An introduction to Discrete de Rham (DDR) methods

The well-posedness of relevant physical models expressed in terms of partial differential equations (PDE) hinges on subtle analytical, homological, and algebraic properties underlying Hilbert complexes [1]. The best-known example is the de Rham complex which, in practically relevant situations, can be expressed through vector proxies as the sequence of Hilbert spaces H^1, $\boldsymbol{H}(\mathbf{curl})$, $\boldsymbol{H}(\mathrm{div})$, and L^2 connected by the vector calculus operators grad, curl, div.

In the first part of this presentation, we illustrate the role of the de Rham complex in the well-posedness of several PDE problems. The design of efficient numerical methods for such problems is challenging for several reasons: on one hand, stability requires to mimic, at the discrete level, the homological and analytical properties of the de Rham complex (leading to the notion of “compatible method”); on the other hand, the complicated geometrical features of the domain and behaviours of the solution require a great flexibility in terms of supported meshes and approximation orders. In the second part of this presentation we provide an introduction to the recently introduced Discrete de Rham (DDR) paradigm [3,4] for the design and analysis of compatible discretization methods supporting general polyhedral meshes and arbitrary orders.

The general principle of DDR methods is to replace both spaces and operators by discrete counterparts designed so as to be compatible with the cohomology properties of the continuous complex. Specifically:

- The discrete spaces are spanned by vectors of polynomials with components attached to mesh entities in order to mimic, through their single-valuedness, global (complete or partial) continuity properties of the continuous spaces. The local polynomial spaces can be either full or incomplete;
- The discrete operators are obtained in two steps: first, reconstructions in full polynomial spaces are built mimicking an approximate version of the Stokes formula; second, whenever needed, the L^2-orthogonal projection on the appropriate incomplete polynomial space is taken.
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A full set of results for DDR methods have been recently proved [3,4], including cohomology-related properties, Poincaré inequalities, as well as primal and adjoint consistency of the discrete vector calculus operators. An overview of such results, along with examples of applications, is provided. In [2], the algebraic results have been generalized through the used of differential forms, leading to a Polytopal Exterior Calculus framework.

References

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