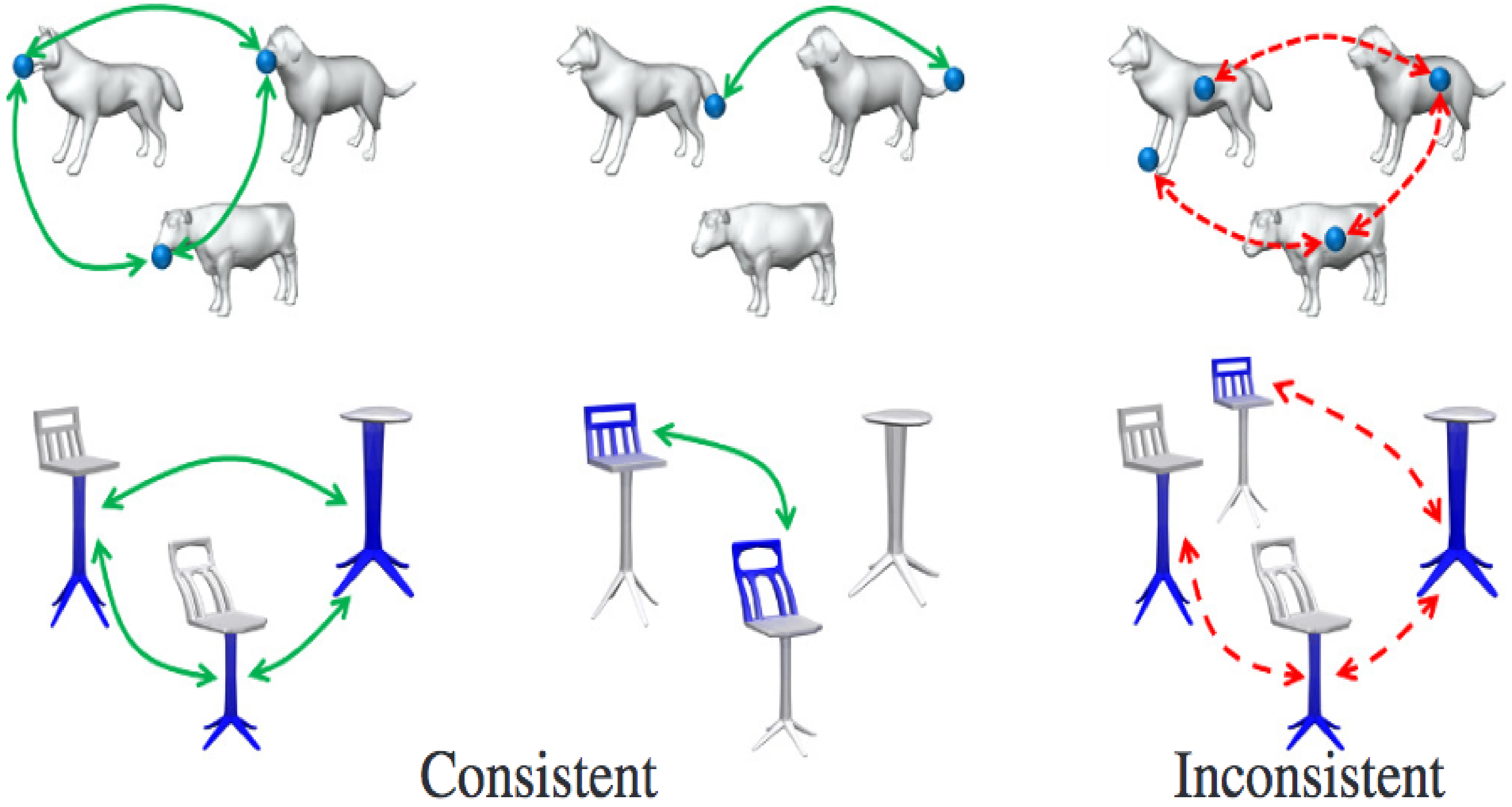
Focuses on aligning function values

Linear map.



Related work

Huang et al., 2014 - using consistency to sharpen functional maps.



Affinity Matrix

Input: $S_1 = \{p_1, \cdots, p_{n_1}\}$, $S_2 = \{q_1, \cdots, q_{n_2}\}$, $K > 1$
$f_i : S_1 \to \mathbb{R}$, $d_i : S_2 \to \mathbb{R}$, with $1 \leq i \leq N$

$w_{ij} = 1$ if $p_i \in S_1$ and $q_j \in S_2$ have similar rank

$N = 1$

$$\begin{array}{cc} & \xleftrightarrow{\qquad\qquad n_1 \qquad\qquad} \\ n_2 \updownarrow & \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

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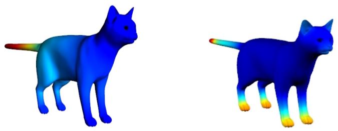
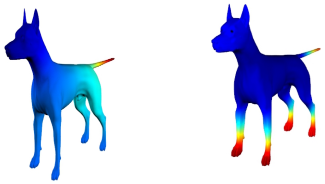


$$\begin{array}{cc} & \xleftrightarrow{\quad\quad\quad n_1 \quad\quad\quad} \\ n_2 \updownarrow & \begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 & 1 & 0 \end{pmatrix} \end{array}$$



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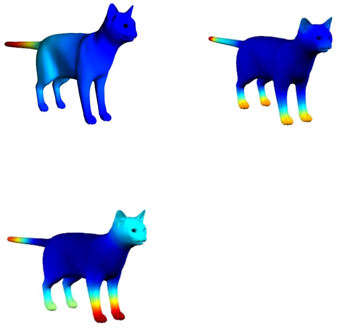

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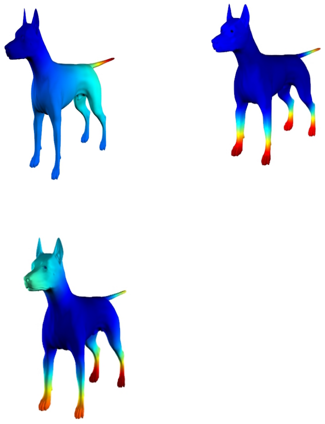

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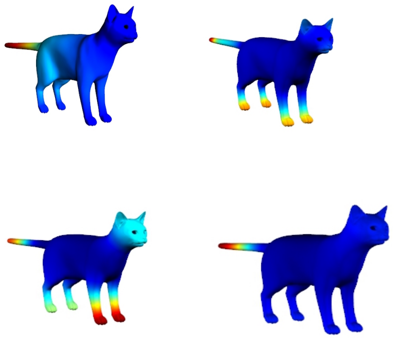
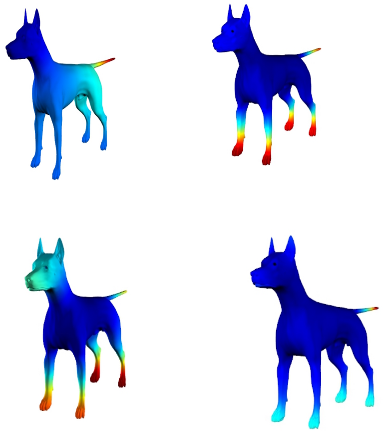


$$\begin{array}{cc} & \xleftrightarrow{\quad\quad\quad\quad n_1 \quad\quad\quad\quad} \\ n_2 \updownarrow & \begin{pmatrix} 1 & 4 & 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 3 & 1 & 1 \\ 4 & 1 & 1 & 0 & 0 & 2 \\ 2 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 2 & 2 & 3 & 1 \\ 0 & 2 & 2 & 1 & 2 & 1 \end{pmatrix} \end{array}$$



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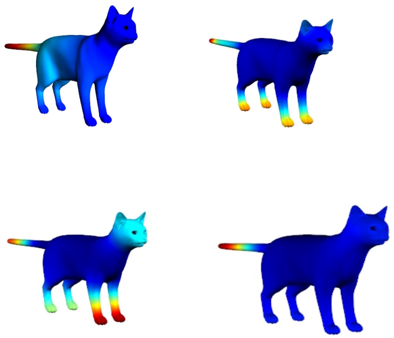
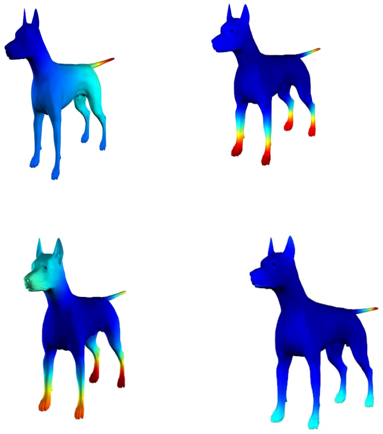
\# of features where p_i of S_2 and q_j of S_1 are binned together!

$N > 1$



$$\begin{array}{c|c} & \xleftrightarrow{\quad\quad\quad n_1 \quad\quad\quad} \\ \hline n_2 & \begin{pmatrix} 1 & \textcircled{4} & 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 3 & 1 & 1 \\ 4 & 1 & 1 & 0 & 0 & 2 \\ 2 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 2 & 2 & 3 & 1 \\ 0 & 2 & 2 & 1 & 2 & 1 \end{pmatrix} \end{array}$$



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$$W_N = \sum_{i=1}^{N} \sum_{j=1}^{K} 1_{C_{i,j}} 1^t_{C'_{i,j}}$$

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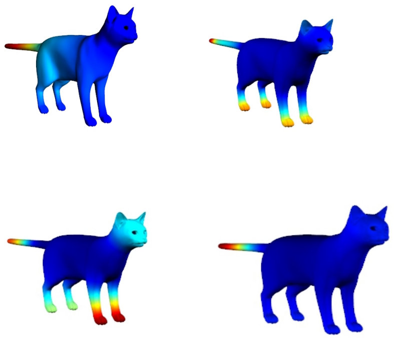

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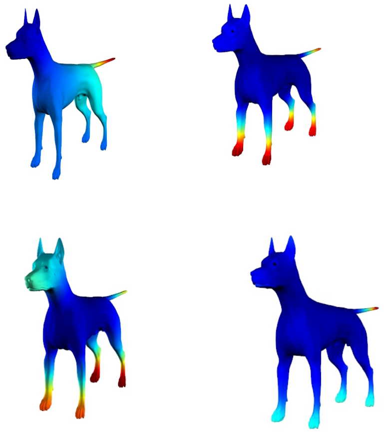

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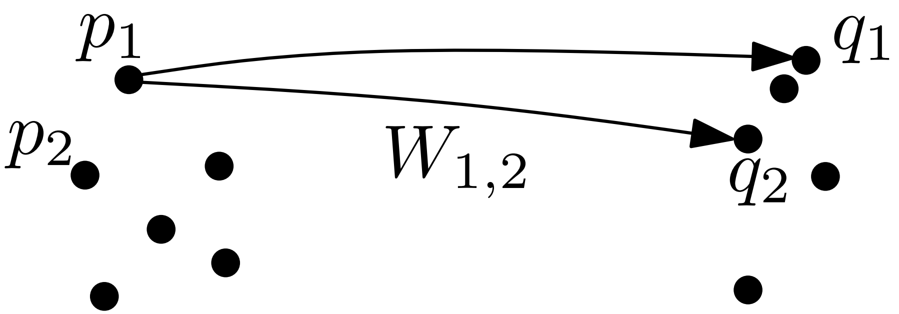
$$W_N = \sum_{i=1}^{N} \sum_{j=1}^{K} 1_{C_{i,j}} 1^t_{C'_{i,j}}$$

$$W = K/(N n_1 n_2) W_N.$$

W_N is a transport plan between uniform probability measures



- $W_{i,j} \geq 0$
- $\sum_j W_{i,j} = 1/n_1$
- $\sum_i W_{i,j} = 1/n_2$

Extraction of information from W

$W : f \in \mathcal{F}(S_1) \to \mathcal{F}(S_2)$ seen as a linear map.

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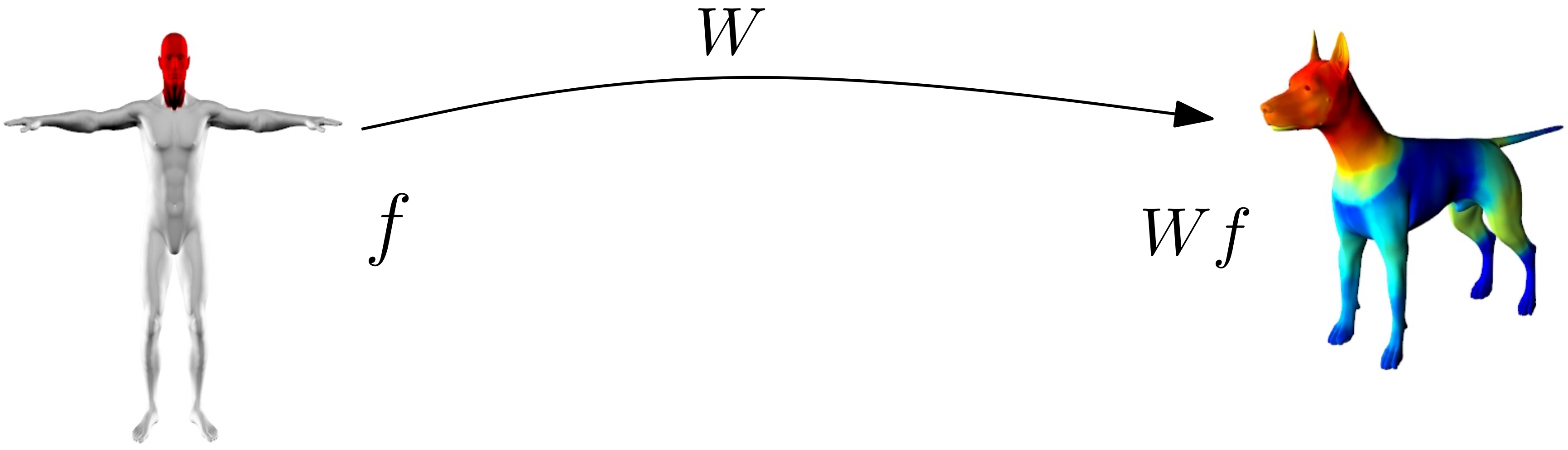

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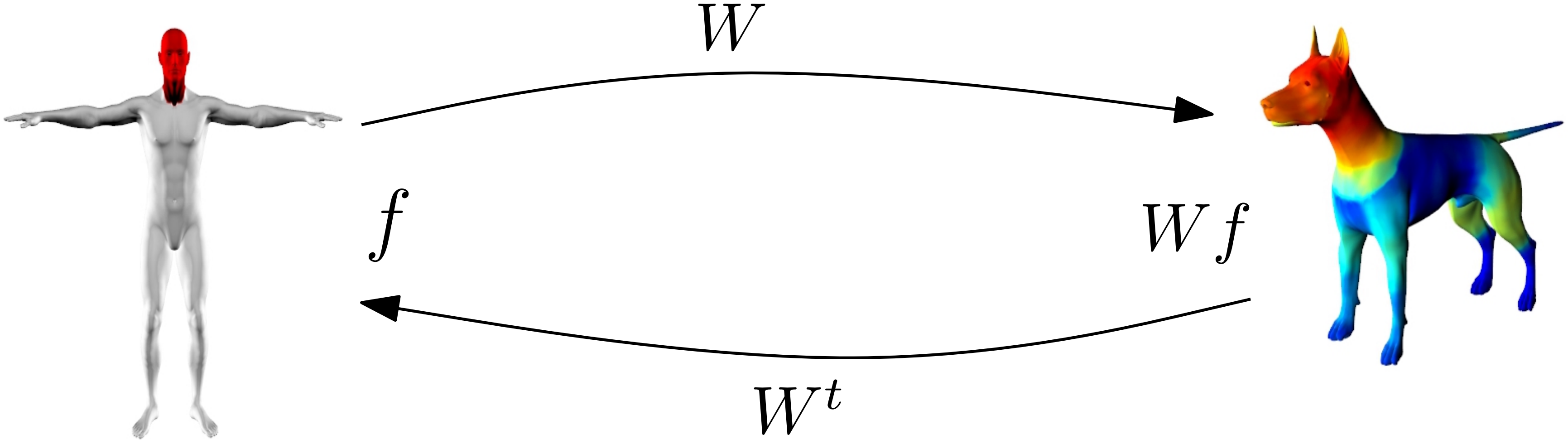

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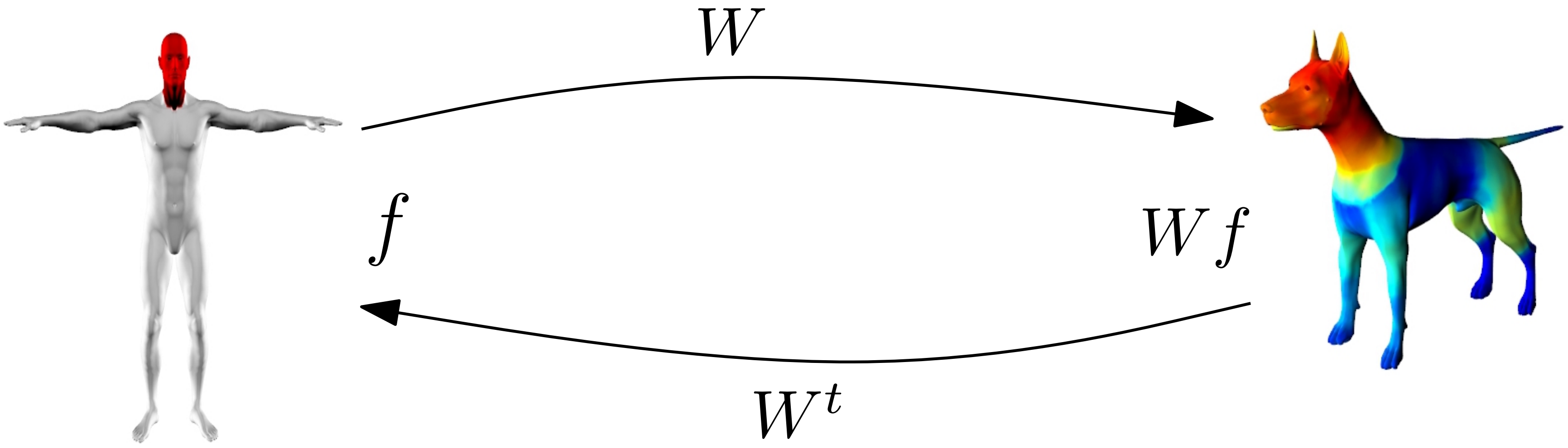

- Stable parts: $W^tWf = \lambda f \rightsquigarrow$ eigenvector
- Maximizes $W \rightsquigarrow$ highest eigenvalue

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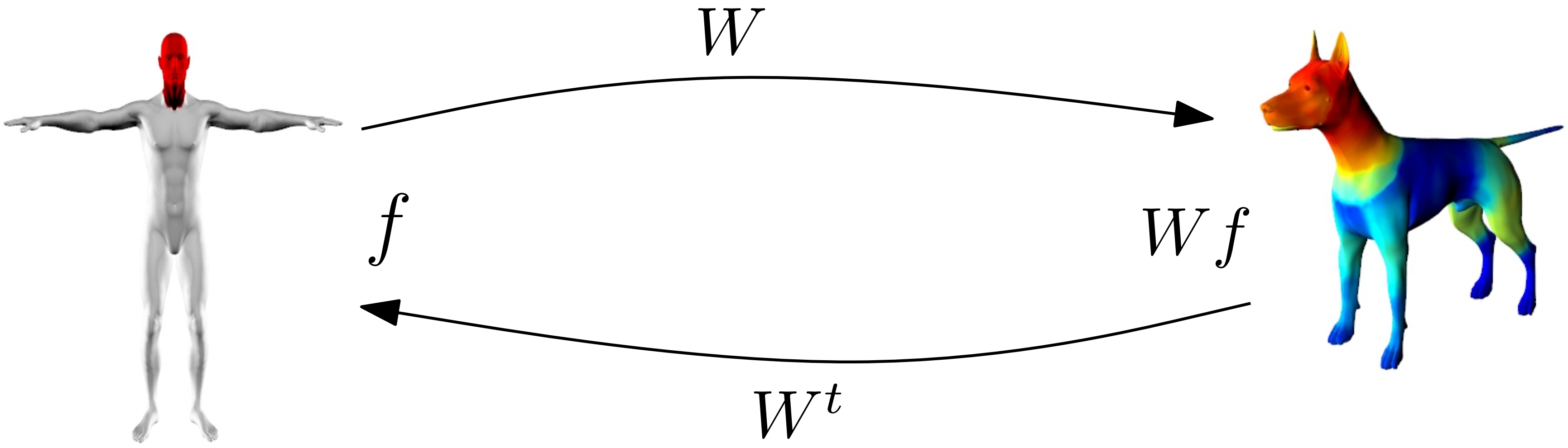

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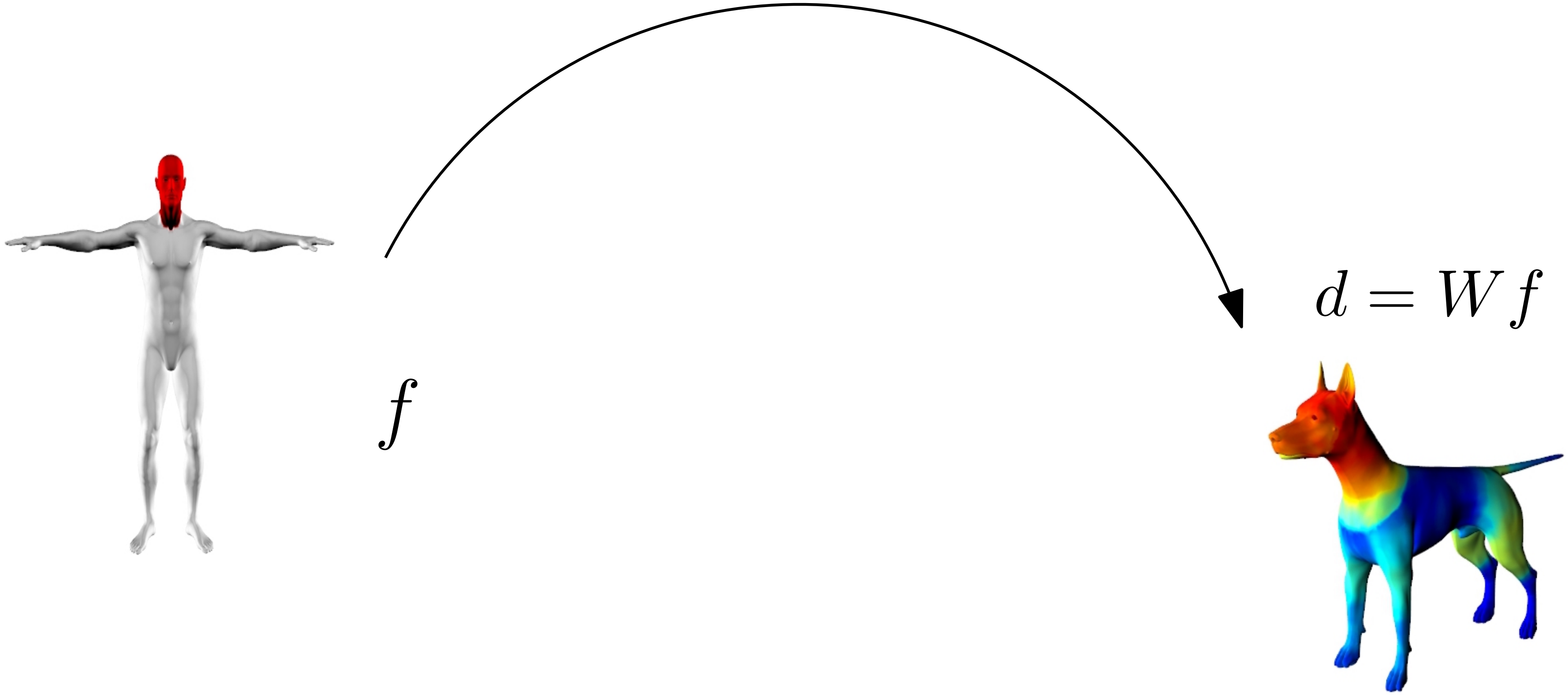
Problem: the solution is S_1 and S_2

$\rightarrow$ We introduce non linearity

Truncated Power Iteration Algorithm



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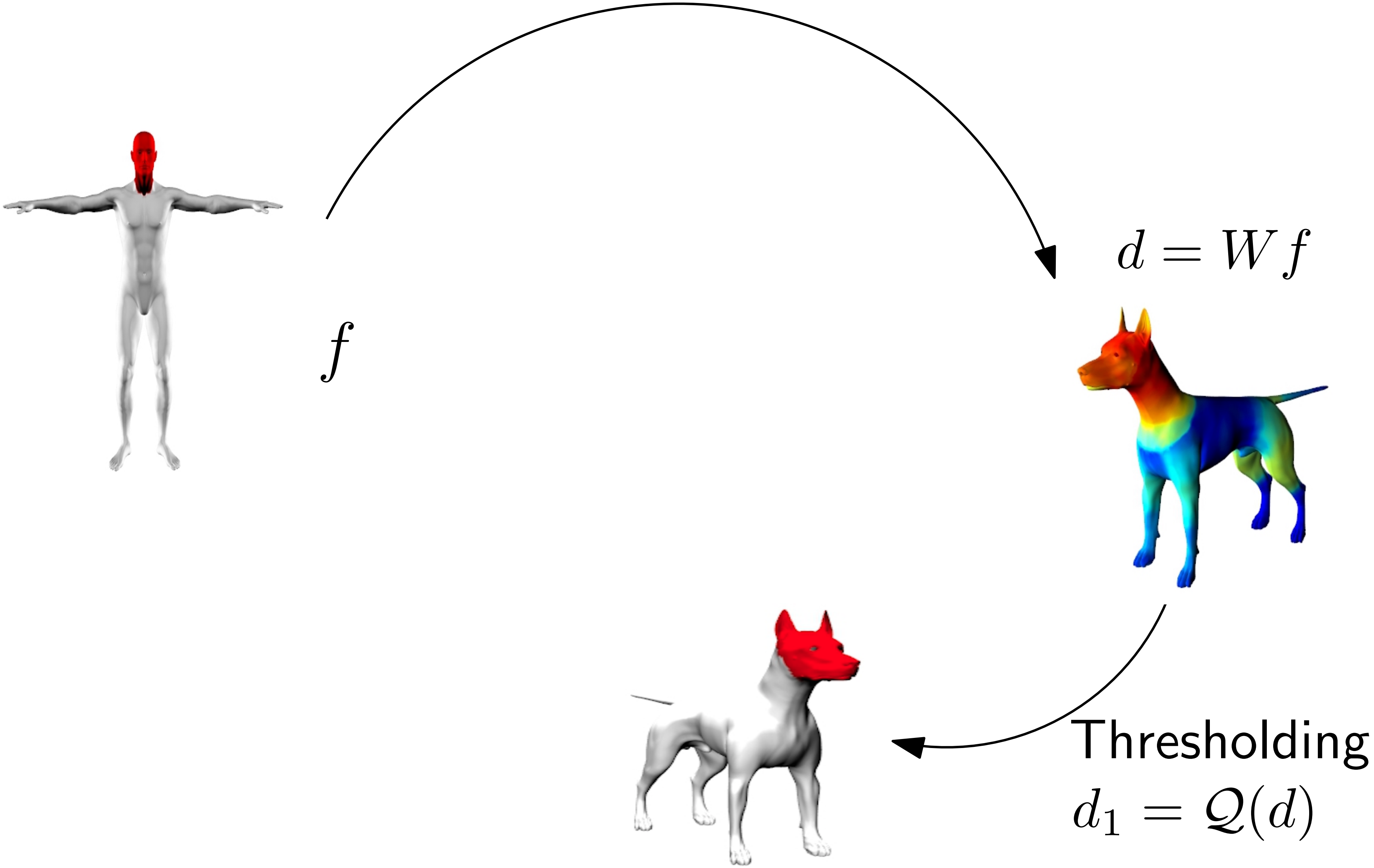

f

$d = Wf$

Thresholding

$d_1 = \mathcal{Q}(d)$

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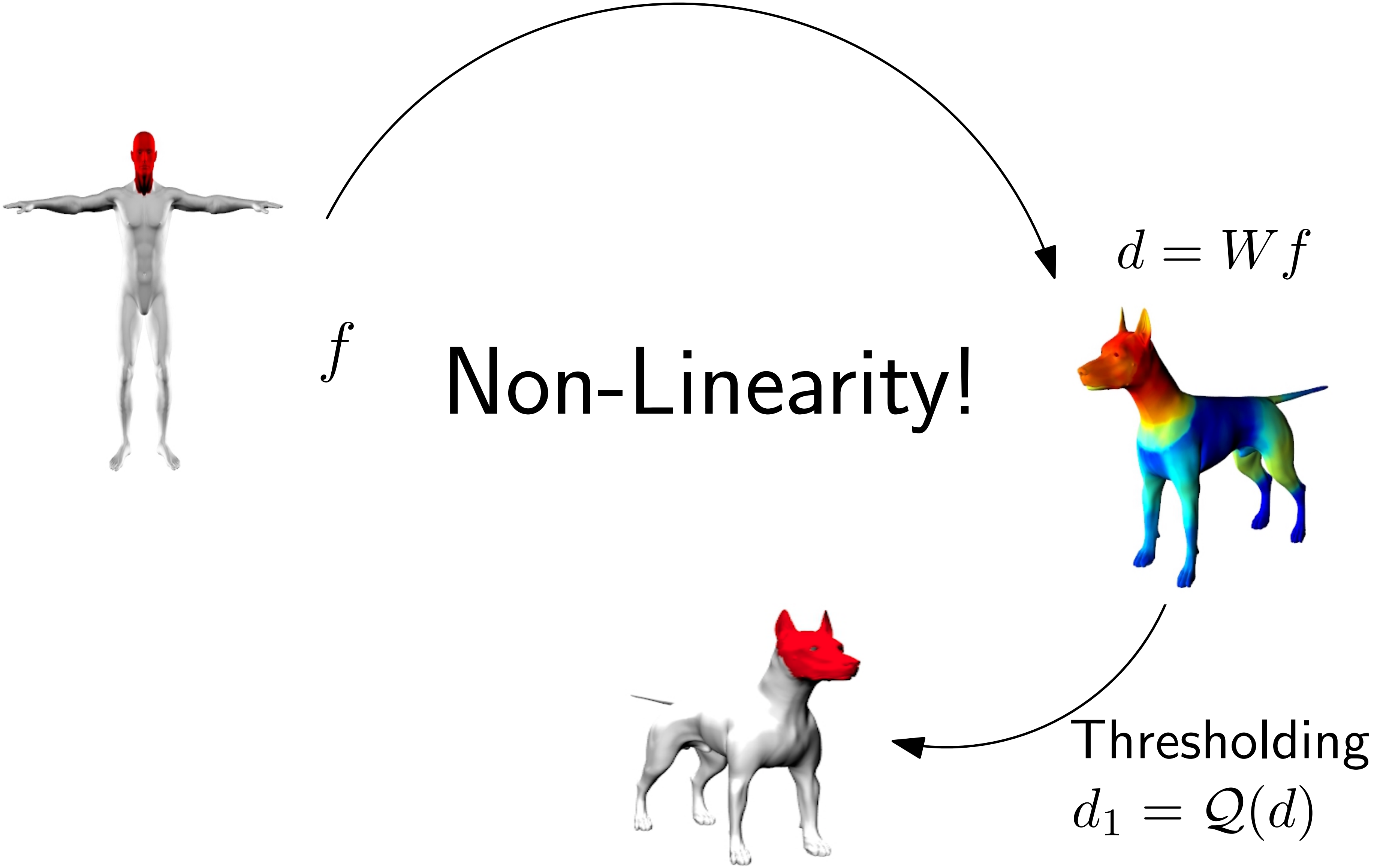

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Non-Linearity!

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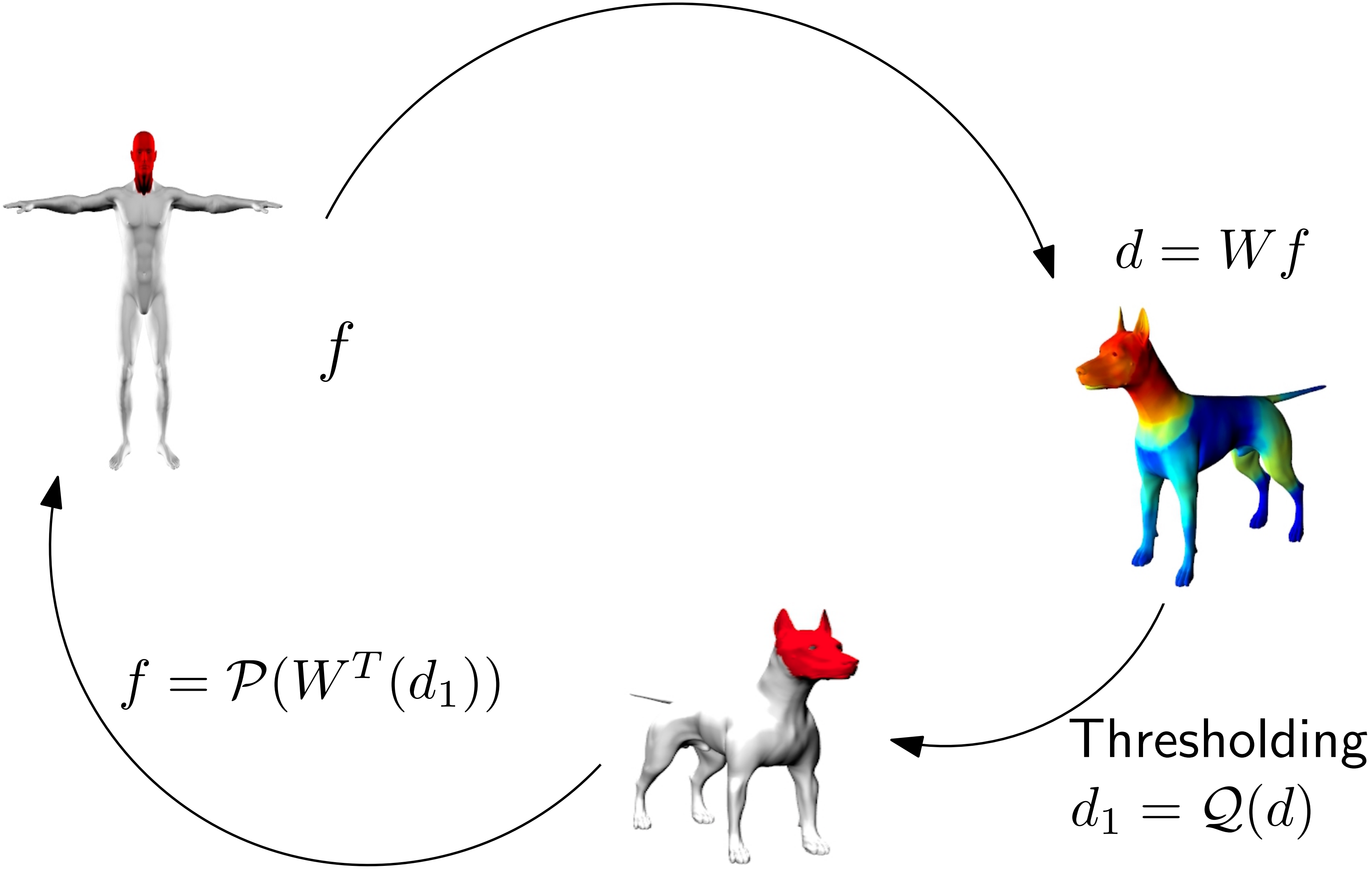

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$d = Wf$

Thresholding

$d_1 = \mathcal{Q}(d)$

$f = \mathcal{P}(W^T(d_1))$

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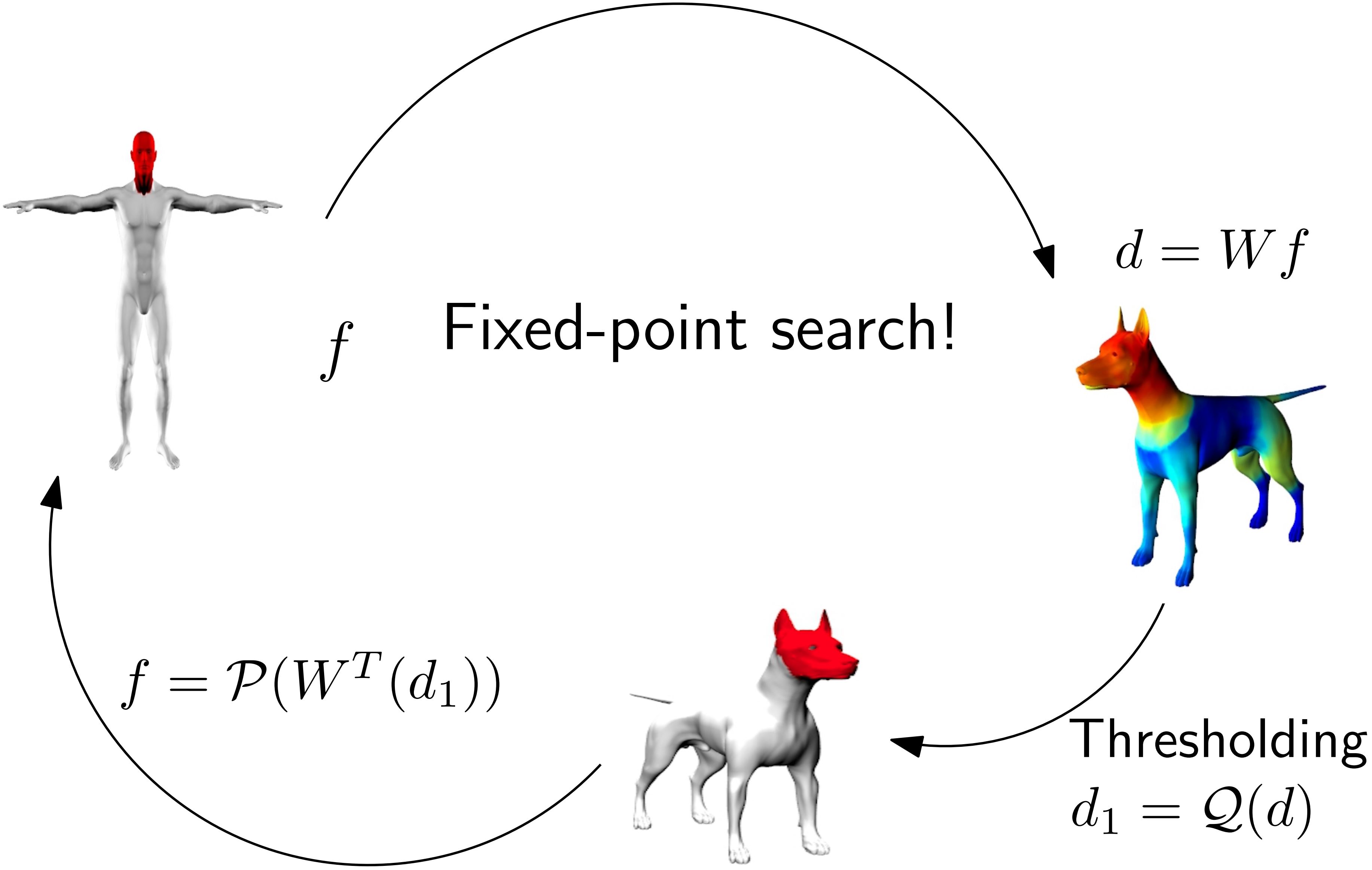

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Fixed-point search!

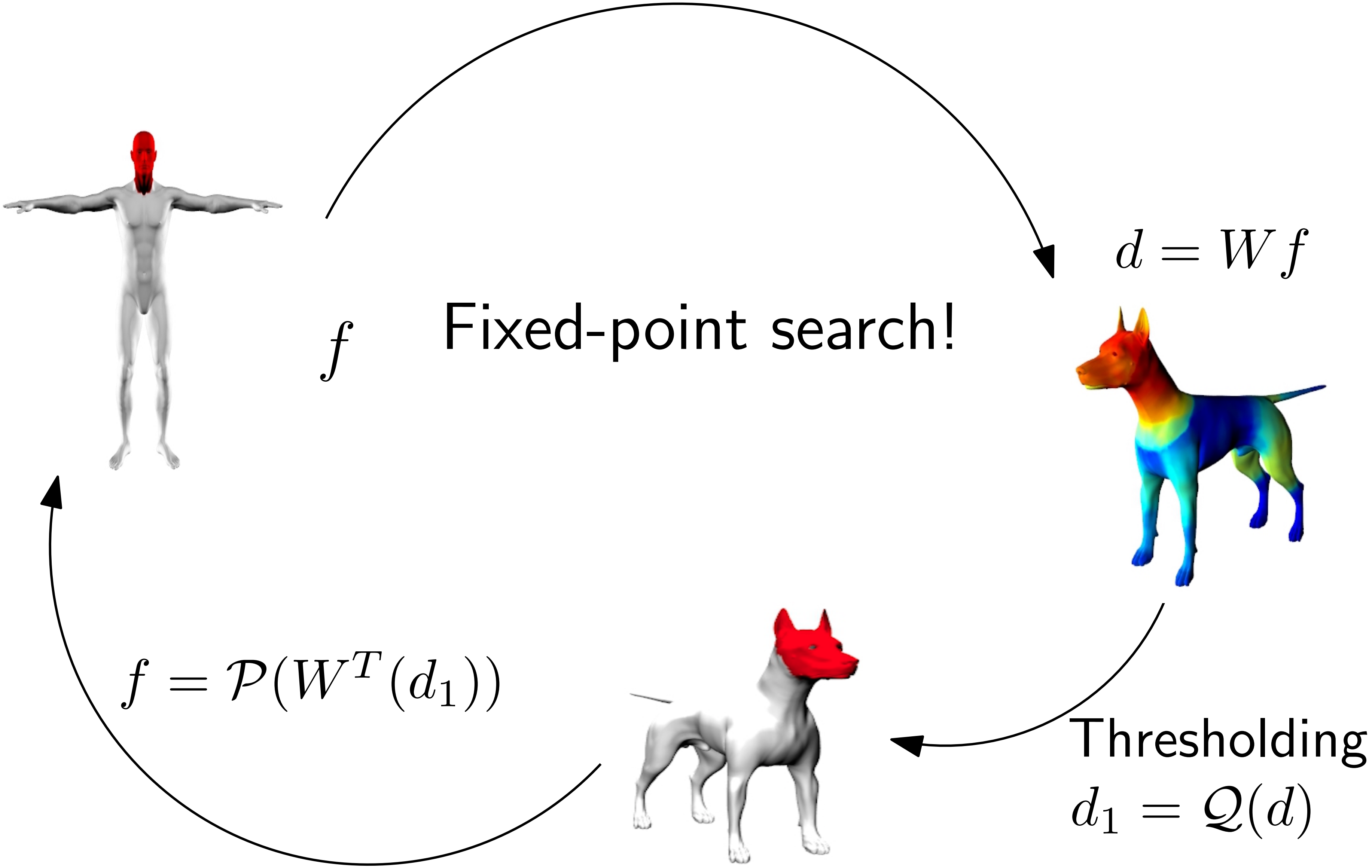
$d = Wf$

Thresholding

$d_1 = \mathcal{Q}(d)$

$f = \mathcal{P}(W^T(d_1))$

Truncated Power Iteration Algorithm



f

Fixed-point search!

$d = Wf$

Thresholding

$d_1 = \mathcal{Q}(d)$

$f = \mathcal{P}(W^T(d_1))$

In practice, converges (almost always) in less than 10 iterations.

Optimization formulation

Optimization problem:

$$\text{argmax}_{x,y} y^T W x$$

$$\text{s.t. } x \in \{0,1\}^{d_1}, y \in \{0,1\}^{d_2},\ ||x||_1 = p, ||y||_1 = q.$$

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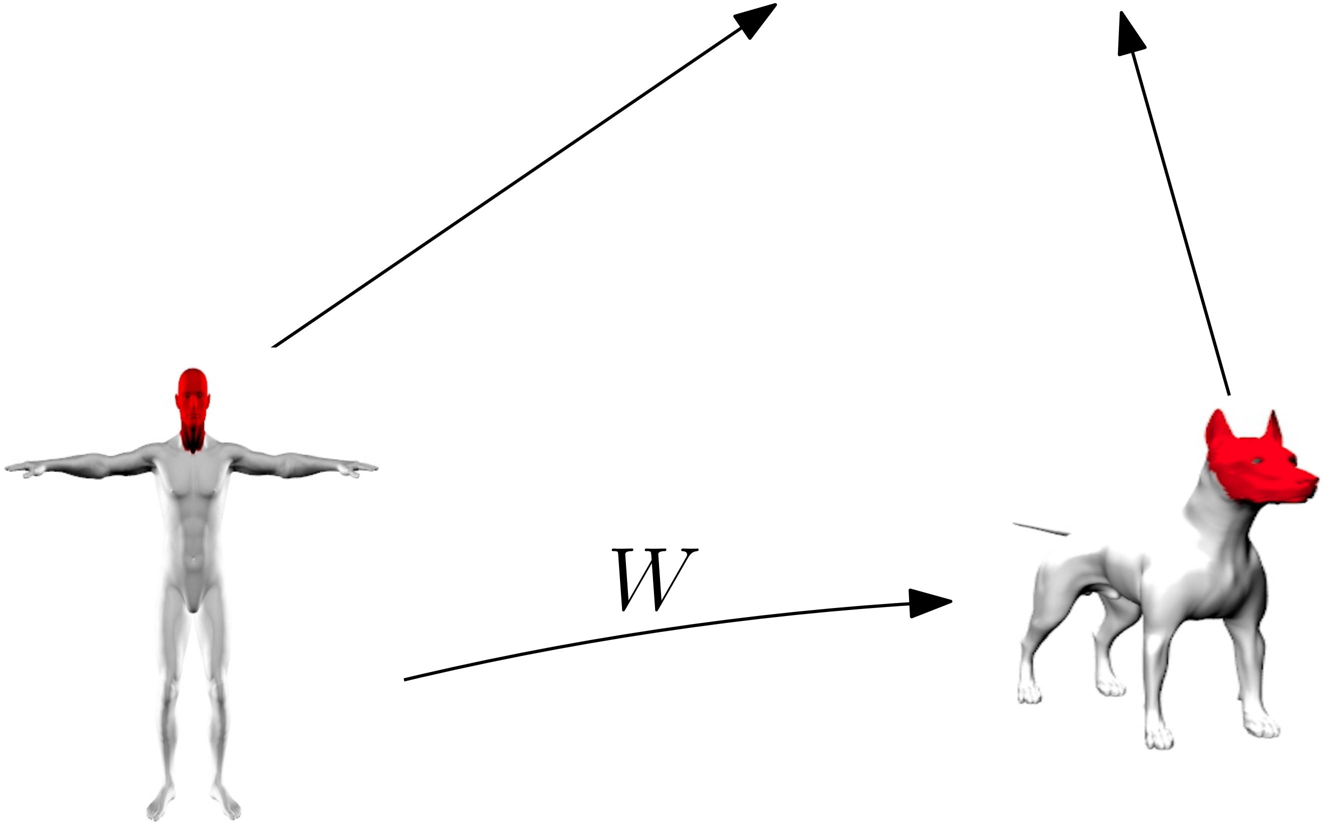

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Remark

$$(Pb_1) \Leftrightarrow argmax_A \|A\|_{1,1}$$

where A is a sub-matrix of size $q \times p$ of W.

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where A is a sub-matrix of size $q \times p$ of W.

$$W = \begin{pmatrix} W_{\Omega_1,\Omega_2} & W_{S_1\backslash\Omega_1,\Omega_2} \\ W_{\Omega_1,S_2\backslash\Omega_2} & W_{S_1\backslash\Omega_1,S_2\backslash\Omega_2} \end{pmatrix}$$

Definition. Let $\Omega_i \subset S_i$

(Ω_1, Ω_2) stable part $\Leftrightarrow W_{\Omega_1,\Omega_2}$ local maximum of $\| \cdot \|_{1,1}$

Optimization formulation

Optimization problem:

$$(Pb_1) \quad \begin{array}{c} \text{argmax}_{x,y} y^T W x \\ \text{s.t. } x \in \{0,1\}^{d_1}, y \in \{0,1\}^{d_2}, \ ||x||_1 = p, ||y||_1 = q. \end{array}$$

Proposition: (Ω_1, Ω_2) is a stable pair

$\Leftrightarrow \exists \, \mathcal{P}, \mathcal{Q}$ s.t $\mathcal{Q}(W 1_{\Omega_1}) = 1_{\Omega_2}$ and $\mathcal{P}(W^T 1_{\Omega_2}) = 1_{\Omega_1}$.

Optimization formulation

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highest p non-zero values

highest q non-zero values

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fixed point of $\mathcal{P} \circ W^T \circ \mathcal{Q} \circ W$

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Optimization problem:

$$(Pb_1) \quad \begin{array}{l} \text{argmax}_{x,y} y^T W x \\ \text{s.t. } x \in \{0,1\}^{d_1}, y \in \{0,1\}^{d_2},\ ||x||_1 = p, ||y||_1 = q. \end{array}$$

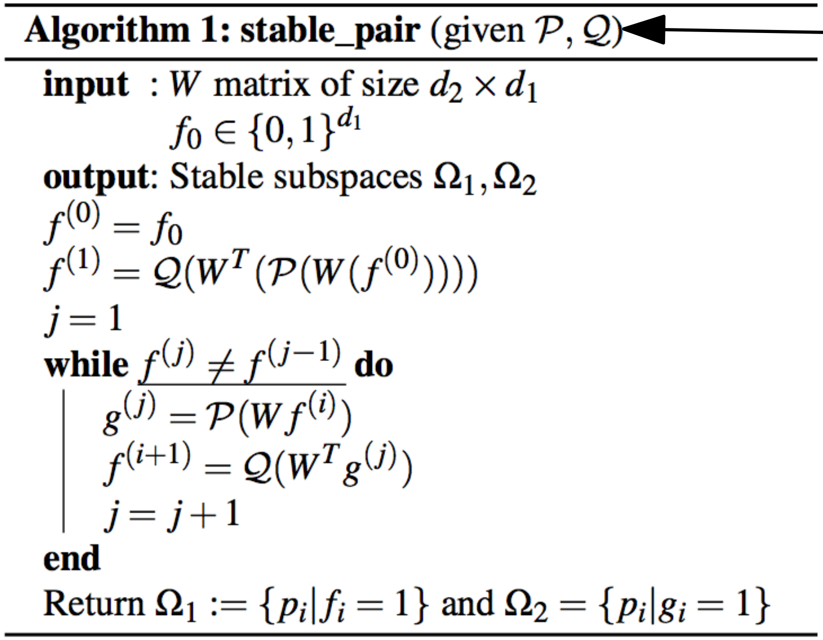
Proposition: (Ω_1, Ω_2) is a stable pair

$\Leftrightarrow \exists\, \mathcal{P}, \mathcal{Q}$ s.t $\mathcal{Q}(W 1_{\Omega_1}) = 1_{\Omega_2}$ and $\mathcal{P}(W^T 1_{\Omega_2}) = 1_{\Omega_1}$.

fixed point of $\mathcal{P} \circ W^T \circ \mathcal{Q} \circ W$

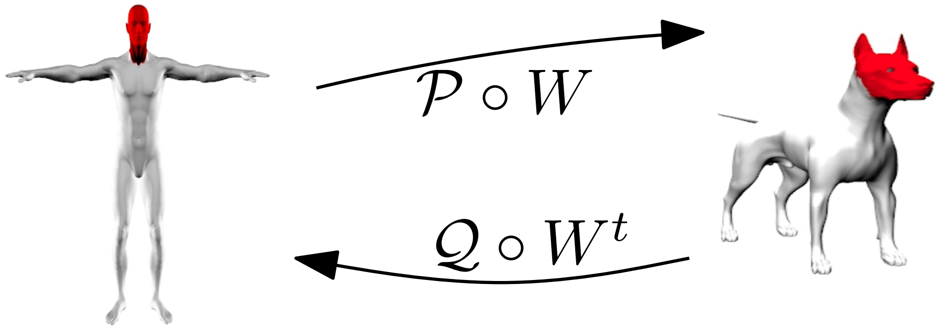
$\Rightarrow$ The algorithm provides a stable pair

Algorithm



```
Algorithm 1: stable_pair (given $\mathcal{P}$, $\mathcal{Q}$)
  input  : $W$ matrix of size $d_2 \times d_1$
           $f_0 \in \{0,1\}^{d_1}$
  output: Stable subspaces $\Omega_1, \Omega_2$
  $f^{(0)} = f_0$
  $f^{(1)} = \mathcal{Q}(W^T(\mathcal{P}(W(f^{(0)}))))$
  $j = 1$
  while $f^{(j)} \neq f^{(j-1)}$ do
      $g^{(j)} = \mathcal{P}(W f^{(i)})$
      $f^{(i+1)} = \mathcal{Q}(W^T g^{(j)})$
      $j = j + 1$
  end
  Return $\Omega_1 := \{p_i | f_i = 1\}$ and $\Omega_2 = \{p_i | g_i = 1\}$
```

Thresholding functions



Remark : $\|W_{\Omega_1,\Omega_2}\|_{1,1}$ strictly increases at each step.

Proposition: If $\mathcal{P}$ and $\mathcal{Q}$ select the q and p largest values, then the algorithm terminates in a finite number of steps.

Robustness to noisy features

Given n pair of functions on S_1 and S_2
$N-n$ pair of noisy functions $\phi_k : S_1 \to \mathbb{R}$
$\psi_k : S_2 \to \mathbb{R}$

we have $W_N^{all} = \kappa\, W_n^{init} + (1-\kappa) W_{N-n}^{noise}$, with $\kappa = \frac{n}{N}$

Proposition
If (ϕ_k) i.i.d, (ψ_k) i.i.d, ϕ_k and ψ_l independent, then $\forall \delta > 0$

$$P(\|W_N^{all} - \kappa\, W_n^{init} - C 1_{d_2,d_1}\|_{\mathcal{F}} < \delta) \geq 1 - \frac{(1-\kappa)\sigma^2}{N\delta^2},$$

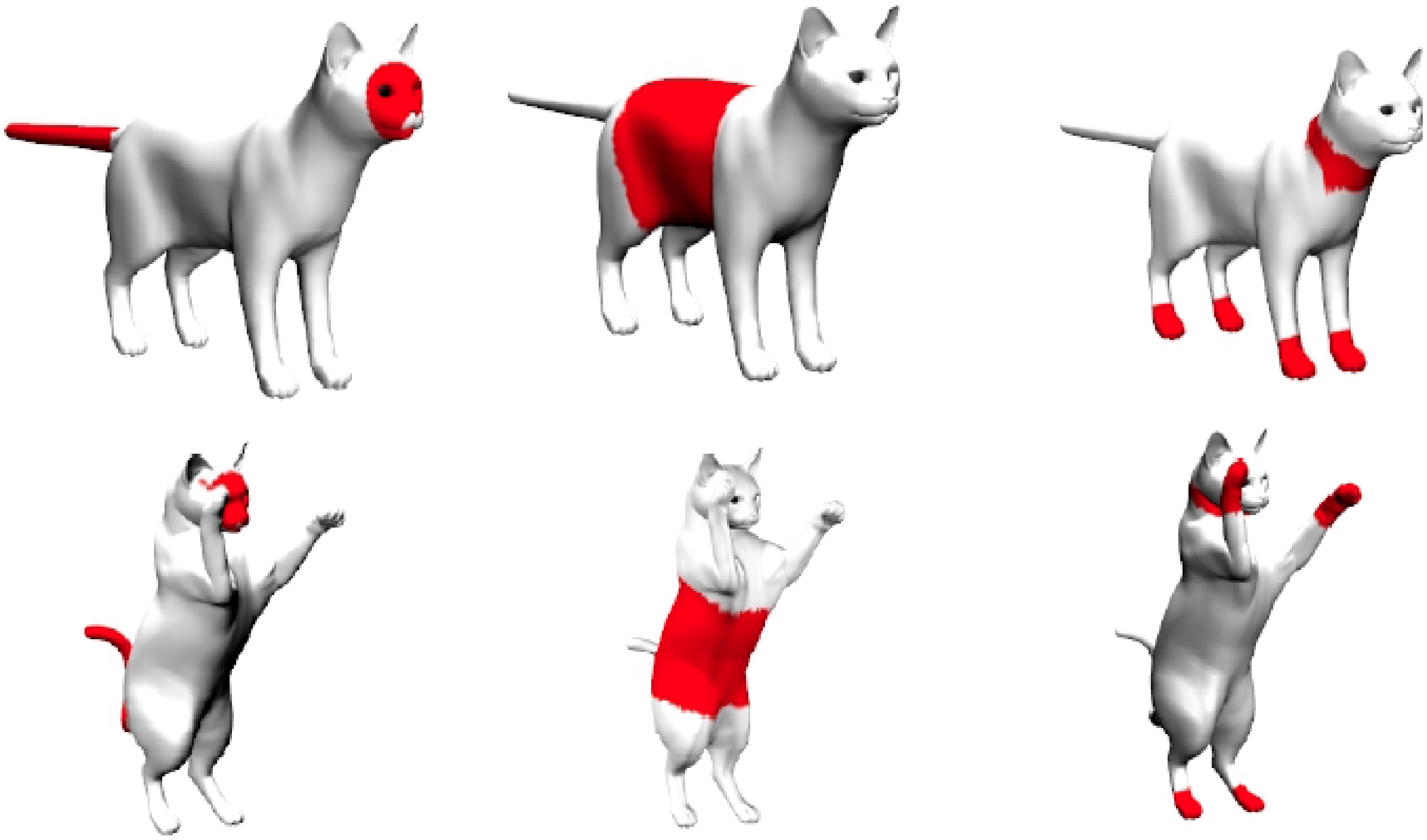
constants

Corollary With high probability, if W_{N-n}^{noise} is δ-independent.

$$(\Omega_1, \Omega_2)\ W_N^{all} - \text{stable} \quad \Leftrightarrow \quad (\Omega_1, \Omega_2)\ W_N^{init} - \text{stable}.$$

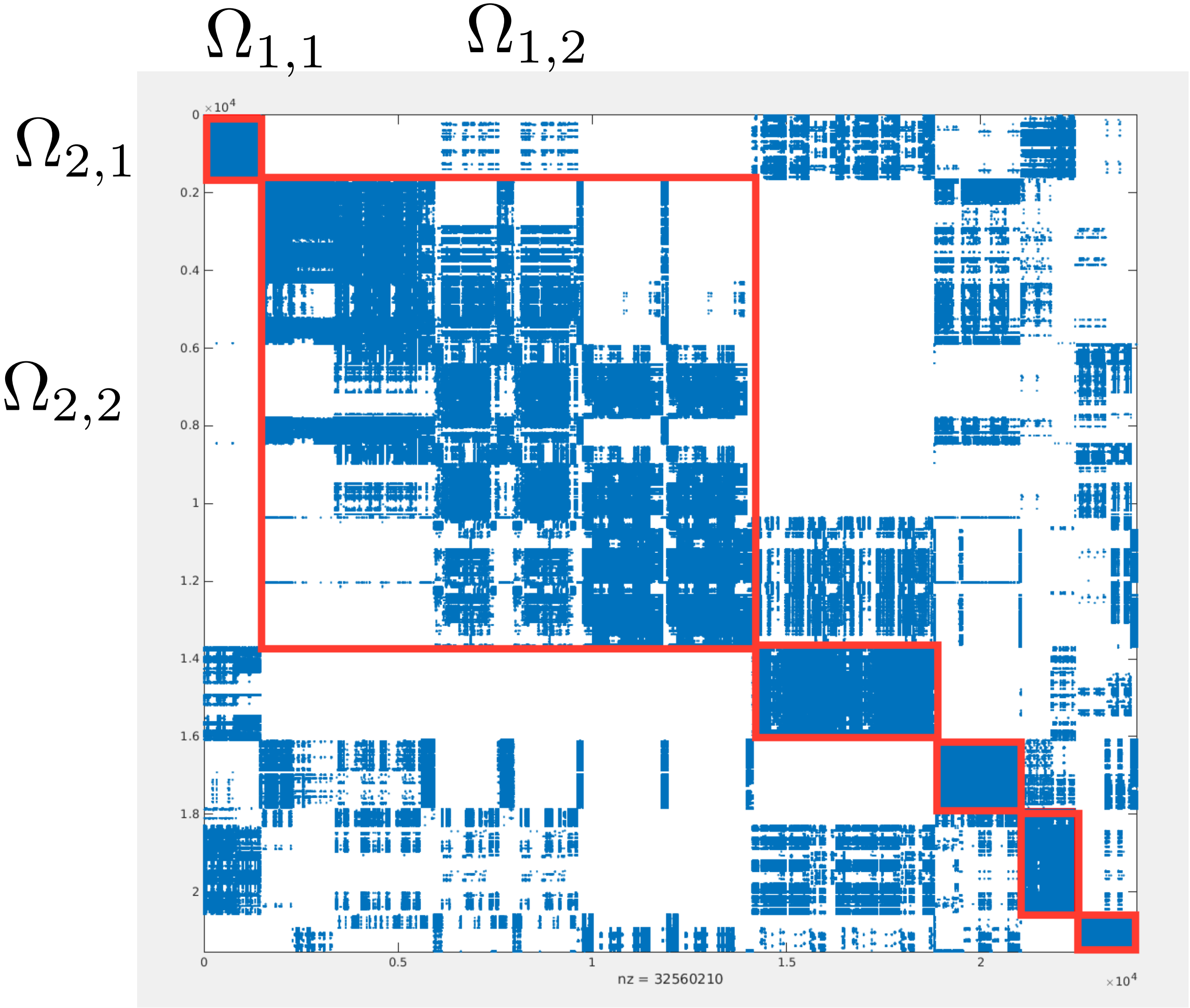
Iterative algorithm

$\Omega_{1,1}$ $\Omega_{1,2} \subset S_1 \setminus \Omega_{1,1}$ $\Omega_{1,3}$



$\Omega_{2,1}$ $\Omega_{2,2} \subset S_2 \setminus \Omega_{2,2}$ $\Omega_{2,3}$

Diagonal by block structure



A remark

Optimization problem:

$$\text{argmax}_{x,y} y^T W x$$

$$\text{s.t. } x \in \{0,1\}^{d_1}, y \in \{0,1\}^{d_2},\ ||x||_1 = p, ||y||_1 = q.$$

Relaxed (continuous) problem:

$$\max_{\|x\|_2 \leq 1, \|y\|_2 \leq 1} y^t W x = \max_{\|x\|_2 \leq 1} \|Wx\|_2$$

A remark

Optimization problem:

$$\text{argmax}_{x,y} y^T W x$$

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Relaxed (continuous) problem:

$$\begin{aligned}\max_{\|x\|_2 \leq 1, \|y\|_2 \leq 1} y^t W x &= \max_{\|x\|_2 \leq 1} \|Wx\|_2 \\ &= \sqrt{\rho(W^T W)}\end{aligned}$$

A remark

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use power iteration method to solve it!

A remark

truncated version of power iteration!

Optimization problem:

$$\text{argmax}_{x,y} y^T W x$$

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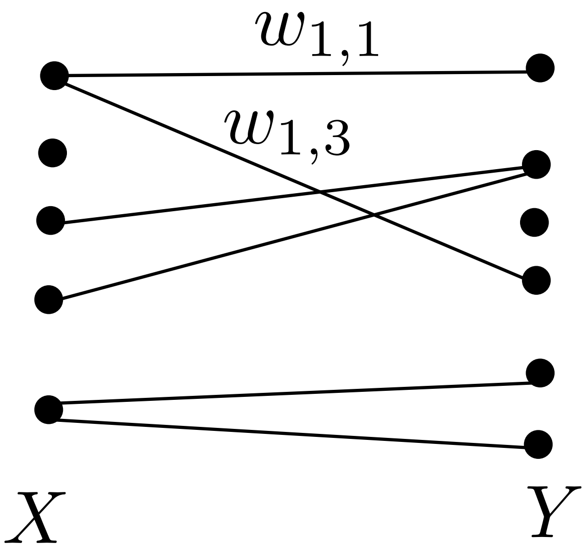
Another problem: Biclustering

We consider a bipartite graph



$$W = (w_{i,j})$$

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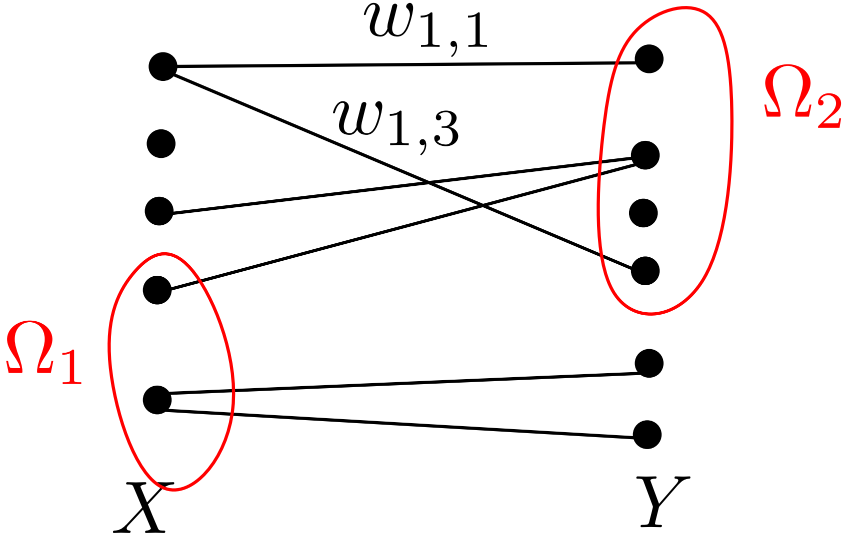

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$$W = \begin{pmatrix} W_{\Omega_1,\Omega_2} & W_{\Omega_1^c,\Omega_2} \\ W_{\Omega_1,\Omega_2^c} & W_{\Omega_1^c,\Omega_2^c} \end{pmatrix}$$

Another problem: Biclustering

[Zha, Ding, Gu,2002]

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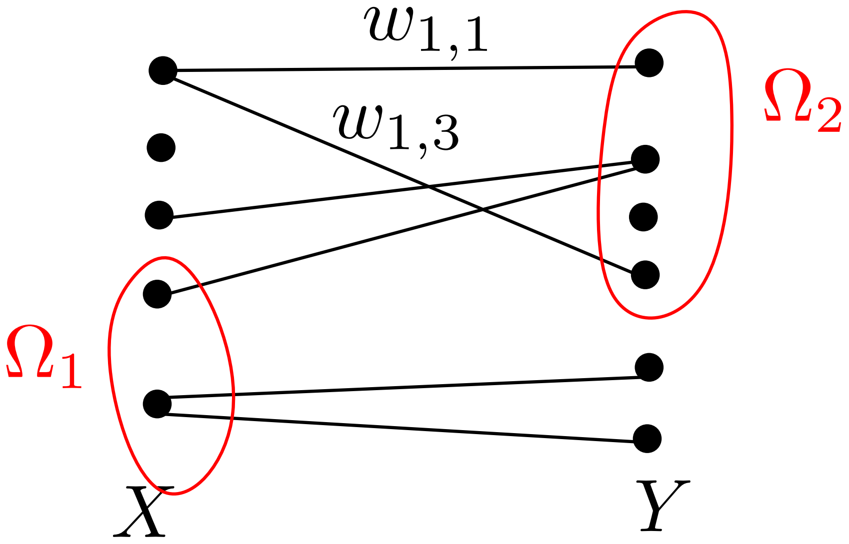

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Biclustering problem [Zha, Ding, Gu,2002]

$$\mathrm{argmin}_{\Omega_1,\Omega_2} NCut(\Omega_1,\Omega_2)$$

with $NCut(\Omega_1,\Omega_2) = \frac{\|W_{\Omega_1,\Omega_2^c}\|_{1,1}+\|W_{\Omega_1^c,\Omega_2}\|_{1,1}}{\|W_{\Omega_1,Y}\|_{1,1}+|W_{X,\Omega_2}\|_{1,1}} + \frac{\|W_{\Omega_1,\Omega_2^c}\|_{1,1}+\|W_{\Omega_1^c,\Omega_2}\|_{1,1}}{\|W_{\Omega_1^c,Y}\|_{1,1}+|W_{X,\Omega_2^c}\|_{1,1}}$

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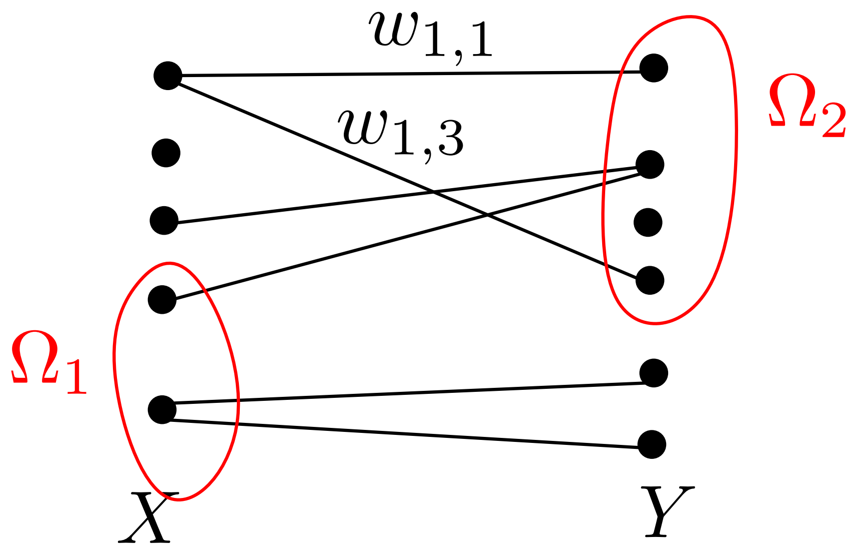

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normalization

Another problem: Biclustering

[Zha, Ding, Gu,2002]

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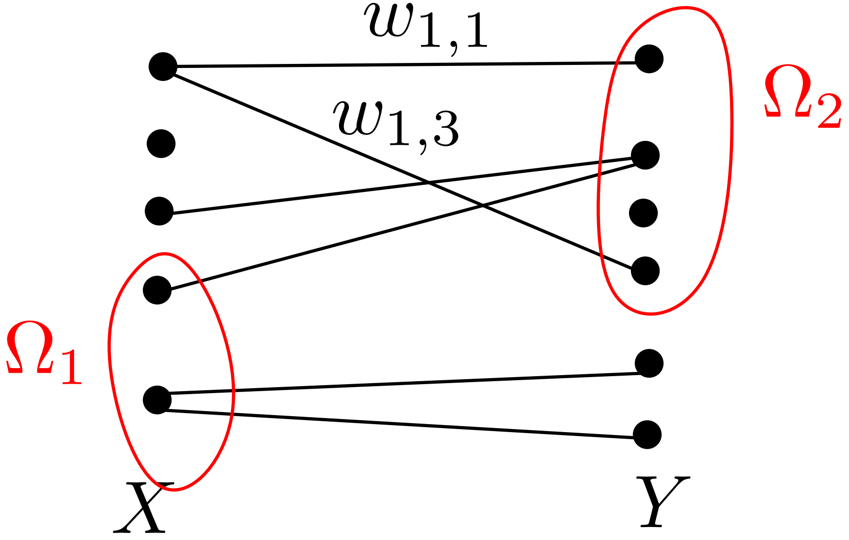

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normalization

symmetry

Biclustering problem

[Zha, Ding, Gu,2002]

Proposition

$$\min_{\Omega_1,\Omega_2} NCut(\Omega_1,\Omega_2) = 1 - \max_{x,y} \frac{2x^tWy}{x^tD_Xx+y^tD_Yy}$$

where $x^tD_X1 + y^tD_Y1 = 0$ and $x_i, y_i \in \{2-2p, -2p\}$

Biclustering problem

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Relaxed problem

Calculate the second largest left and right singular vectors of

$$D_X^{-1/2} W D_Y^{1/2}$$

Biclustering problem

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Relaxed problem

Calculate the second largest left and right singular vectors of

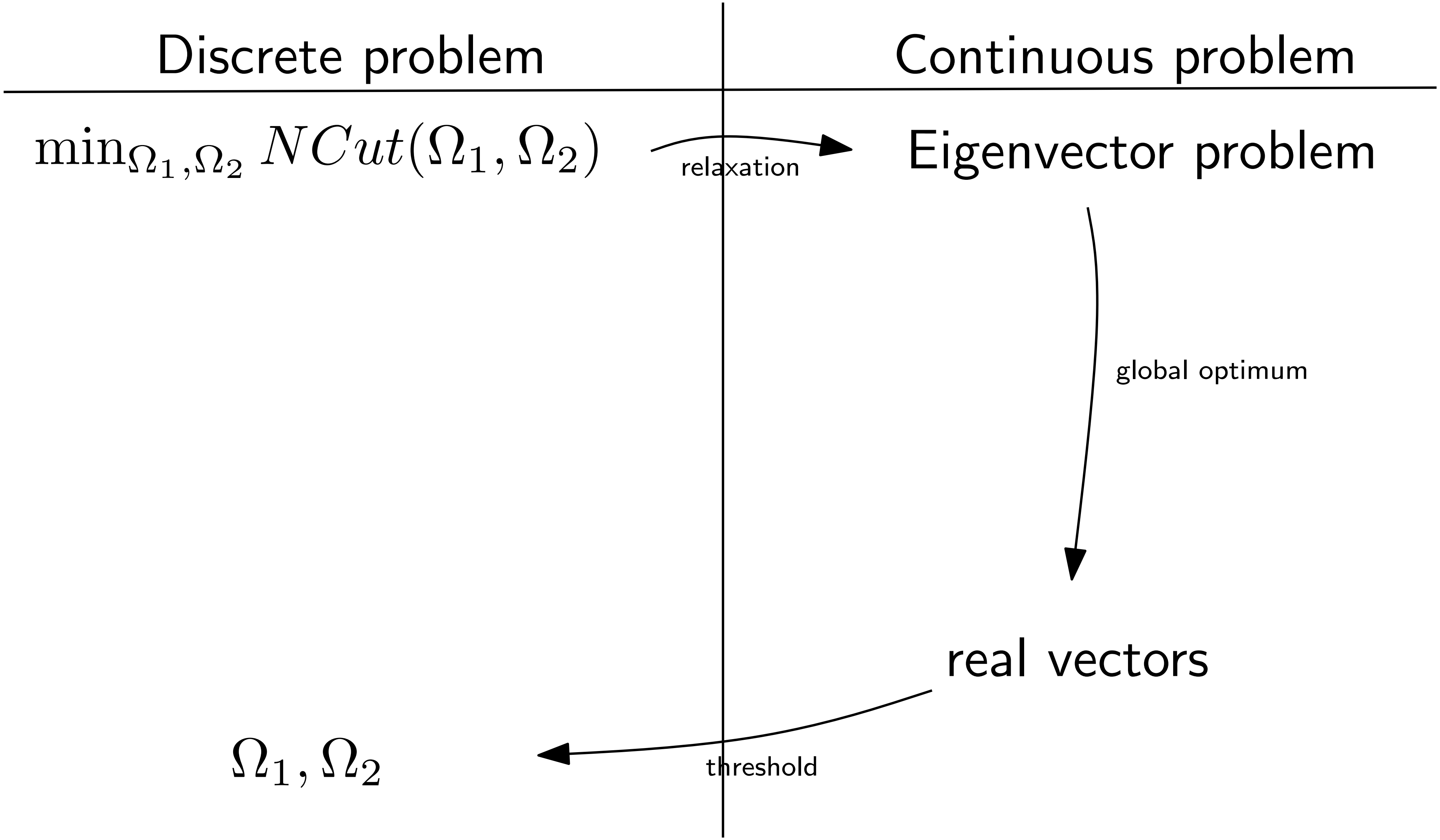
$$D_X^{-1/2}WD_Y^{1/2}$$

Algorithm [Zha, Ding, Gu,2002]

- Compute the left and right second largest singular vectors
- $\Omega_1 = \{i, x_i \geq x_c\}$ and $\Omega_2 = \{j, x_j \geq y_c\}$

where c_x and c_y are cutoffs

Biclustering problem



Discrete problem

Continuous problem

$\min_{\Omega_1,\Omega_2} NCut(\Omega_1,\Omega_2)$

relaxation

Eigenvector problem

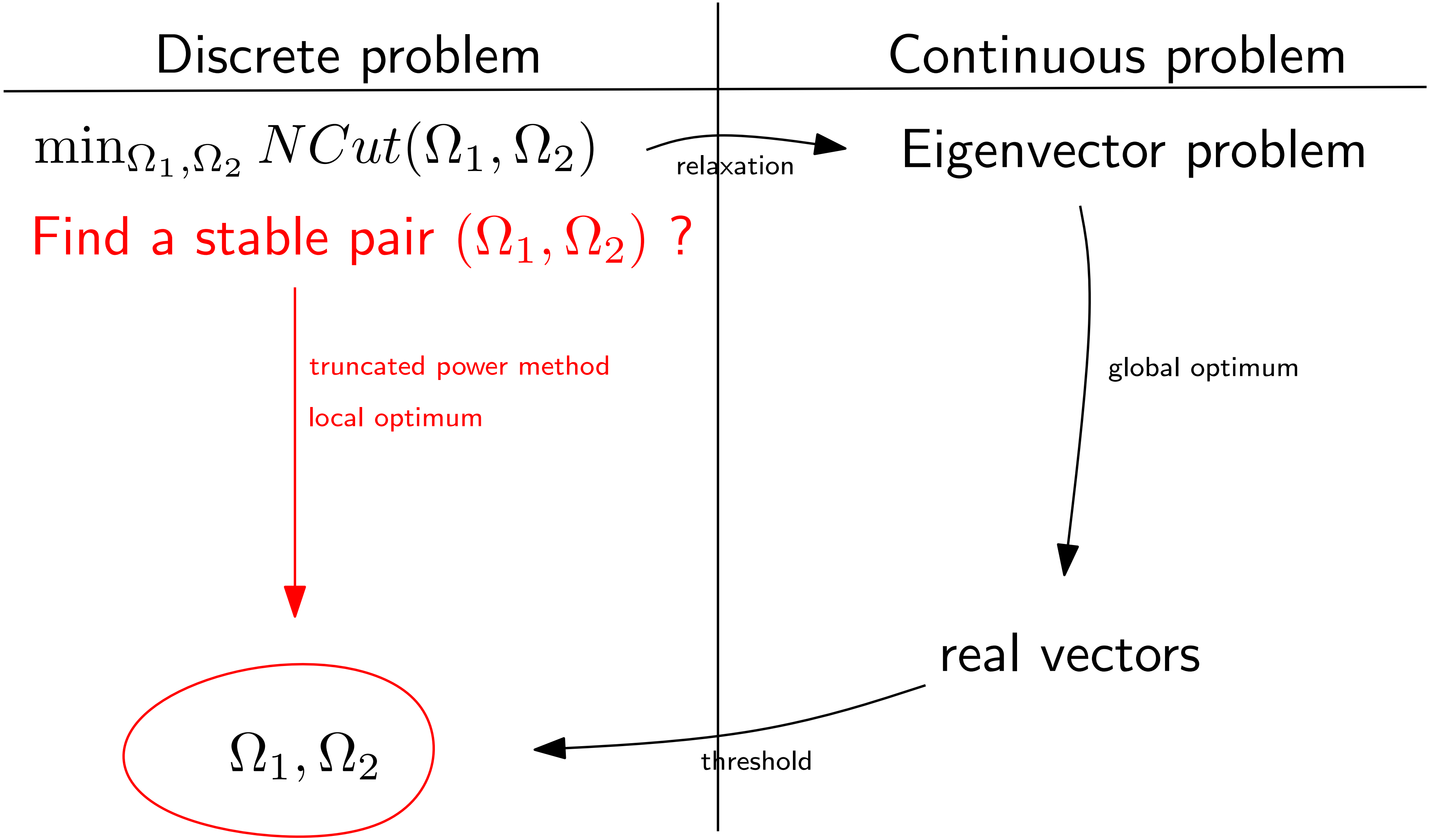
global optimum

real vectors

threshold

Ω_1, Ω_2

Biclustering problem



Discrete problem

Continuous problem

$\min_{\Omega_1,\Omega_2} NCut(\Omega_1, \Omega_2)$

relaxation

Eigenvector problem

Find a stable pair (Ω_1, Ω_2) ?

truncated power method

local optimum

global optimum

real vectors

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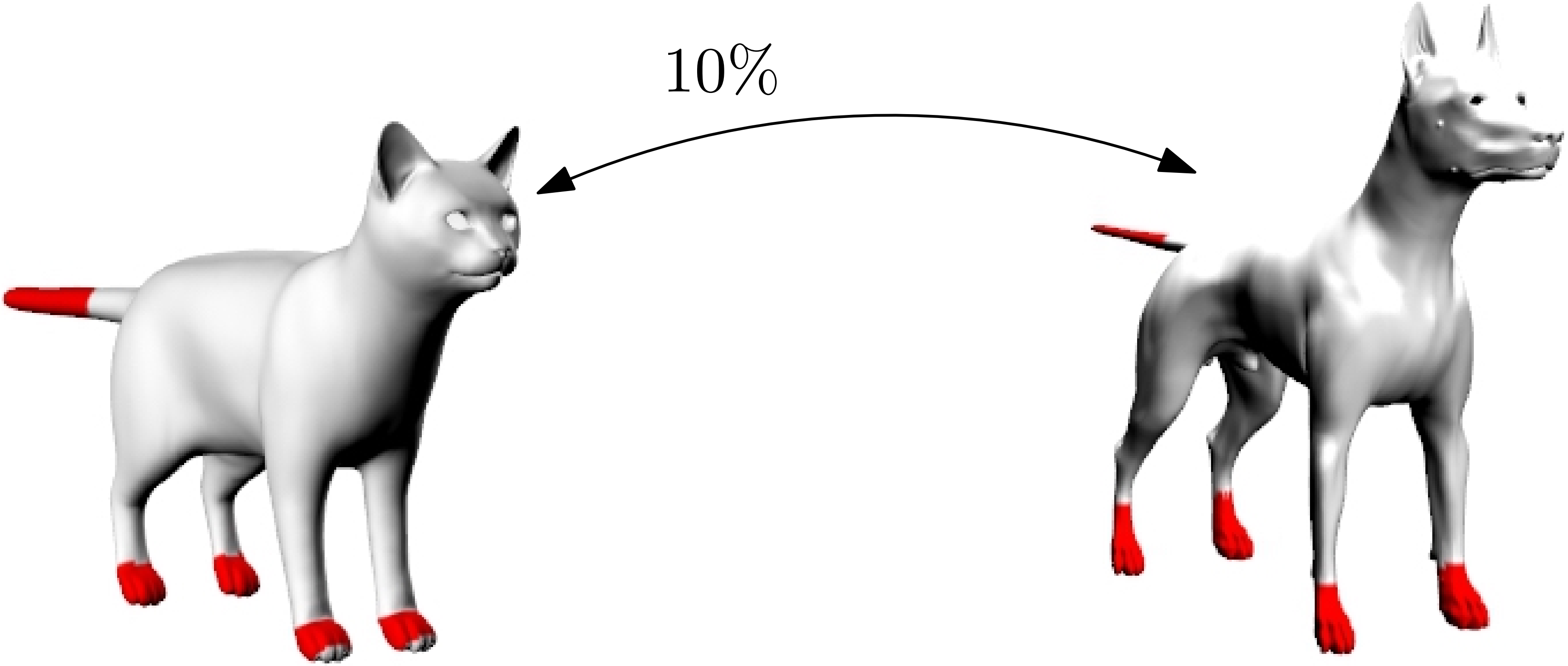
Evolution

Evolution



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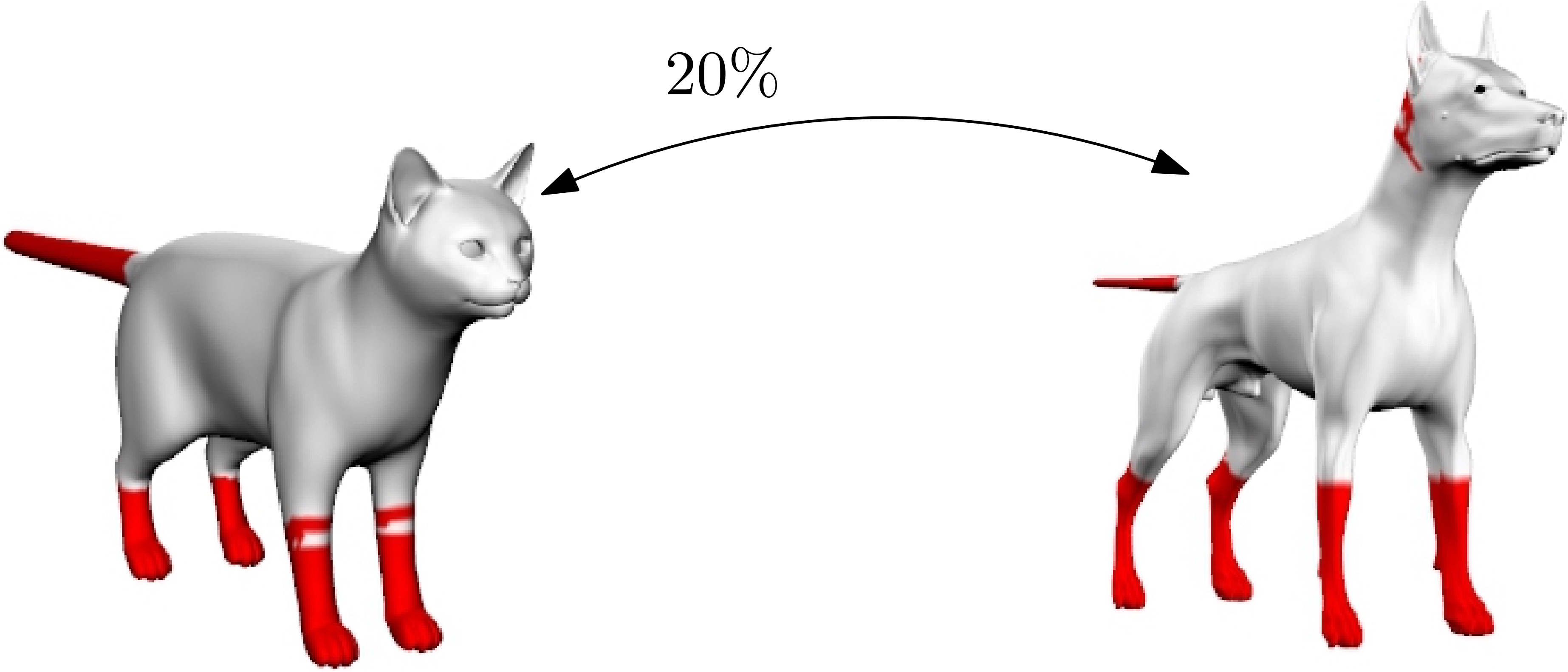

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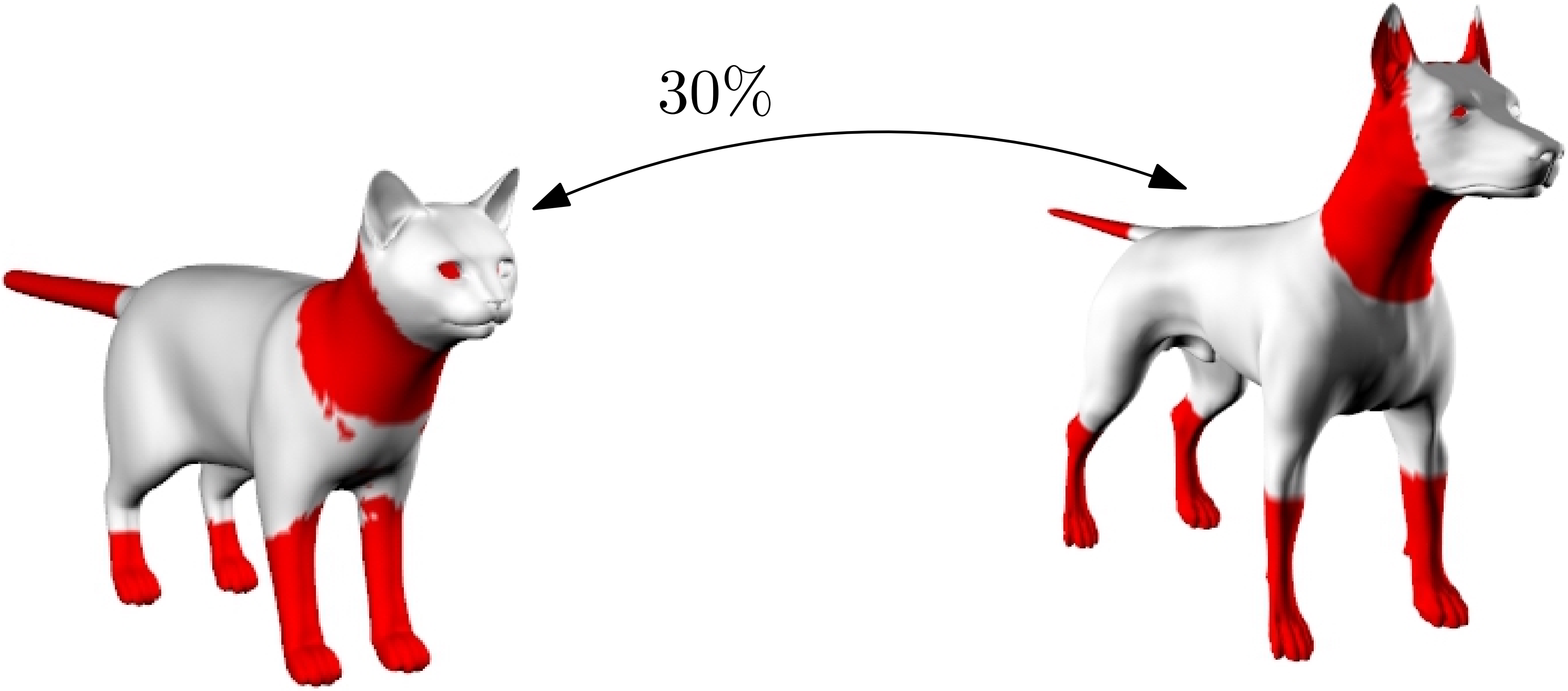

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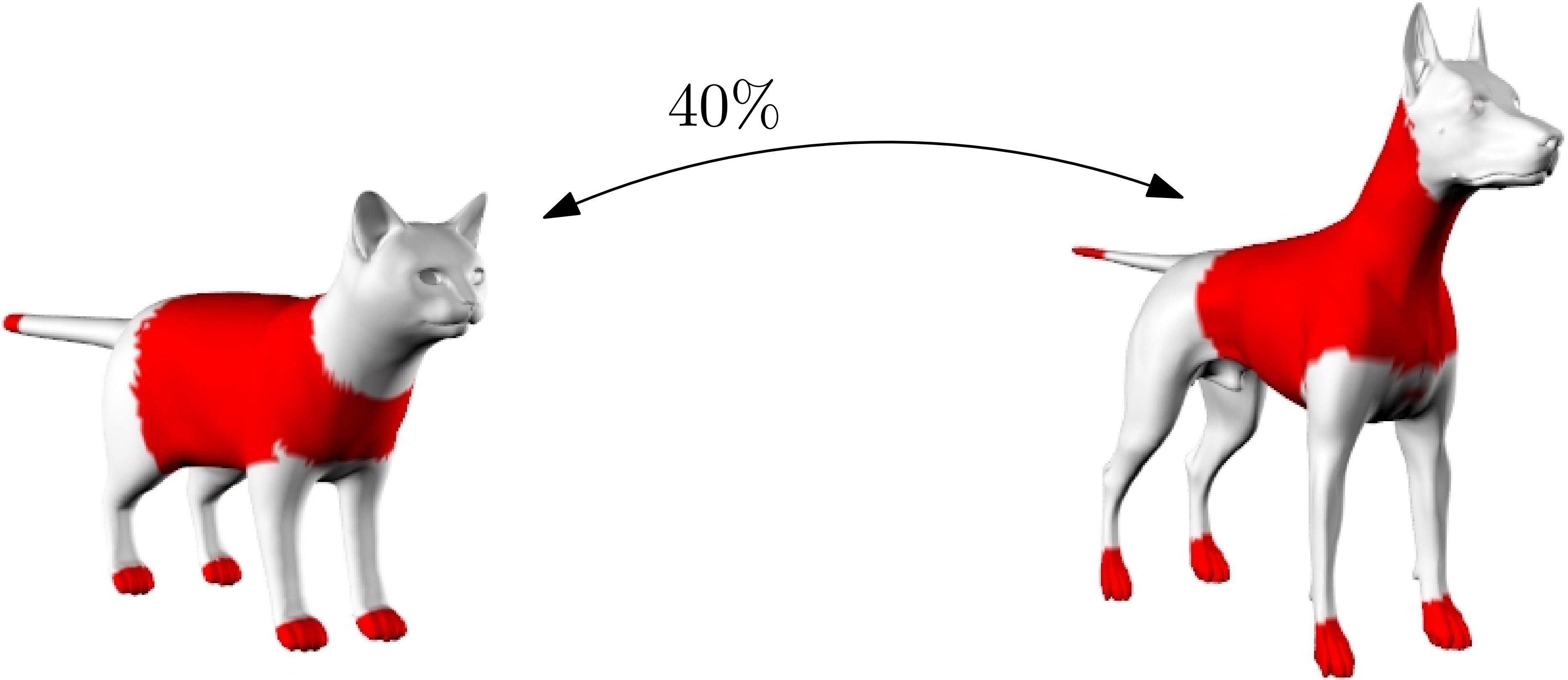

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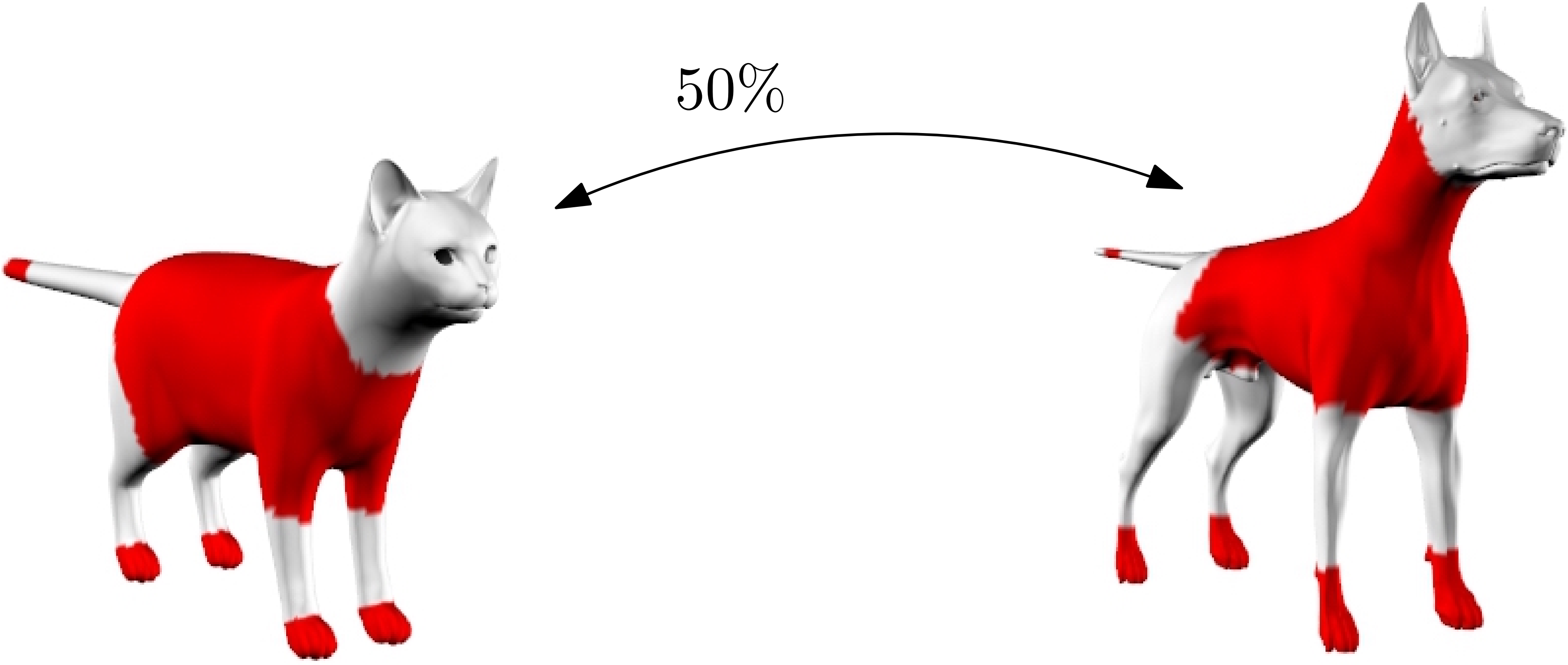
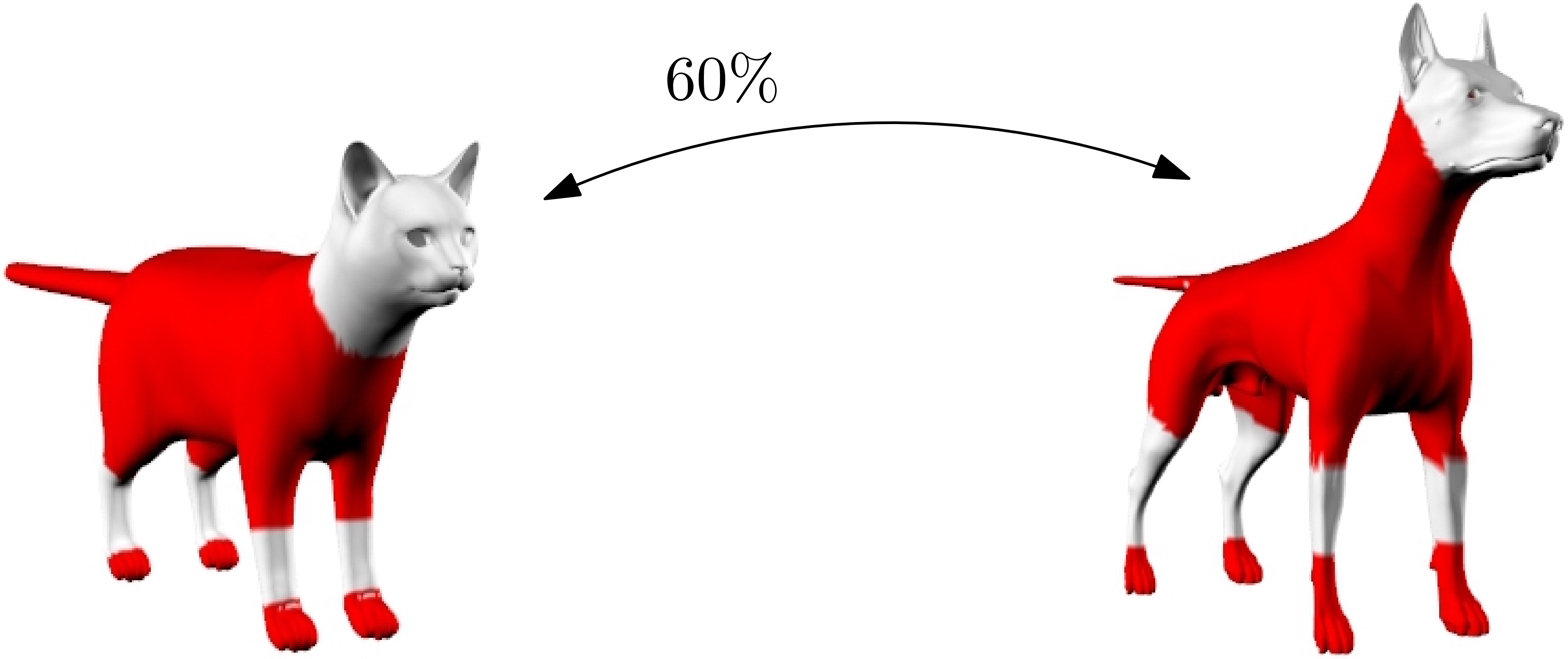


Evolution



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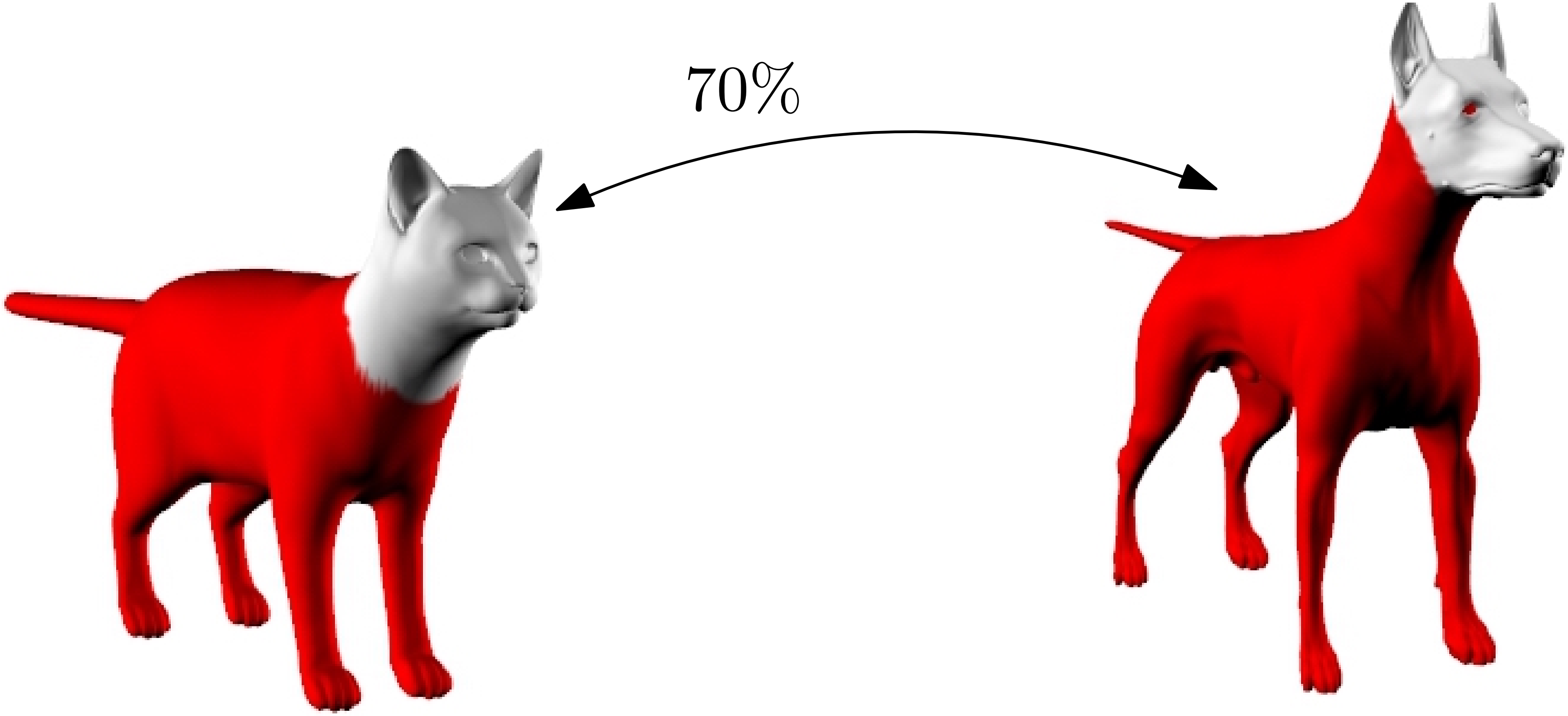

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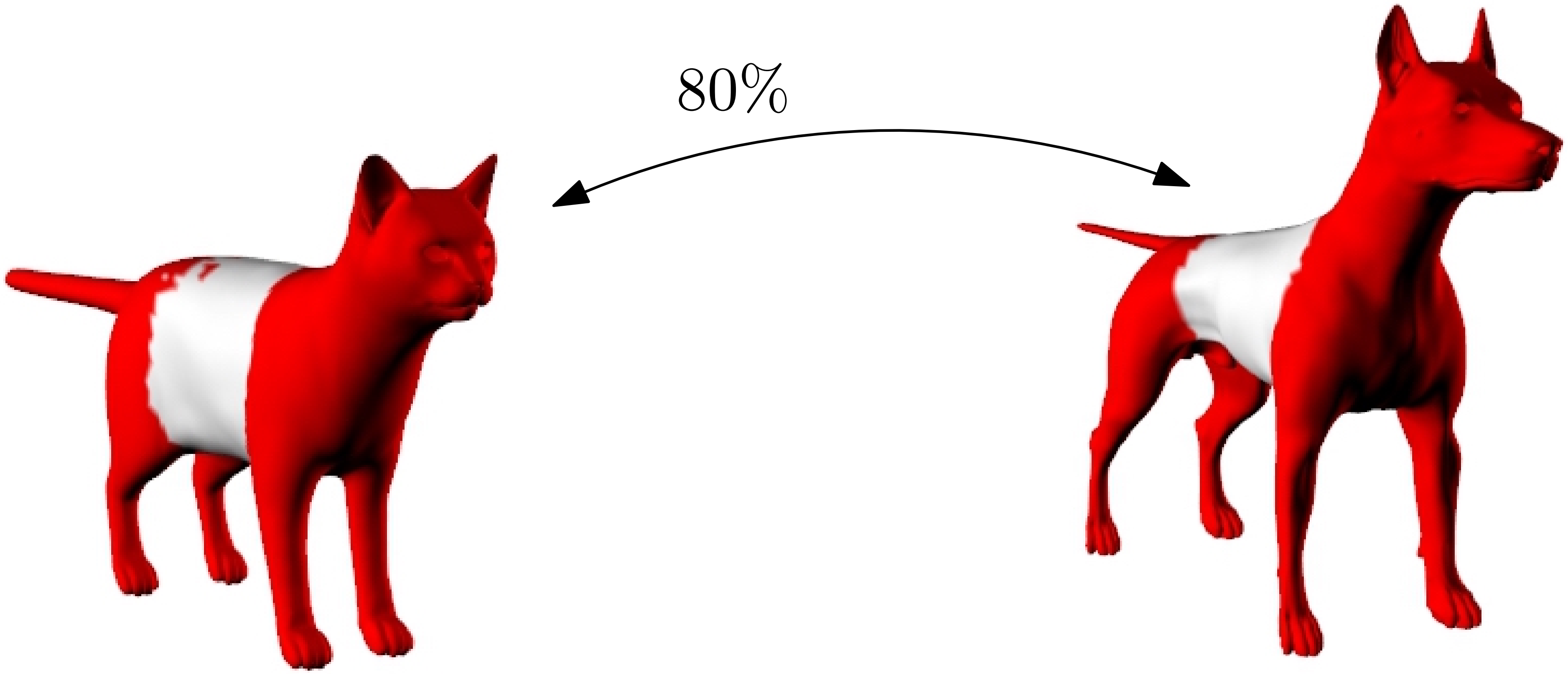

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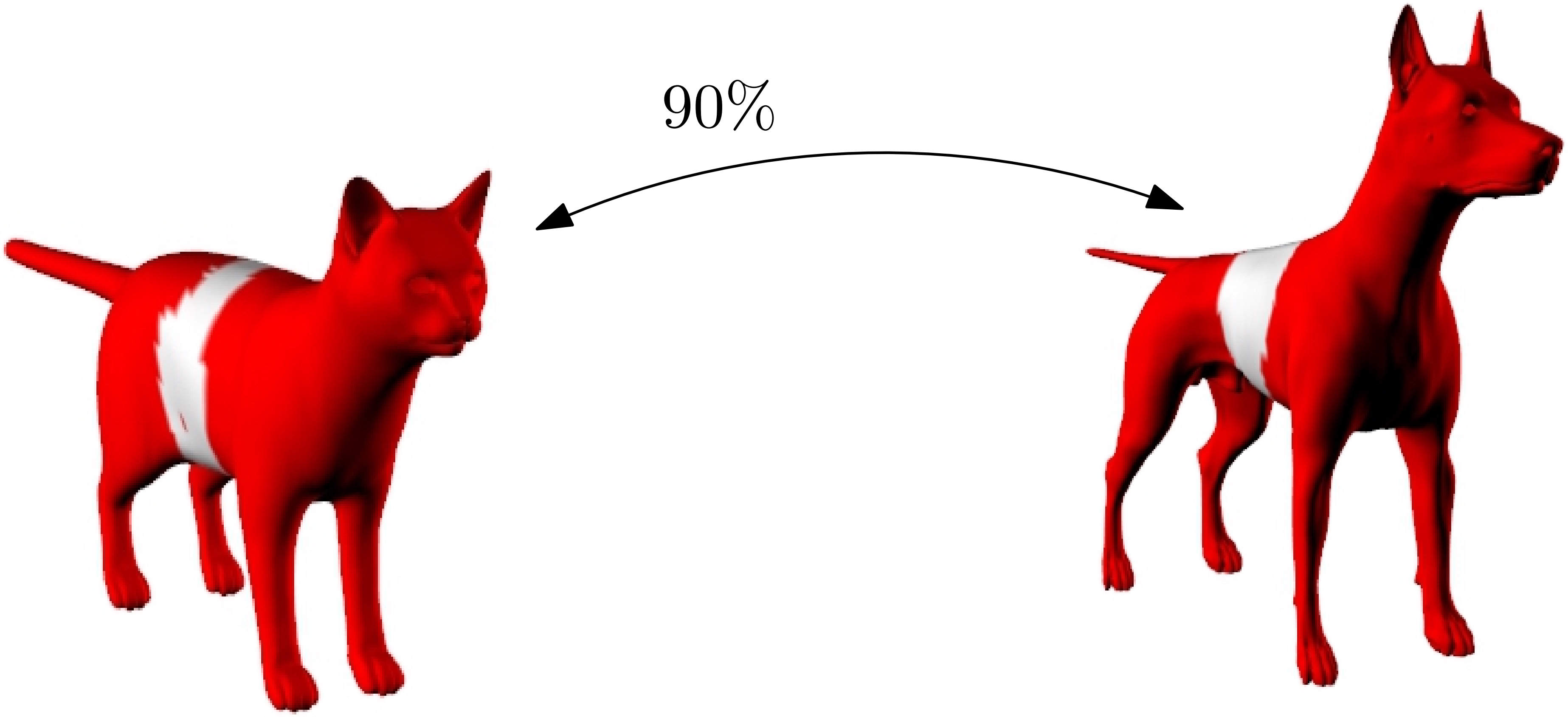

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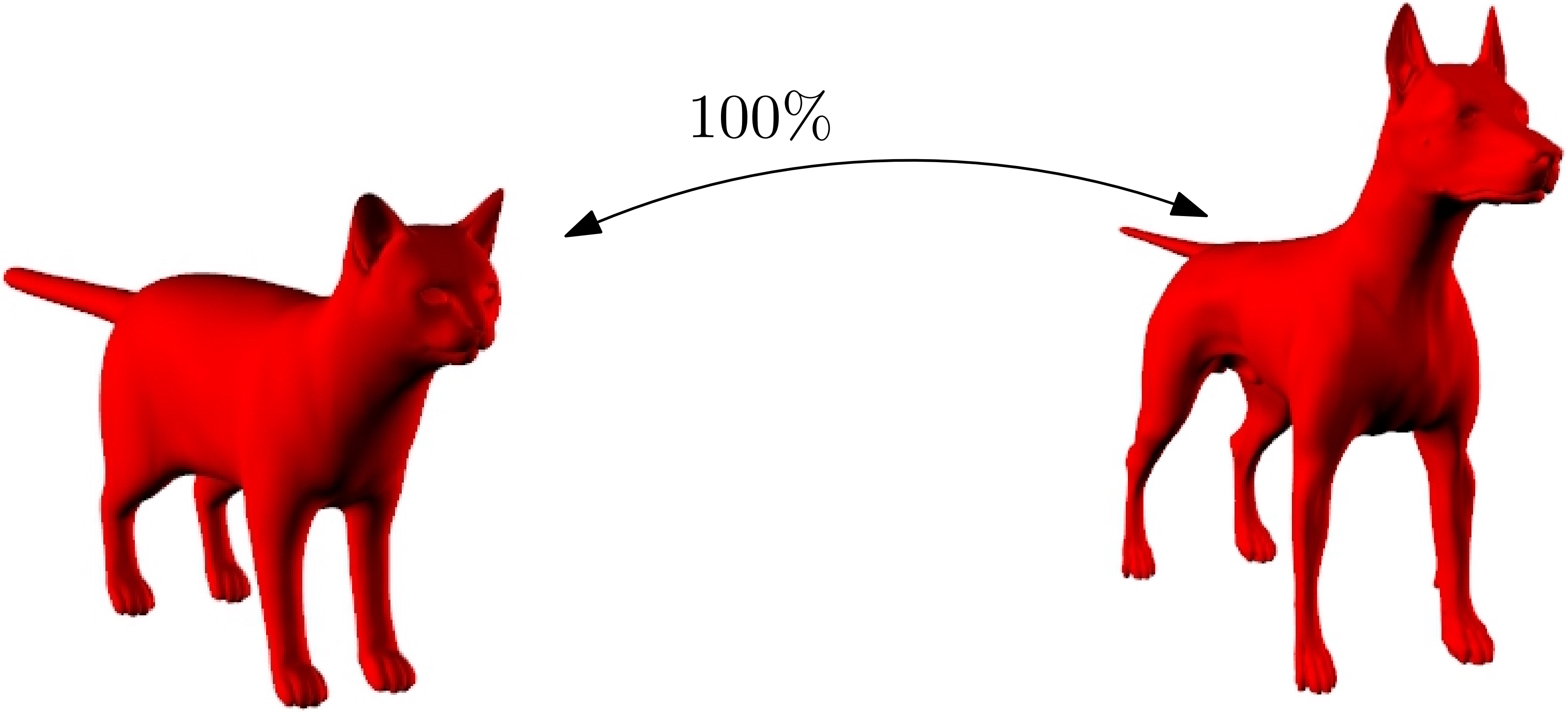

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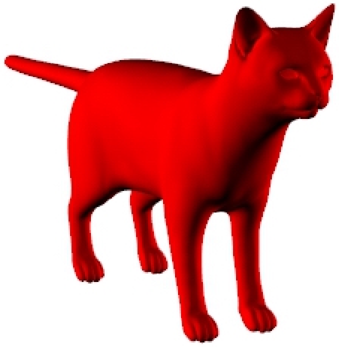
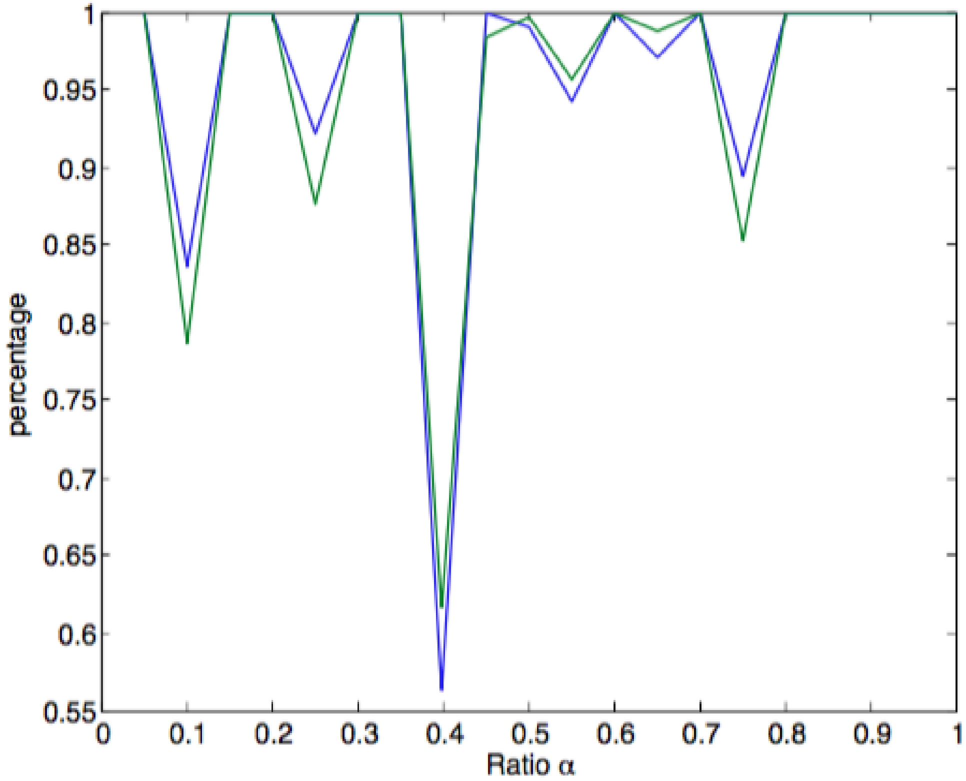
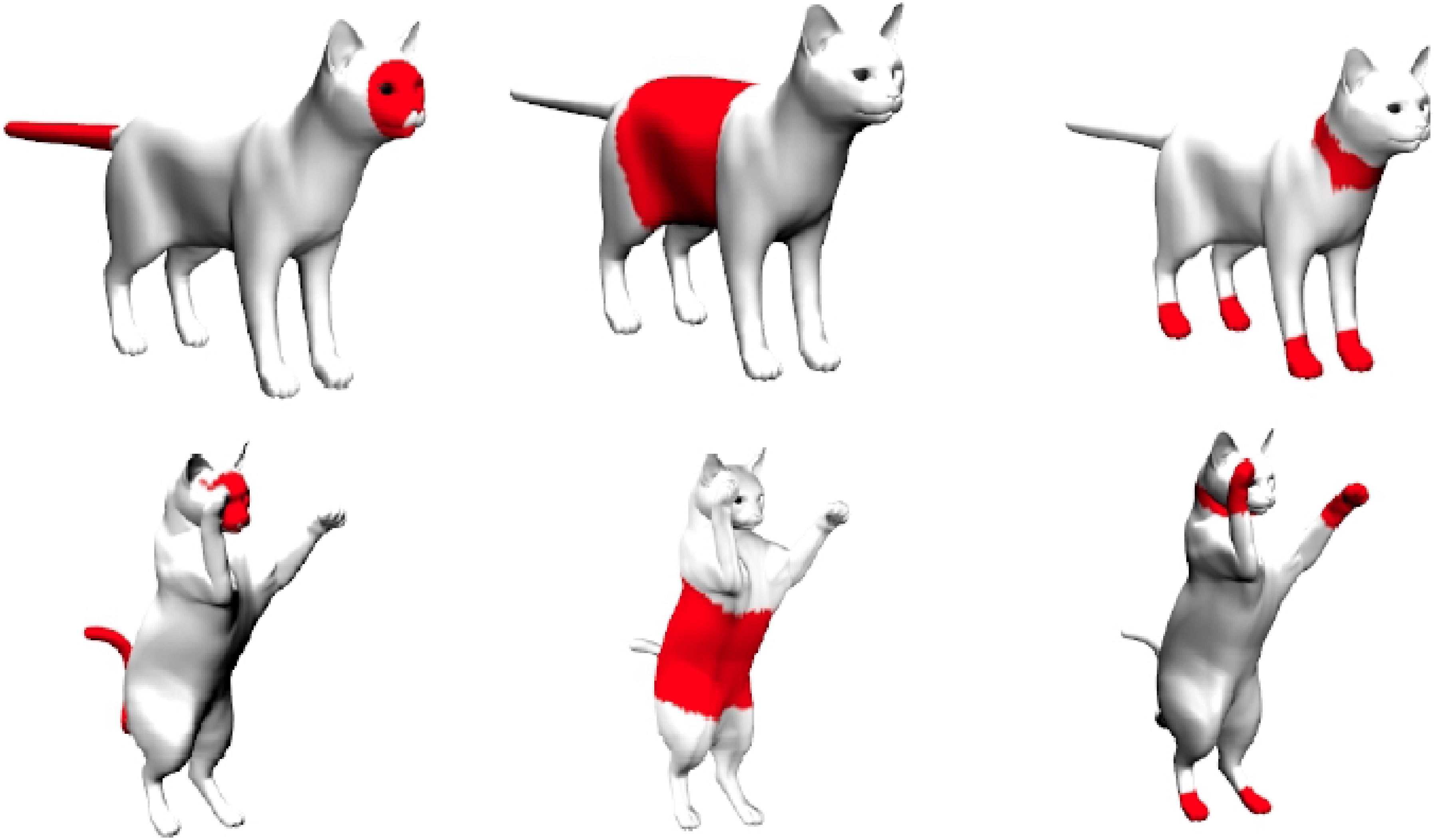


100%

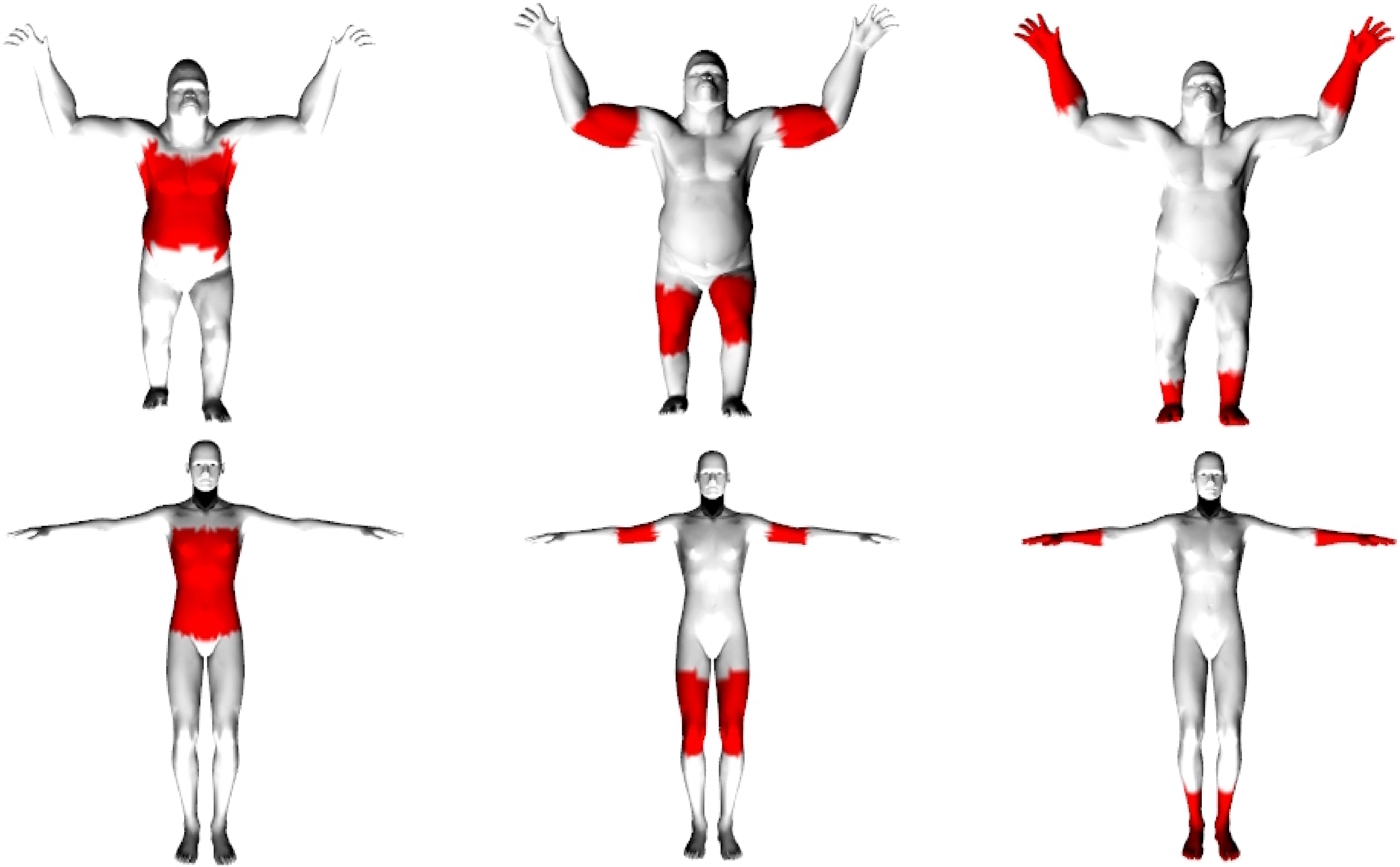
$$\frac{A_i \cap A_{i+1}}{A_i}$$



Isometric Shape Correspondences



Non-Isometric Shape Correspondences



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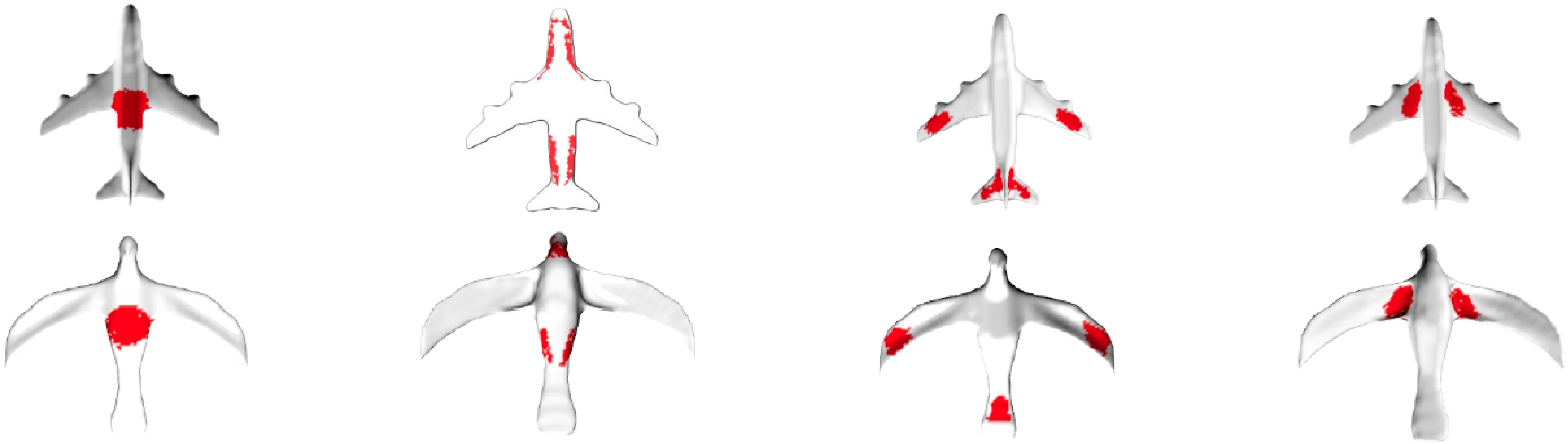

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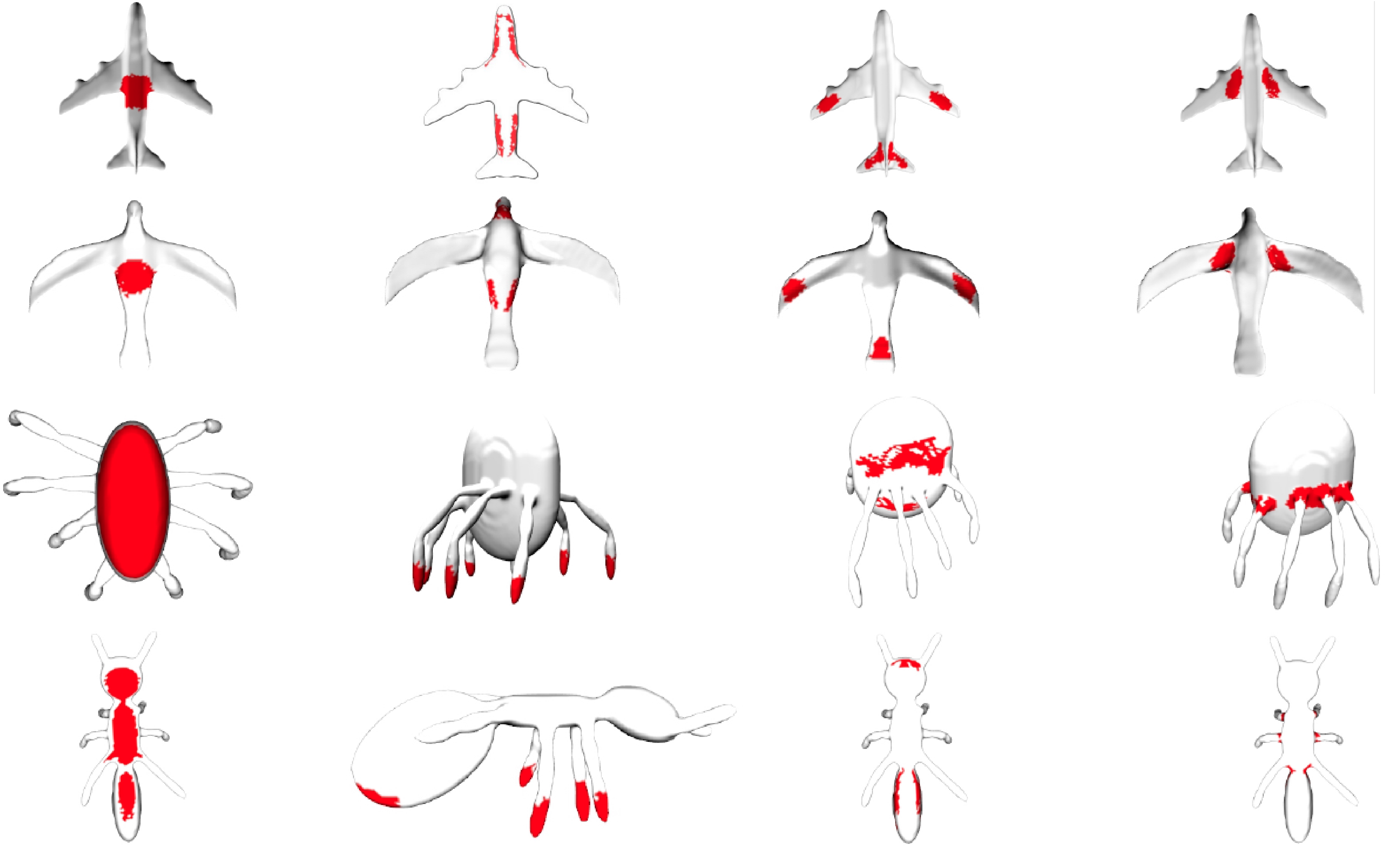

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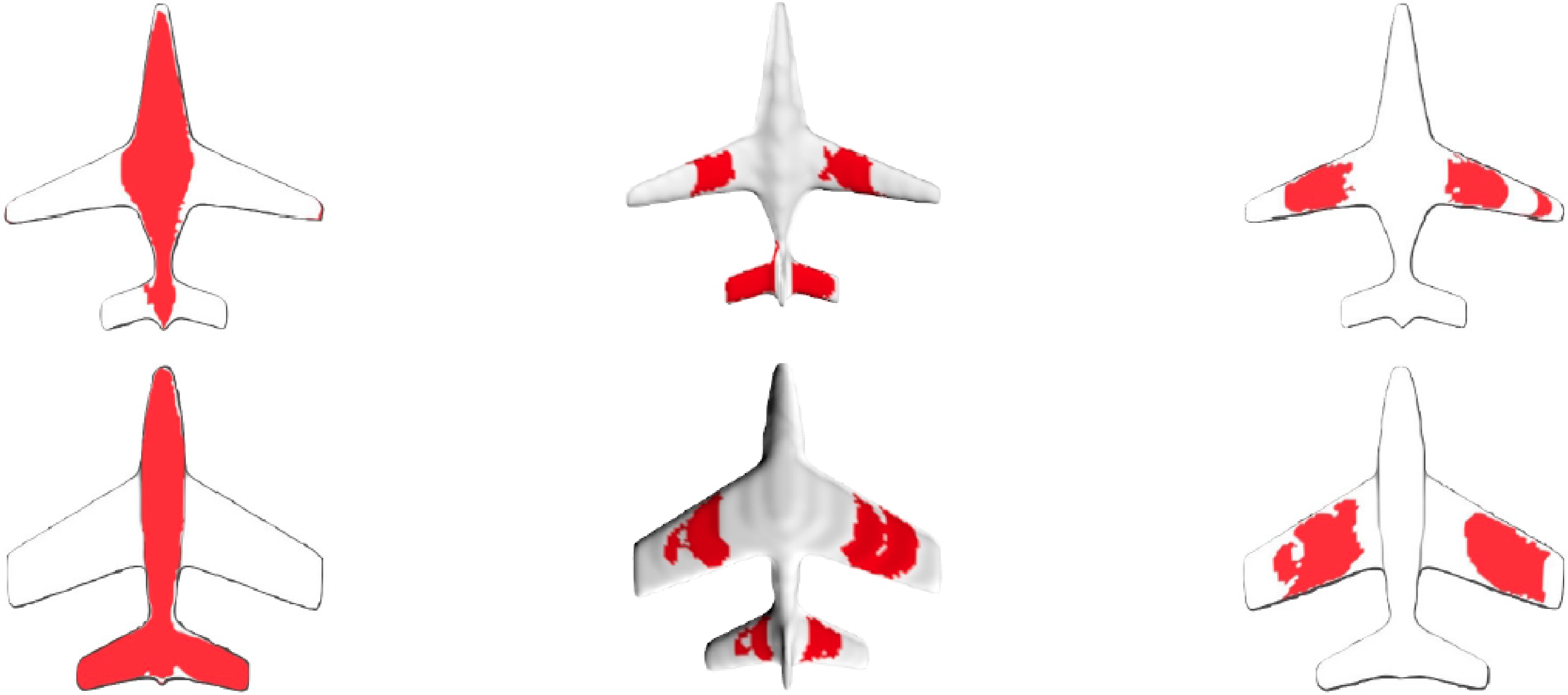

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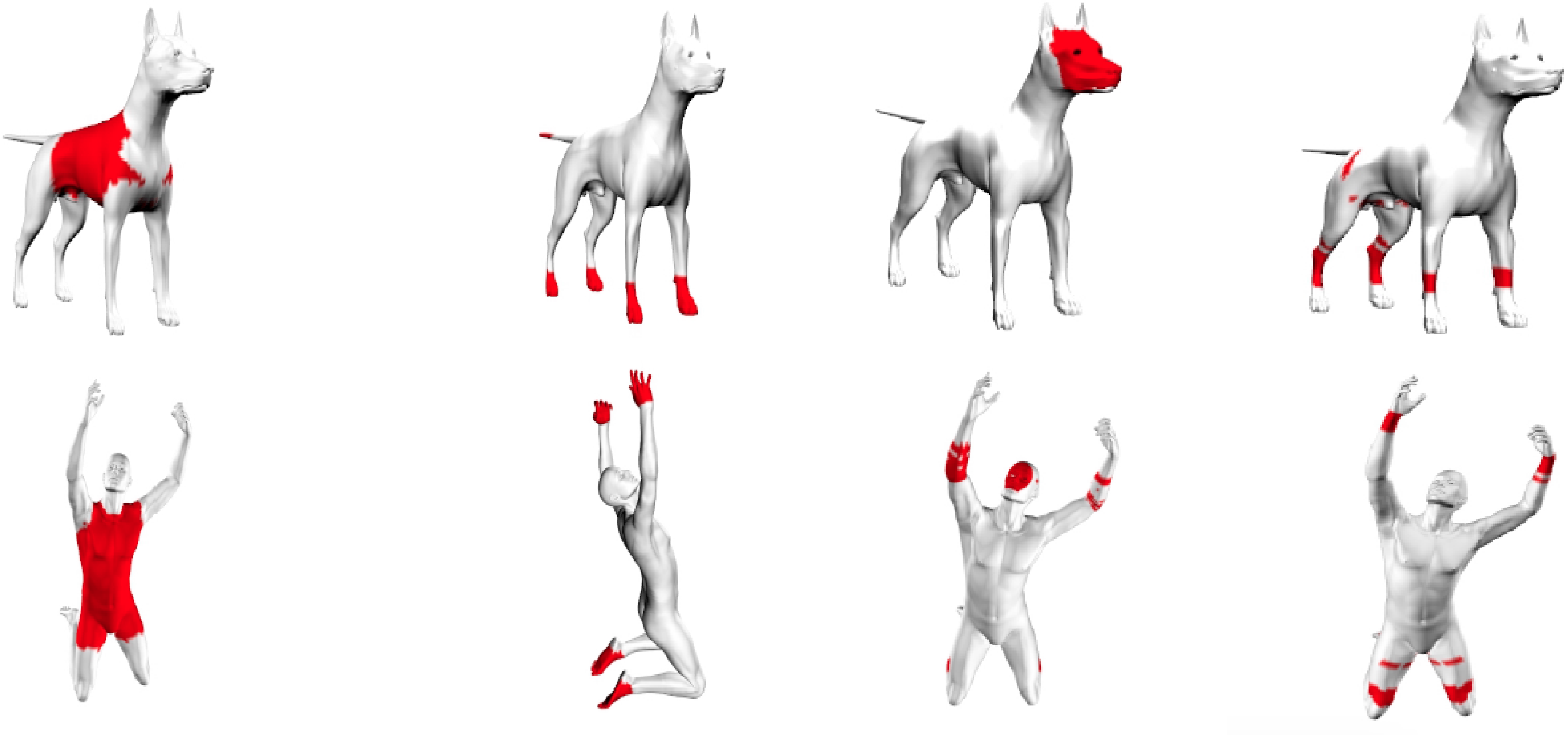


Conclusion

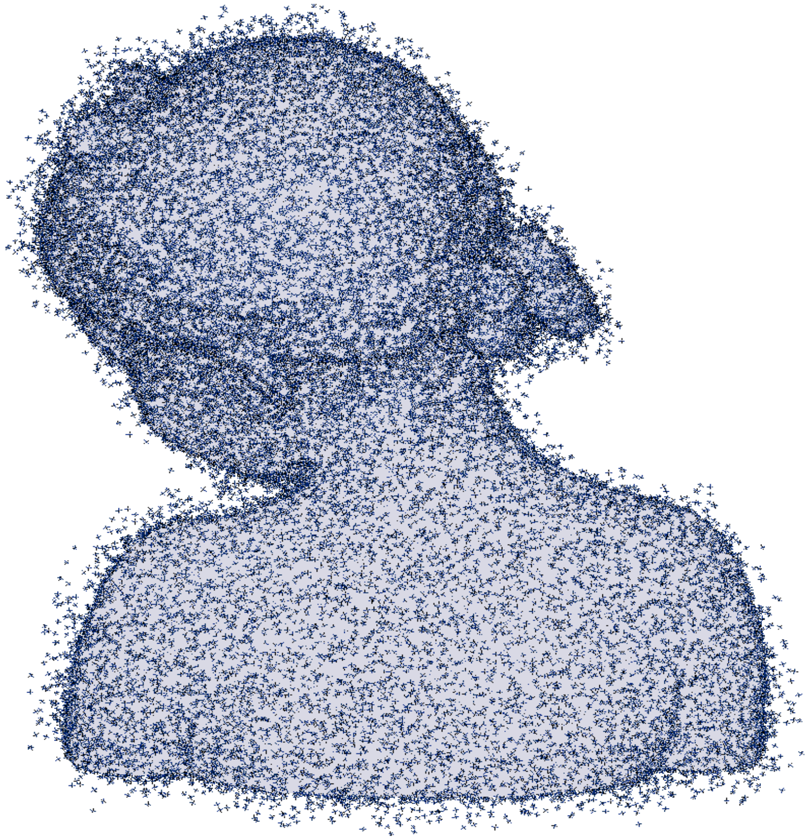
Conclusion

- matching regions, using the idea of sorting function values
- propose a biclustering algorithm, stays in a discrete framework, local warranties
- analyze the set of local minimum (the results depend on the initialization but seem coherent), link with diagonal by block matrices

Generalized Voronoi Covariance Measure

Boris Thibert

with L. Cuel, J.-0. Lachaud, Q Mérigot



TopData and Gudhi workshop, October 17,2016

Distance-like functions

Tool 1. Generalized notion of distance. [Chazal, Cohen-Steiner, Mérigot, 10]

Definition: $\delta : \mathbb{R}^d \longrightarrow \mathbb{R}$ is distance-like if:

$\lim_{\|x\|\to\infty} \delta(x) = +\infty$

δ^2 is 1-semiconcave

Regularity properties of distance functions

Distance-like functions

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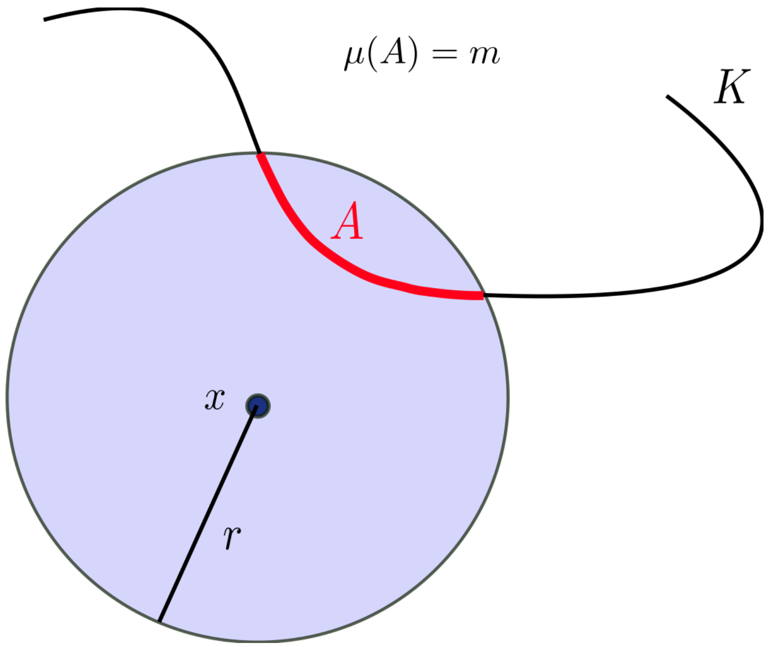
Regularity properties of distance functions

Exemple: The distance to the measure μ

$$d^2_{\mu,m_0}(x) = \frac{1}{m_0} \int_0^{m_0} \delta^2_{\mu,m}(x) dm$$

où $\delta_{\mu,m}(p) = inf\{r \geq 0,\ \mu(B(p,r)) \geq m\}$



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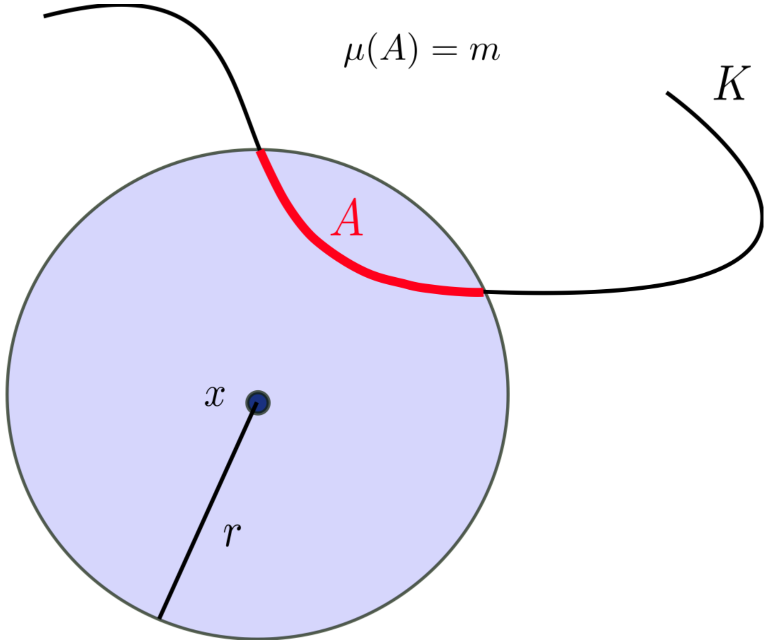
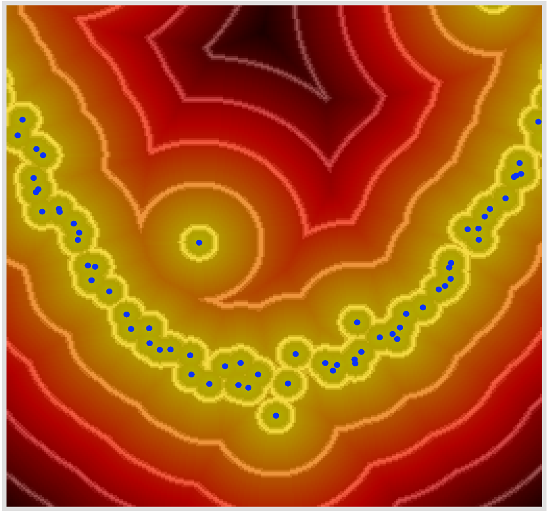
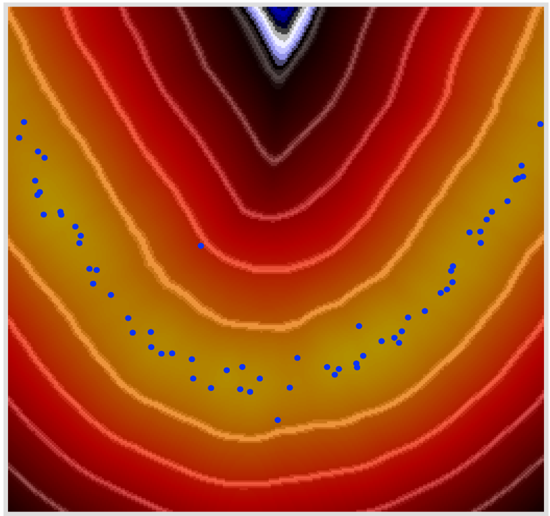


- d_{μ,m_0} is robust to outliers [Chazal, Cohen-Steiner, Mérigot, 10]





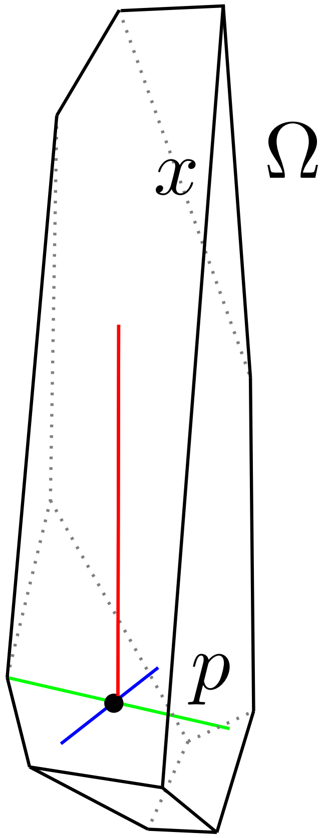
Stable

VCM

Tool 2. Voronoi Covariance Measure. [Mérigot, Ovsianikov, Guibas, 10']

- The Covariance measure of a set Ω

$$\int_{\Omega}(x-p)\otimes(x-p)\mathrm{d}x.$$



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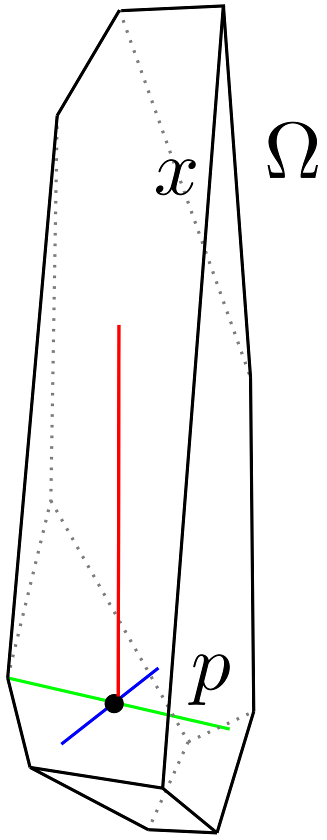


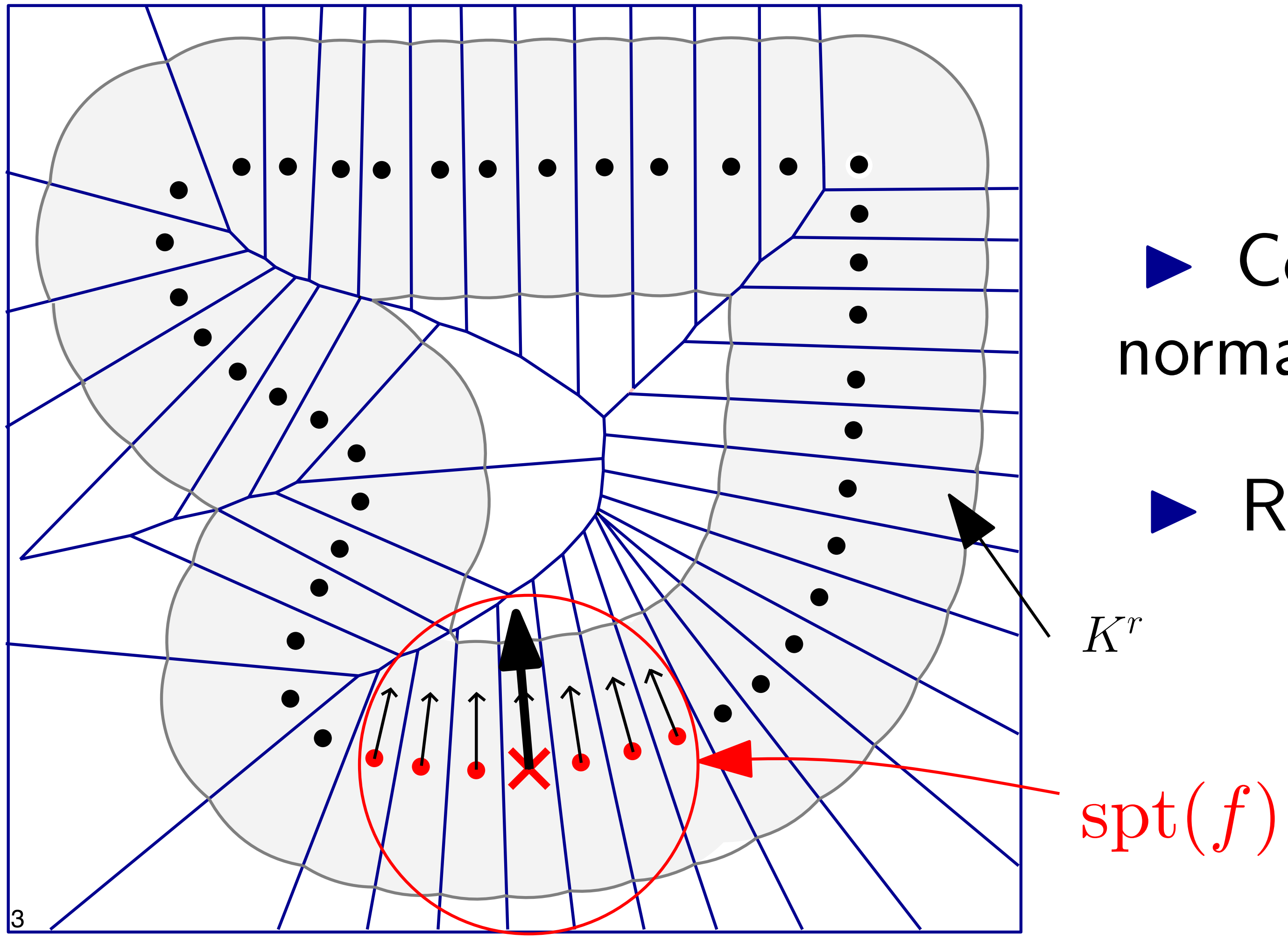
- The Covariance measure of a set Ω

$$\int_{\Omega}(x-p)\otimes(x-p)\mathrm{d}x.$$

- The VCM associates to $f:\mathbb{R}^d\to\mathbb{R}$ the matrix:

$$\mathcal{V}_{d_K,r}(f)=\int_{K^r}(x-p_K(x))\otimes(x-p_K(x)).f(p_K(x))\mathrm{d}x.$$

K: finite point set.



- Contains geometric information normals, principal directions,...

- Robust to a Hausdorff perturbation

Generalized VCM

Definition (Generalized Voronoi Covariance Measure).
Let $\delta : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a distance-like function. The δ-VCM associates to $f : \mathbb{R}^d \to \mathbb{R}$ the matrix

$$\mathcal{V}_{\delta,r}(f) := \int_{\delta^r} \mathbf{n}_\delta(x) \otimes \mathbf{n}_\delta(x) . f\left(x - \mathbf{n}_\delta(x)\right) \mathrm{d}x,$$

replaces $x - p_K(x)$

where $\delta^r := \delta^{-1}((-\infty, r])$ and $\mathbf{n}_\delta(x) := \frac{1}{2}\nabla\delta^2(x)$.

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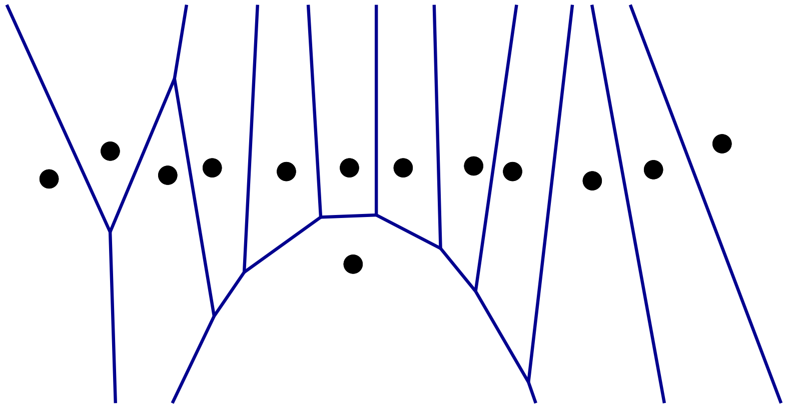
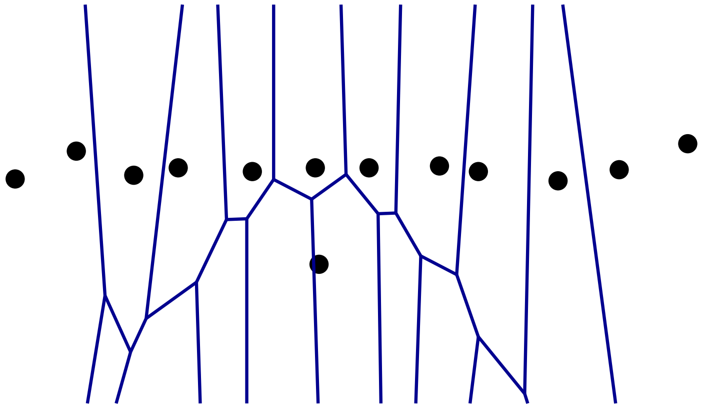
Point cloud and k-distance

VCM



Intersection Voronoi cells with balls

k-VCM



Intersection power cells with balls

$\sum_{p\in P}\chi(p)\int_{\mathrm{Pow}_P(p)\cap\delta_P^R}(x-p)\otimes(x-p)\mathrm{d}x.$

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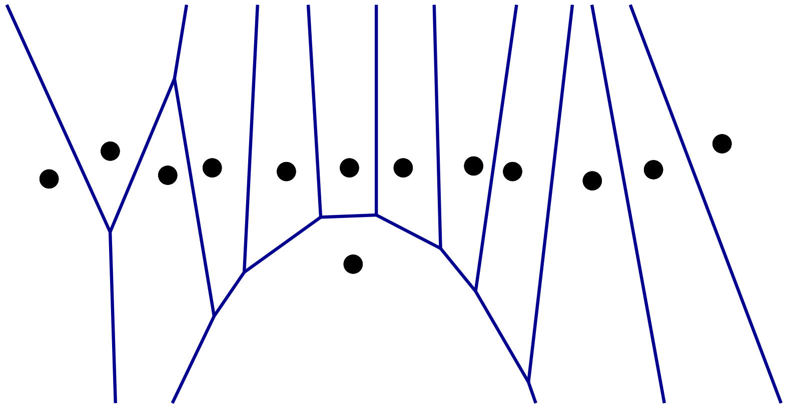
VCM



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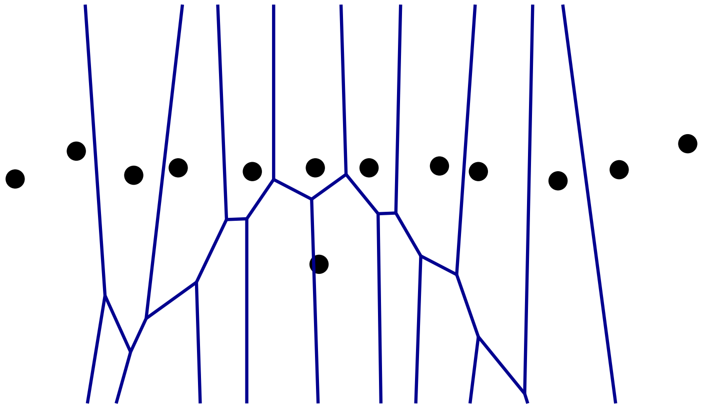
k-VCM



Intersection power cells with balls

$\sum_{p\in P} \chi(p) \int_{\mathrm{Pow}_P(p)\cap\delta_P^R} (x-p)\otimes(x-p)\mathrm{d}x.$

- Works with power distance (witness k-distance)

$$\delta_P(x) := \left(\min_{p\in P}\left(\|x-p\|^2 + \omega_p\right)\right)^{1/2}.$$

- Computations with power diagrams (CGAL)

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replaces $x - p_K(x)$

Theorem Let K be a compact set, δ a distance-like function.
For any bounded Lipschitz function $\chi : \mathbb{R}^d \to \mathbb{R}^+$,

$$\|\mathcal{V}_{\delta,R}(\chi) - \mathcal{V}_{d_K,R}(\chi)\| \leq C_1\ \|\chi\|_{\mathrm{BL}} \|\delta - d_K\|_\infty^{\frac{1}{2}},$$

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Corollary (Cuel, Lachaud, Mérigot, T., 15') Let S be a surface of $\mathbb{R}^3$ and P a point cloud. There exists $m > 0$, s.t. for any $\chi : \mathbb{R}^d \longrightarrow \mathbb{R}$

$$\|\mathcal{V}_{\mathrm{d}_{\mu_P,m},r}(f) - \mathcal{V}_{\mathrm{d}_S,r}(f)\|_{\mathrm{op}} \leq C \, \|f\|_{\mathrm{BL}} \; W_2(\mu_S, \mu_P)^{\frac{1}{4}}$$

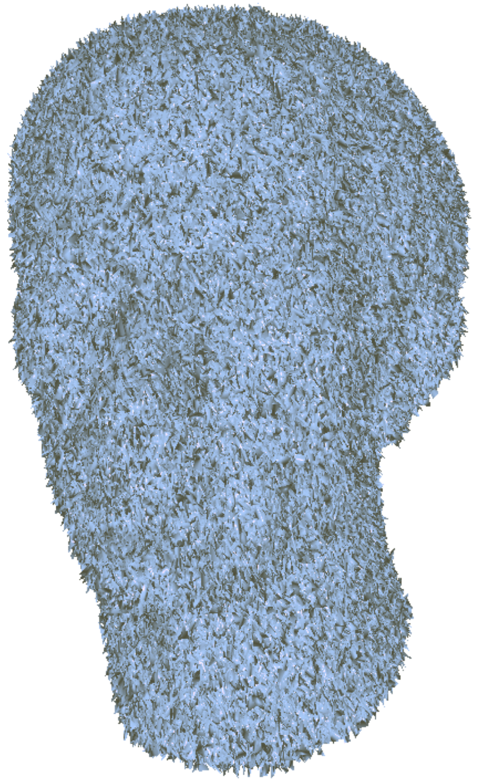
distance to μ_P

distance to S

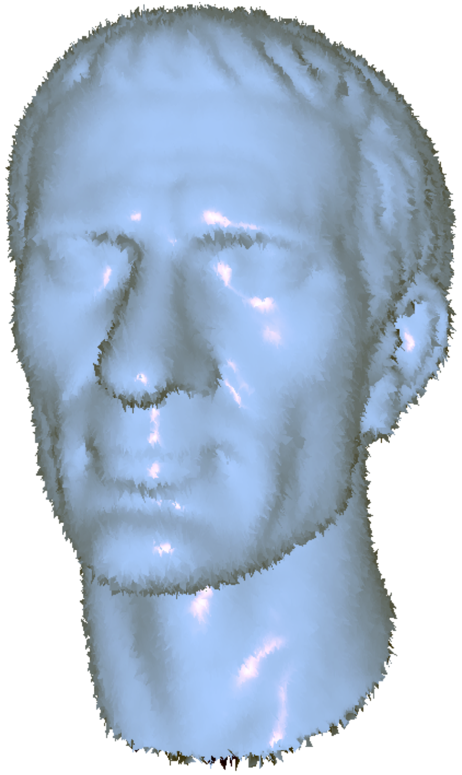
uniform probability measures on S and P

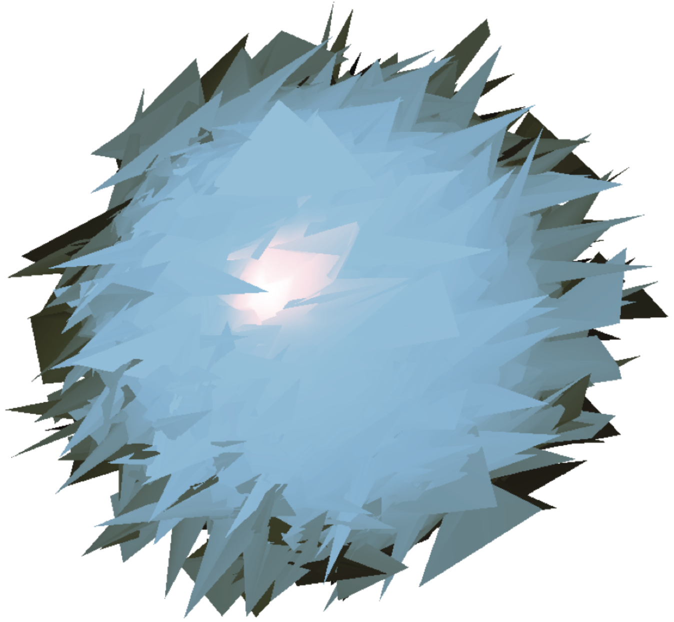
Generalized VCM

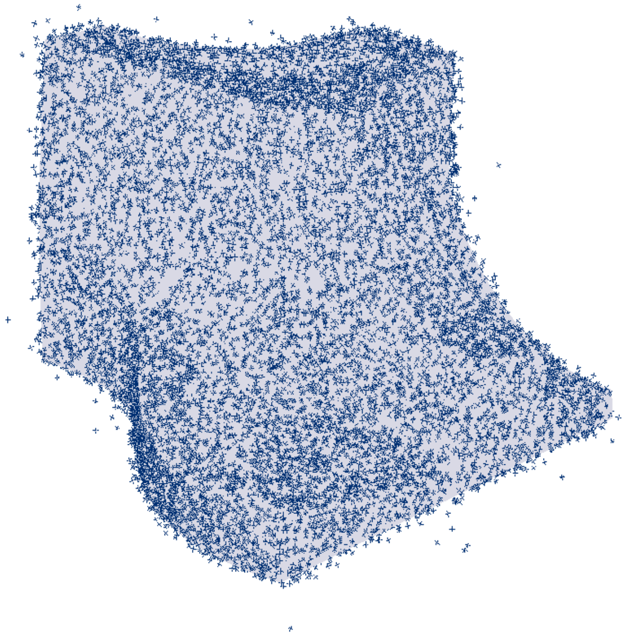
geometric normal



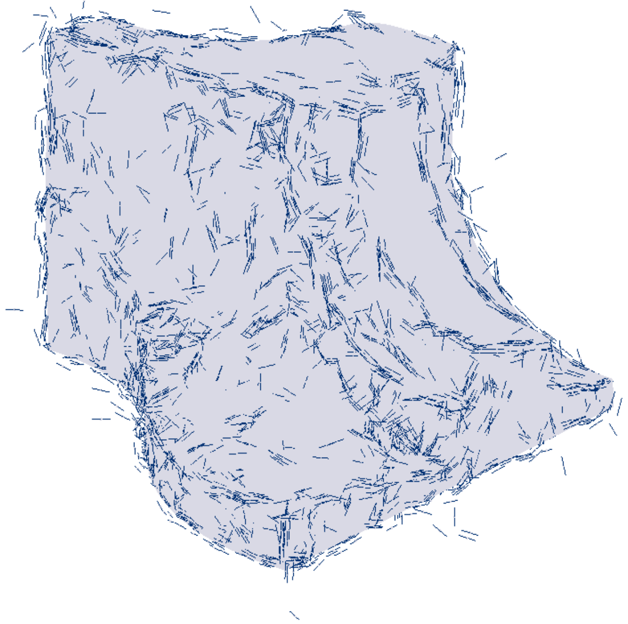
δ-VCM normal



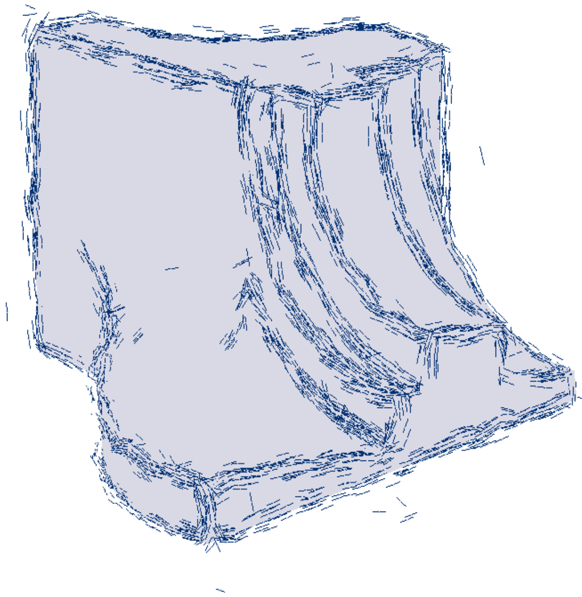




Input



VCM



δ-VCM