T-coercivity: a practical tool for the study of variational formulations

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Outline

What is T-coercivity?

A tool to study variational formulations [Chesnel-PC'13]

Abstract framework: *Find* $u \in V$ *s.t.* $\forall w \in W,\ a(u,w) = {}_{W'}\langle f, w\rangle_W$.

Approximate framework: *Find* $u_\delta \in V_\delta$ *s.t.* $\forall w_\delta \in W_\delta,\ a(u_\delta, w_\delta) = {}_{W'}\langle f, w_\delta\rangle_W$.

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1. First, analyse the variational formulation theoretically:
   - prove well-posedness ;
   - existence, uniqueness and continuous dependence of the solution with respect to the data.

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Within the framework of T-coercivity, steps 1 and 2 are very strongly correlated!

What is T-coercivity?
As an abstract tool

Let

- V, W be Hilbert spaces ;
- $a(\cdot,\cdot)$ be a continuous sesquilinear form on $V \times W$;
- f be an element of W', the dual space of W.

Solve

$$\text{(VF)} \quad \textit{Find } u \in V \textit{ s.t. } \forall w \in W,\ a(u,w) = {}_{W'}\langle f, w\rangle_W .$$

[Banach-Nečas-Babuška] The *inf-sup condition* writes

$$\text{(isc)} \quad \exists \alpha > 0,\ \forall v \in V,\ \sup_{w \in W\setminus\{0\}} \frac{|a(v,w)|}{\|w\|_W} \geq \alpha\, \|v\|_V .$$

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Definition (T-coercivity)

The form $a(\cdot,\cdot)$ is T-*coercive* if

$$\exists \mathtt{T} \in \mathcal{L}(V,W) \text{ bijective},\ \exists \underline{\alpha} > 0,\ \forall v \in V,\ |a(v,\mathtt{T}v)| \geq \underline{\alpha}\,\|v\|_V^2.$$

NB. In other words, the form $a(\cdot,\mathtt{T}\cdot)$ is coercive on $V \times V$.

Let

- V, W be Hilbert spaces;
- $a(\cdot,\cdot)$ be a continuous sesquilinear form on $V \times W$;
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Solve

$$\text{(VF)} \quad \textit{Find } u \in V \textit{ s.t. } \forall w \in W,\ a(u,w) = {}_{W'}\langle f, w\rangle_W .$$

Theorem (Well-posedness)

The three assertions below are equivalent:

(i) *the Problem (VF) is well-posed;*

(ii) *the form $a(\cdot,\cdot)$ satisfies (isc) and $\{w \in W \mid \forall v \in V,\ a(v,w) = 0\} = \{0\}$;*

(iii) *the form $a(\cdot,\cdot)$ is $\mathtt{T}$-coercive.*

The operator $\mathtt{T}$ realises the inf-sup condition (isc) explicitly: $w = \mathtt{T}u$ works!

What is T-coercivity?

As an abstract tool (simplified)

Let

- V be a Hilbert space ;
- $a(\cdot,\cdot)$ be a continuous, sesquilinear, *hermitian* form on $V \times V$;
- f be an element of V', the dual space of V.

Solve

$$\text{(VF)} \quad \textit{Find } u \in V \textit{ s.t. } \forall w \in V,\ a(u,w) = {}_{V'}\langle f, w\rangle_V.$$

Let

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Definition (T-coercivity, hermitian case)

The form $a(\cdot,\cdot)$ is T-*coercive* if

$$\exists \mathtt{T} \in \mathcal{L}(V),\ \exists \underline{\alpha} > 0,\ \forall v \in V,\ |a(v, \mathtt{T}v)| \geq \underline{\alpha}\, \|v\|_V^2 .$$

Let

- V be a Hilbert space ;
- $a(\cdot,\cdot)$ be a continuous, sesquilinear, *hermitian* form on $V \times V$;
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Solve

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Theorem (Well-posedness, hermitian case)

The three assertions below are equivalent:

(i) *the Problem (VF) is well-posed ;*

(ii) *the form $a(\cdot,\cdot)$ satisfies (isc) ;*

(iii) *the form $a(\cdot,\cdot)$ is $\mathtt{T}$-coercive (hermitian case).*

The operator $\mathtt{T}$ realises the inf-sup condition (isc) explicitly.

Let

- $(V_\delta)_\delta$ be a family of finite dimensional subspaces of V ;
- $(W_\delta)_\delta$ be a family of finite dimensional subspaces of W.

Assume that $\dim(V_\delta) = \dim(W_\delta)$ for all $\delta > 0$.
Solve

$$(\mathrm{VF})_\delta \quad \textit{Find } u_\delta \in V_\delta \textit{ s.t. } \forall w_\delta \in W_\delta,\ a(u_\delta, w_\delta) = {}_{W'}\langle f, w_\delta\rangle_W.$$

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[Banach-Nečas-Babuška] The *uniform discrete inf-sup condition* writes

$$(\mathrm{udisc}) \quad \exists \alpha_\dagger > 0,\ \forall \delta > 0,\ \forall v_\delta \in V_\delta,\ \sup_{w_\delta \in W_\delta \setminus \{0\}} \frac{|a(v_\delta, w_\delta)|}{\|w_\delta\|_W} \geq \alpha_\dagger \|v_\delta\|_V .$$

NB. When (udisc) is fulfilled, $(\mathrm{VF})_\delta$ is well-posed for all $\delta > 0$.

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$$(\mathsf{VF})_\delta \quad \textit{Find } u_\delta \in V_\delta \textit{ s.t. } \forall w_\delta \in W_\delta,\ a(u_\delta, w_\delta) = {}_{W'}\langle f, w_\delta\rangle_W.$$

Definition (uniform $\mathtt{T}_\delta$-coercivity)

The form a is *uniformly* $\mathtt{T}_\delta$*-coercive* if

$$\exists \underline{\alpha}_\dagger, \underline{\beta}_\dagger > 0,\ \forall \delta > 0,\ \exists \mathtt{T}_\delta \in \mathcal{L}(V_\delta, W_\delta),\ |||\mathtt{T}_\delta||| \le \underline{\beta}_\dagger \text{ and } \forall v_\delta \in V_\delta,\ |a(v_\delta, \mathtt{T}_\delta v_\delta)| \ge \underline{\alpha}_\dagger \|v_\delta\|_V^2.$$

NB. When a is uniformly $\mathtt{T}_\delta$-coercive, $(\mathsf{VF})_\delta$ is well-posed for all $\delta > 0$.

Let

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Theorem (Céa's lemma)

Assume that the family $(V_\delta)_\delta$ fulfills the basic approximability property in V.
In addition, assume that

(i) *either, the form $a(\cdot,\cdot)$ satisfies (udisc) ;*

(ii) *or, the form $a(\cdot,\cdot)$ is uniformly T_δ-coercive.*

Then, $\lim_{\delta\to 0} \|u - u_\delta\|_V = 0$.

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(i) *either, the form $a(\cdot,\cdot)$ satisfies (udisc) ;*

(ii) *or, the form $a(\cdot,\cdot)$ is uniformly T_δ-coercive.*

Then, $\lim_{\delta\to 0} \|u - u_\delta\|_V = 0$. And error estimates whenever possible...

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Two key ideas [Chesnel-PC'13]

[1st Key Idea] Use the knowledge on operator $\mathtt{T}$ to derive the discrete operators $(\mathtt{T}_\delta)_\delta$!

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Two key ideas [Chesnel-PC'13]

[1st Key Idea] Use the knowledge on operator $\mathtt{T}$ to derive the discrete operators $(\mathtt{T}_\delta)_\delta$!

[2nd Key Idea] Discretize the variational formulation with (bijective) operator $\mathtt{T}$:

$$(\text{VF})_{\mathtt{T}} \quad \textit{Find } u \in V \textit{ s.t. } \forall v \in V,\ a(u, \mathtt{T}v) = {}_{W'}\langle f, \mathtt{T}v \rangle_W \ !$$

Given $\delta > 0$, let $N = \dim(V_\delta)$. $(\mathrm{VF})_\delta$ is equivalent to
Solve

$$\textit{Find } U \in \mathbb{C}^N \textit{ s.t. } \forall W \in \mathbb{C}^N,\ (\mathbb{A}U|W) = (F|W).$$

$$\textit{Or, find } U \in \mathbb{C}^N \textit{ s.t. } \mathbb{A}U = F.$$

What is T-coercivity?

As an approximation tool (solving the equivalent linear system)

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According to the uniform $\mathtt{T}_\delta$-coercivity assumption

$$\forall V \in \mathbb{C}^N,\ |(\mathbb{T}^*\mathbb{A}V|V)| \geq \underline{\alpha}_\dagger(\mathbb{M}V|V).$$

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[2nd Key Idea] Use $\mathbb{T}$ associated with $\mathtt{T}$ for the approximation of (VF)$_\mathtt{T}$...

What is T-coercivity?

Can be applied to various types of variational formulations

† = Abstract T-coercivity only.

1. Coercive plus compact formulations. See for instance:
   - integral equations: Buffa-Costabel-Schwab'02 [Θ-coercivity]; Buffa-Christiansen'03; Buffa-Christiansen'05; Buffa'05; Unger'21; Levadoux (2022, HAL report) [τ-coercivity].
   - volume equations: Hiptmair'02 ["$(X+S)$-coercivity"]; Buffa'05; PC'12 ["elementary" proofs]; Hohage-Nannen'15 [S-coercivity]; Sayas-Brown-Hassell'19†; Halla'21 ["generalized" proofs].
2. Formulations with sign-changing coefficients. See for instance:
   - for scalar models: BonnetBenDhia-PC-Zwölf'10; BonnetBenDhia-Chesnel-Haddar'11†; Nicaise-Venel'11; BonnetBenDhia-Chesnel-PC'12†; Chesnel-PC'13; Bunoiu-Ramdani'16†; Carvalho-Chesnel-PC'17; BonnetBenDhia-Carvalho-PC'18; Bunoiu-Ramdani-Timofte'21-'22-'23†; Carvalho-Moitier'23; Halla-Hohage-Oberender (2024, ArXiv report).
   - for EM models: BonnetBenDhia-Chesnel-PC'14† (2D-3D); PC'22 (3D); Halla'23 (2D); Yang-Wang-Mao'23 (3D).
3. Mixed formulations.
   - for the Stokes model: see below!
   - for diffusion models: Jamelot-PC'13; PC-Jamelot-Kpadonou'17; see below!
   - for static models in electromagnetism: Barré-PC (to appear, 2023); PC-Jamelot'24; see below!

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Outline

Further remarks

1. Let Ω be a simply connected domain of $\mathbb{R}^3$ with a connected boundary. The magnetostatic equations write

$$\begin{cases} \mathbf{curl}(\mu^{-1}\boldsymbol{B}) = \boldsymbol{J} \text{ in } \Omega \\ \operatorname{div} \boldsymbol{B} = 0 \text{ in } \Omega \\ \boldsymbol{B} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega, \end{cases}$$

for some uniformly elliptic, bounded tensor $\boldsymbol{x} \mapsto \mu(\boldsymbol{x})$ (magnetic permeability).

1. Assuming that $\boldsymbol{J} \in \boldsymbol{H}(\operatorname{div} 0; \Omega)$, one analyses mathematically the model

$$(\mathrm{MSt})_{\boldsymbol{B}} \qquad \begin{cases} \text{Find } \boldsymbol{B} \in \boldsymbol{L}^2(\Omega) \text{ such that} \\ \mathbf{curl}(\mu^{-1}\boldsymbol{B}) = \boldsymbol{J} \text{ in } \Omega \\ \operatorname{div} \boldsymbol{B} = 0 \text{ in } \Omega \\ \boldsymbol{B} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega. \end{cases}$$

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Since $\boldsymbol{B} \in \boldsymbol{H}_0(\operatorname{div} 0; \Omega)$, there exists one, and only one, $\boldsymbol{A} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega) \cap \boldsymbol{H}(\operatorname{div} 0; \Omega)$ such that $\boldsymbol{B} = \mathbf{curl}\, \boldsymbol{A}$ in Ω. We study the model in the vector potential $\boldsymbol{A}$.

1. Assuming that $\boldsymbol{J} \in \boldsymbol{H}(\operatorname{div} 0; \Omega)$, one analyses mathematically the model

$$(\mathrm{MSt})_{\boldsymbol{A}} \qquad \begin{cases} \text{Find } \boldsymbol{A} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega) \text{ such that} \\ \mathbf{curl}(\mu^{-1}\, \mathbf{curl}\, \boldsymbol{A}) = \boldsymbol{J} \text{ in } \Omega \\ \operatorname{div} \boldsymbol{A} = 0 \text{ in } \Omega. \end{cases}$$

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2. The equivalent variational formulation writes

$$(\text{FV-MSt})_{\boldsymbol{A}} \quad \begin{cases} \text{Find } \boldsymbol{A} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \text{ such that} \\ \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega), \quad \displaystyle\int_\Omega \mu^{-1}\,\mathbf{curl}\,\boldsymbol{A}\cdot\mathbf{curl}\,\boldsymbol{v}\,d\Omega = \int_\Omega \boldsymbol{J}\cdot\boldsymbol{v}\,d\Omega \\ \forall q \in H^1_0(\Omega), \quad \displaystyle\int_\Omega \boldsymbol{A}\cdot\nabla q\,d\Omega = 0. \end{cases}$$

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2. Let $\gamma > 0$. Introducing an artificial pressure p, another equivalent variational formulation is

$$(\text{FV-MSt})^{\gamma}_{\boldsymbol{A}} \begin{cases} \text{Find } \boldsymbol{A} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega),\ p \in H^1_0(\Omega) \text{ such that} \\ \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega), \quad \displaystyle\int_\Omega \mu^{-1}\,\mathbf{curl}\,\boldsymbol{A} \cdot \mathbf{curl}\,\boldsymbol{v}\, d\Omega \\ \qquad\qquad\qquad\qquad \displaystyle +\gamma \int_\Omega \boldsymbol{v} \cdot \nabla p\, d\Omega = \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v}\, d\Omega \\ \forall q \in H^1_0(\Omega), \quad \displaystyle\gamma \int_\Omega \boldsymbol{A} \cdot \nabla q\, d\Omega = 0. \end{cases}$$

Taking $\boldsymbol{v} = \nabla p$, one finds that $p = 0$!

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$$(\text{FV-MSt})^{\gamma}_{\boldsymbol{A}} \begin{cases} \text{Find } (\boldsymbol{A}, p) \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \times H^1_0(\Omega) \text{ such that} \\ \forall(\boldsymbol{v}, q) \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \times H^1_0(\Omega), \quad \displaystyle\int_\Omega \mu^{-1}\,\mathbf{curl}\,\boldsymbol{A}\cdot\mathbf{curl}\,\boldsymbol{v}\,d\Omega \\ \qquad\qquad + \gamma\displaystyle\int_\Omega \boldsymbol{v}\cdot\nabla p\,d\Omega + \gamma\int_\Omega \boldsymbol{A}\cdot\nabla q\,d\Omega = \int_\Omega \boldsymbol{J}\cdot\boldsymbol{v}\,d\Omega. \end{cases}$$

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2. Let $\gamma > 0$. Introducing an artificial pressure p, another equivalent variational formulation is

$$(\text{FV-MSt})^{\gamma}_{\boldsymbol{A}} \begin{cases} \text{Find } (\boldsymbol{A}, p) \in \boldsymbol{H}_0(\mathbf{curl}; \Omega) \times H^1_0(\Omega) \text{ such that} \\ \forall (\boldsymbol{v}, q) \in \boldsymbol{H}_0(\mathbf{curl}; \Omega) \times H^1_0(\Omega), \quad \int_\Omega \mu^{-1}\, \mathbf{curl}\, \boldsymbol{A} \cdot \mathbf{curl}\, \boldsymbol{v}\, d\Omega \\ \qquad + \gamma \int_\Omega \boldsymbol{v} \cdot \nabla p\, d\Omega + \gamma \int_\Omega \boldsymbol{A} \cdot \nabla q\, d\Omega = \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v}\, d\Omega. \end{cases}$$

Question: how to prove well-posedness "easily"?

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Question: how to prove well-posedness "easily"?

Use T-coercivity for the magnetostatics model!

Let

- $V = \boldsymbol{H}_0(\mathbf{curl};\Omega) \times H^1_0(\Omega)$, endowed with $\|(\boldsymbol{v},q)\|_V = (\|\boldsymbol{v}\|^2_{\boldsymbol{H}(\mathbf{curl};\Omega)} + |q|^2_{1,\Omega})^{1/2}$;
- $a((\boldsymbol{v},q),(\boldsymbol{w},r)) = \int_\Omega \mu^{-1}\,\mathbf{curl}\,\boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{w}\,d\Omega + \gamma\int_\Omega \boldsymbol{w}\cdot\nabla q\,d\Omega + \gamma\int_\Omega \boldsymbol{v}\cdot\nabla r\,d\Omega$;
- ${}_{V'}\langle f,(\boldsymbol{w},r)\rangle_V = \int_\Omega \boldsymbol{J}\cdot\boldsymbol{w}\,d\Omega$.

Let

- $V = \boldsymbol{H}_0(\mathbf{curl};\Omega) \times H_0^1(\Omega)$, endowed with $\|(\boldsymbol{v},q)\|_V = (\|\boldsymbol{v}\|^2_{\boldsymbol{H}(\mathbf{curl};\Omega)} + |q|^2_{1,\Omega})^{1/2}$;
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The first goal is to prove that the form $a(\cdot,\cdot)$ is $\mathtt{T}$-coercive.
NB. The form a is not coercive, because $a((0,q),(0,q)) = 0$ for $q \in H_0^1(\Omega)$.

Let

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Given $(\boldsymbol{v},q)\in V$, we look for $(\boldsymbol{w}^\star,r^\star)\in V$ with linear dependence such that

$$|a((\boldsymbol{v},q),(\boldsymbol{w}^\star,r^\star))| \geq \underline{\alpha}\,\|(\boldsymbol{v},q)\|_V^2,$$

with $\underline{\alpha}>0$ independent of $(\boldsymbol{v},q)$. In other words, T is defined by $\mathtt{T}((\boldsymbol{v},q)) = (\boldsymbol{w}^\star,r^\star)$.

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with $\underline{\alpha}>0$ independent of $(\boldsymbol{v},q)$. Three steps:

1. $\boldsymbol{v}=0$;
2. $q=0$;
3. General case.

Recall $a((\boldsymbol{v},q),(\boldsymbol{w},r)) = \int_\Omega \mu^{-1}\,\mathbf{curl}\,\boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{w}\,d\Omega + \gamma\int_\Omega \boldsymbol{w}\cdot\nabla q\,d\Omega + \gamma\int_\Omega \boldsymbol{v}\cdot\nabla r\,d\Omega$.

1. $a((0,q),(\boldsymbol{w},r)) = \gamma\int_\Omega \boldsymbol{w}\cdot\nabla q\,d\Omega$: so choosing $(\boldsymbol{w}^\star,r^\star) = (\nabla q, 0)$ yields

$$|a((0,q),(\boldsymbol{w}^\star,r^\star))| = \gamma\int_\Omega |\nabla q|^2\,d\Omega = \gamma\,\|(0,q)\|_V^2.$$

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1. $a((0,q),(\boldsymbol{w},r)) = \gamma\int_\Omega \boldsymbol{w}\cdot\nabla q\,d\Omega$: choose $(\boldsymbol{w}^\star,r^\star) = (\nabla q, 0)$.
2. $a((\boldsymbol{v},0),(\boldsymbol{w},r)) = \int_\Omega \mu^{-1}\,\mathbf{curl}\,\boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{w}\,d\Omega + \gamma\int_\Omega \boldsymbol{v}\cdot\nabla r\,d\Omega$: according to eg. Monk'03, one has the (double) orthogonal Helmholtz decomposition

$$\boldsymbol{H}_0(\mathbf{curl};\Omega) = \boldsymbol{K}_N(\Omega) \overset{\perp}{\oplus} \nabla[H^1_0(\Omega)] \text{ where } \boldsymbol{K}_N(\Omega) = \boldsymbol{H}_0(\mathbf{curl};\Omega)\cap\boldsymbol{H}(\mathrm{div}\,0;\Omega),$$

and $\boldsymbol{k}\mapsto\|\,\mathbf{curl}\,\boldsymbol{k}\|$ defines a norm on $\boldsymbol{K}_N(\Omega)$, equivalent to $\|\cdot\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$.
Let $\boldsymbol{v} = \boldsymbol{k}_v + \nabla\phi_v$, then choosing $(\boldsymbol{w}^\star,r^\star) = (\boldsymbol{k}_v,\phi_v)$ yields

$$|a((\boldsymbol{v},0),(\boldsymbol{w}^\star,r^\star))| = \int_\Omega \mu^{-1}\,\mathbf{curl}\,\boldsymbol{k}_v\cdot\mathbf{curl}\,\boldsymbol{k}_v\,d\Omega + \gamma\int_\Omega |\nabla\phi_v|^2\,d\Omega \gtrsim \|(\boldsymbol{v},0)\|_V^2.$$

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3. General case: the linear combination $(\boldsymbol{w}^\star,r^\star) = (\nabla q + \boldsymbol{k}_v,\phi_v)$ now leads to

$$\begin{aligned} a((\boldsymbol{v},q),(\boldsymbol{w}^\star,r^\star)) &= \int_\Omega \mu^{-1}\,\mathbf{curl}\,\boldsymbol{k}_v\cdot\mathbf{curl}\,\boldsymbol{k}_v\,d\Omega + \gamma\int_\Omega |\nabla q|^2\,d\Omega + \gamma\int_\Omega |\nabla\phi_v|^2\,d\Omega \\ &\gtrsim \|(\boldsymbol{v},q)\|_V^2. \end{aligned}$$

Regarding the proof with T-coercivity, one can make several observations:

1. The (double) orthogonal Helmholtz decomposition plays a crucial role!
2. The operator $\mathtt{T}$ is independent of the chosen value for γ.
3. The approach can be transposed to the approximation, see below!

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The second goal is to prove the uniform discrete inf-sup condition, with the help of the uniform $\mathtt{T}_\delta$-coercivity. Given finite dimensional subspaces $(\boldsymbol{V}_\delta)_\delta$ of $\boldsymbol{H}_0(\mathbf{curl};\Omega)$, resp. $(Q_\delta)_\delta$ of $H^1_0(\Omega)$, one can build an approximation of the magnetostatics model. Question: how to choose them?

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Mimic the previous proof to guarantee uniform $\mathtt{T}_\delta$-coercivity! [1st Key Idea]

The discrete variational formulation writes

$$(\text{FV-MSt})_{\boldsymbol{A}}^{\gamma,\delta} \begin{cases} \text{Find } (\boldsymbol{A}_\delta, p_\delta) \in \boldsymbol{V}_\delta \times Q_\delta \text{ such that} \\ \forall(\boldsymbol{v}_\delta, q_\delta) \in \boldsymbol{V}_\delta \times Q_\delta, \qquad \displaystyle\int_\Omega \mu^{-1}\,\mathbf{curl}\,\boldsymbol{A}_\delta \cdot \mathbf{curl}\,\boldsymbol{v}_\delta\, d\Omega \\ \displaystyle +\gamma \int_\Omega \boldsymbol{v}_\delta \cdot \nabla p_\delta\, d\Omega + \gamma \int_\Omega \boldsymbol{A}_\delta \cdot \nabla q_\delta\, d\Omega = \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v}_\delta\, d\Omega. \end{cases}$$

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Given $(\boldsymbol{v}_\delta, q_\delta) \in \boldsymbol{V}_\delta \times Q_\delta$, we look for $(\boldsymbol{w}_\delta^\star, r_\delta^\star) \in \boldsymbol{V}_\delta \times Q_\delta$ with linear dependence such that

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|a((\boldsymbol{v}_\delta, q_\delta), (\boldsymbol{w}_\delta^\star, r_\delta^\star))| \geq \underline{\alpha}_\dagger \, \|(\boldsymbol{v}_\delta, q_\delta)\|_V^2,
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with $\underline{\alpha}_\dagger > 0$ independent of δ and of $(\boldsymbol{v}_\delta, q_\delta)$.

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with $\underline{\alpha}_\dagger > 0$ independent of δ and of $(\boldsymbol{v}_\delta, q_\delta)$. To mimick the $\mathtt{T}$-coercivity approach, one needs that $\nabla[Q_\delta] \subset \boldsymbol{V}_\delta$, so that a discrete Helmholtz decomposition holds in $\boldsymbol{V}_\delta$:

$$\boldsymbol{V}_\delta = \boldsymbol{K}_\delta \overset{\perp}{\oplus} \nabla[Q_\delta] \text{ where } \boldsymbol{K}_\delta = \left\{\boldsymbol{k}_\delta \in \boldsymbol{V}_\delta \,|\, \forall q_\delta \in Q_\delta,\ (\boldsymbol{k}_\delta, \nabla q_\delta)_{\boldsymbol{L}^2(\Omega)} = 0\right\}.$$

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with $\underline{\alpha}_\dagger > 0$ independent of δ and of $(\boldsymbol{v}_\delta, q_\delta)$. Mimicking the $\mathtt{T}$-coercivity approach, using the discrete decomposition $\boldsymbol{v}_\delta = \boldsymbol{k}_\delta + \nabla \phi_\delta$, one chooses: $(\boldsymbol{w}_\delta^\star, r_\delta^\star) = (\nabla q_\delta + \boldsymbol{k}_\delta, \phi_\delta)$.

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with $\underline{\alpha}_\dagger > 0$ independent of δ and of $(\boldsymbol{v}_\delta, q_\delta)$. Mimicking the $\mathtt{T}$-coercivity approach, using the discrete decomposition $\boldsymbol{v}_\delta = \boldsymbol{k}_\delta + \nabla \phi_\delta$, one chooses: $(\boldsymbol{w}_\delta^\star, r_\delta^\star) = (\nabla q_\delta + \boldsymbol{k}_\delta, \phi_\delta)$.
Difficulty: does $\boldsymbol{k}_\delta \mapsto \|\,\mathbf{curl}\,\boldsymbol{k}_\delta\|$ define a norm on $\boldsymbol{K}_\delta$, uniformly equivalent to $\|\cdot\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$?

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Browsing `Monk'03`, a classical choice is:

Nédélec FE (1st family) of order $k \geq 1$ for $\boldsymbol{V}_\delta$, resp. Lagrange FE of order $k \geq 1$ for Q_δ.

The proof is "elementary"! Convergence and error estimates follow...

Given $\gamma > 0$, the variational formulation is

$$(\text{FV-MSt})^{\gamma}_{\boldsymbol{A}} \left\{ \begin{array}{l} \text{Find } (\boldsymbol{A}, p) \in \boldsymbol{H}_0(\mathbf{curl}; \Omega) \times H^1_0(\Omega) \text{ such that} \\ \forall (\boldsymbol{v}, q) \in \boldsymbol{H}_0(\mathbf{curl}; \Omega) \times H^1_0(\Omega), \quad \displaystyle\int_\Omega \mu^{-1}\, \mathbf{curl}\, \boldsymbol{A} \cdot \mathbf{curl}\, \boldsymbol{v}\, d\Omega \\ \qquad\qquad + \gamma \displaystyle\int_\Omega \boldsymbol{v} \cdot \nabla p\, d\Omega + \gamma \int_\Omega \boldsymbol{A} \cdot \nabla q\, d\Omega = \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v}\, d\Omega. \end{array} \right.$$

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Replace the test-fields $(\boldsymbol{v}, q) = (\boldsymbol{k}_v + \nabla \phi_v, q)$ by $\mathtt{T}(\boldsymbol{v}, q) = (\boldsymbol{k}_v + \nabla q, \phi_v)$.
An equivalent variational formulation is

$$(\text{FV-MSt})^{\gamma}_{\mathtt{T}} \quad \begin{cases} \text{Find } (\boldsymbol{A}, p) \in \boldsymbol{H}_0(\mathbf{curl}; \Omega) \times H^1_0(\Omega) \text{ such that} \\ \forall (\boldsymbol{v}, q) \in \boldsymbol{H}_0(\mathbf{curl}; \Omega) \times H^1_0(\Omega), \quad \displaystyle\int_\Omega \mu^{-1} \, \mathbf{curl}\, \boldsymbol{A} \cdot \mathbf{curl}\, \boldsymbol{v} \, d\Omega \\ \qquad + \gamma \displaystyle\int_\Omega \nabla q \cdot \nabla p \, d\Omega + \gamma \int_\Omega \nabla \phi_A \cdot \nabla \phi_v \, d\Omega = \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v} \, d\Omega. \end{cases}$$

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The equivalent variational formulation also writes

$$(\text{FV-MSt})^{\gamma}_{\mathtt{T}} \begin{cases} \text{Find } \boldsymbol{A} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega),\ p \in H^1_0(\Omega) \text{ such that} \\ \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega), \quad \int_\Omega \mu^{-1} \,\mathbf{curl}\, \boldsymbol{A} \cdot \mathbf{curl}\, \boldsymbol{v} \, d\Omega \\ \qquad + \gamma \int_\Omega \nabla \phi_A \cdot \nabla \phi_v \, d\Omega = \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v} \, d\Omega \\ \forall q \in H^1_0(\Omega), \quad \gamma \int_\Omega \nabla q \cdot \nabla p \, d\Omega = 0. \end{cases}$$

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Replace the test-fields $(\boldsymbol{v}, q) = (\boldsymbol{k}_v + \nabla \phi_v, q)$ by $\mathtt{T}(\boldsymbol{v}, q) = (\boldsymbol{k}_v + \nabla q, \phi_v)$.
The equivalent variational formulation also writes

$$(\text{FV-MSt})_{\mathtt{T}}^{\gamma} \left\{ \begin{array}{l} \text{Find } \boldsymbol{A} \in \boldsymbol{H}_0(\mathbf{curl};\Omega),\ p \in H_0^1(\Omega) \text{ such that} \\ \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega), \quad \displaystyle\int_\Omega \mu^{-1} \,\mathbf{curl}\, \boldsymbol{A} \cdot \mathbf{curl}\, \boldsymbol{v} \, d\Omega \\ \qquad\qquad \displaystyle + \gamma \int_\Omega \nabla \phi_A \cdot \nabla \phi_v \, d\Omega = \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v} \, d\Omega \\ \forall q \in H_0^1(\Omega), \quad \displaystyle \gamma \int_\Omega \nabla q \cdot \nabla p \, d\Omega = 0 \quad \Longrightarrow \boxed{p = 0}. \end{array} \right.$$

Given $\gamma > 0$, the (simplified) variational formulation to be solved is

$$\text{(FV-MSt)}_{\mathtt{T}}^{\gamma} \left\{ \begin{array}{l} \text{Find } \boldsymbol{A} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \text{ such that} \\ \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega), \quad b_\gamma(\boldsymbol{A},\boldsymbol{v}) = \displaystyle\int_\Omega \boldsymbol{J}\cdot\boldsymbol{v}\,d\Omega, \end{array} \right.$$

with $b_\gamma(\boldsymbol{v},\boldsymbol{w}) = \displaystyle\int_\Omega \mu^{-1}\,\mathbf{curl}\,\boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{w}\,d\Omega + \gamma\int_\Omega \nabla\phi_v\cdot\nabla\phi_w\,d\Omega.$

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To approximate $(\text{FV-MSt})_{\mathrm{T}}^{\gamma}$:

- either one can evaluate simply the second term in the expression of $b_\gamma(\cdot,\cdot)$, that is evaluate the gradient part in the (discrete) Helmholtz decomposition ;
- or, one has to modify this second term.

We study next the second option.

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$$c_\gamma(\boldsymbol{v},\boldsymbol{w}) = b_\gamma(\boldsymbol{v},\boldsymbol{w}) + \gamma\int_\Omega \boldsymbol{k}_v\cdot\boldsymbol{k}_w\, d\Omega = \int_\Omega \mu^{-1}\,\mathbf{curl}\,\boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{w}\, d\Omega + \gamma\int_\Omega \boldsymbol{v}\cdot\boldsymbol{w}\, d\Omega.$$

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Magnetostatics

Solving numerically the variational formulation with operator `T` - 3

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Approximate the perturbed variational formulation with ad hoc γ!

Given $\gamma > 0$, the discrete perturbed variational formulation writes

$$
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One has a variant of Céa's lemma, with γ-robust estimates

$$\| \mathbf{curl}(\boldsymbol{A}_\gamma - \boldsymbol{A}^\delta_\gamma) \| \lesssim \inf_{\boldsymbol{v}_\delta \in \boldsymbol{V}_\delta} \left[\gamma^{1/2} \| \boldsymbol{A}_\gamma - \boldsymbol{v}_\delta \| + \| \mathbf{curl}(\boldsymbol{A}_\gamma - \boldsymbol{v}_\delta) \| \right].$$

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Let $\boldsymbol{A}_\delta$ be the solution of the perturbed variational formulation for $\gamma = \gamma(\delta)$: $\boldsymbol{A}_\delta = \boldsymbol{A}^\delta_{\gamma(\delta)}$.

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The method is similar to that of `Reitzinger-Schöberl'02`, `Duan-Li-Tan-Zheng'12` and `PC-Wu-Zou'14`. However the derivation is completely different!

A numerical illustration (©PC-Wu-Zou'14):

- the permeability is $\mu = 1$, the domain Ω is a cube;
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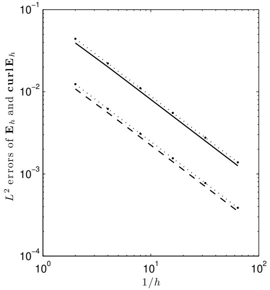
Expected convergence rate is $O(h)$:

- error $\|\boldsymbol{A} - \boldsymbol{A}_\delta\|$ (dashed line) ;
- error $\|\boldsymbol{B} - \mathbf{curl}\,\boldsymbol{A}_\delta\|_{\boldsymbol{H}(\mathrm{div}\,;\Omega)} = \|\mathbf{curl}(\boldsymbol{A} - \boldsymbol{A}_\delta)\|$ (solid line).



Further remarks

Some extensions:

1. Stokes model: see Jamelot (2022, HAL report) for a non-conforming discretisation (Crouzeix-Raviart FE or Fortin-Soulié FE); see master's thesis by MRoueh (2022) for DG discretisation ; see Barré-Grandmont-Moireau'22 for a poromechanics model.
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5. in Banach spaces, T-coercivity implies Hilbert structure, see Ern-Guermont'21-Vol.II.
6. T-coercivity still usable with the Strang lemmas (approximate forms).

Thank you for your attention!