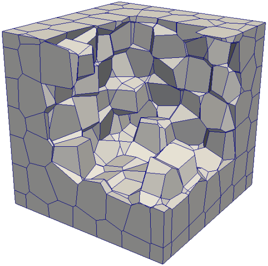
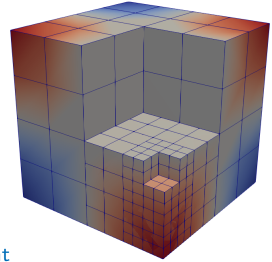
The HIPOTHEC project: whys and wherefores



Simon Lemaire
(Inria, Univ. Lille)



Kick-off workshop - Wissant
March 7, 2024

Main features

- ANR PRCE project, in partnership with EDF
- January 2024 – December 2028 (60 months)
- budget ≈ 605k EUR in total
- 2 academic research poles + EDF R&D

Consortium

- North pole:
  - SL (CR, **coordinator**) + Théophile Chaumont-Frelet (CR quasi-HDR), Centre Inria de l'Université de Lille
  - Serge Nicaise (PR), Université Polytechnique des Hauts-de-France (Valenciennes)
- EDF R&D:
  - Jérémy Dalphin (IRj), EDF Lab Paris-Saclay
  - Jean-Pierre Ducreux (IRs), EDF Lab Paris-Saclay
- South pole:
  - Francesca Rapetti (MCF HDR), Université Côte d'Azur (Nice)
  - Daniele A. Di Pietro (PR), Université de Montpellier

Outline

The whys

The wherefores

Nuclear safety

- Thermally-constrained metallic components: possible formation of cracks (e.g. stress corrosion cracking)
- Non-invasive detection of shallow flaws: based on eddy current testing (ECT)

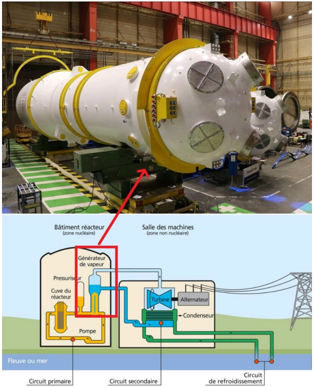


Figure: One of the four steam generators of an EPR (25m high, 510 tons).

Numerical simulation of ECT

- Forward simulator: used to calibrate/qualify ECT probes (*make the measurements fit the simulations*)
- Inverse simulator: used to unravel the anatomy of flaws (*make the simulations fit the measurements*)

Forward model

Find $\boldsymbol{e} : \Omega \to \mathbb{C}^3$ s.t.

$$\begin{cases} \mathbf{curl}(\mu^{-1}\mathbf{curl}\,\boldsymbol{e}) + i\omega\sigma\boldsymbol{e} = -i\omega\boldsymbol{j} & \text{in } \Omega, \\ \mathrm{div}(\varepsilon\boldsymbol{e}) = 0 & \text{in } \Omega_c^c, \\ \boldsymbol{e}\times\boldsymbol{n} = \mathbf{0} & \text{on } \partial\Omega, \end{cases}$$

with electric conductivity

$$\sigma = \begin{cases} 0 & \text{in } \Omega_c^c \\ \sigma_c & \text{in } \Omega_c \end{cases}.$$

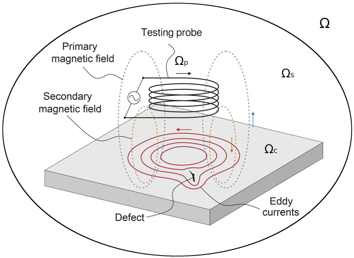


Figure: Sketch of a prototypical ECT setting.

Current limitations of EDF's forward simulator (code_Carmel)

- L1. Magnitude of the numerical error on the control signal
- L2. Modeling of defects and conforming 3D (re)meshing

Numerical simulation of ECT

- Forward simulator: used to calibrate/qualify ECT probes (*make the measurements fit the simulations*)
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Forward model

Find $\boldsymbol{e} : \Omega \to \mathbb{C}^3$, $\langle \varepsilon \boldsymbol{e}_{|\Omega_c^c} \cdot \boldsymbol{n}_c, 1 \rangle_{\partial\Omega_c} = 0$, s.t.

$$\begin{cases} \mathbf{curl}(\mu^{-1}\mathbf{curl}\, \boldsymbol{e}) + i\omega\sigma \boldsymbol{e} = -i\omega \boldsymbol{j} & \text{in } \Omega, \\ \operatorname{div}(\varepsilon \boldsymbol{e}) = 0 & \text{in } \Omega_c^c, \\ \boldsymbol{e} \times \boldsymbol{n} = \mathbf{0} & \text{on } \partial\Omega, \end{cases}$$

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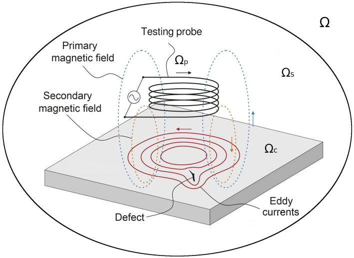
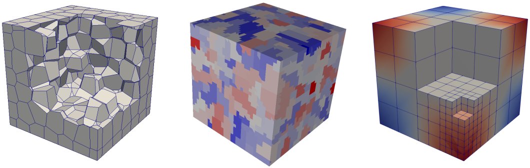


Figure: Sketch of a prototypical ECT setting.

Current limitations of EDF's forward simulator (code_Carmel)

- L1. Magnitude of the numerical error on the control signal
- L2. Modeling of defects and conforming 3D (re)meshing

The ambition of the HIPOTHEC project is to address at once both limitations L1 and L2, taking advantage of new-generation high-order polyhedral methods.



Project workflow

- WP1. Design and a priori analysis of HHO methods for ECT
  - T1. Taming topology ⋆ **Silvano Pitassi** (Lille & Palaiseau)
  - T2. Taming defects ⋆ **PdA** (Lille)
- WP2. Polyhedral a posteriori analysis and multigrid solvers
  - T3. Polyhedral error estimators ⋆ **PhD1** (Valenciennes & Lille), **PdB** (Nice)
  - T4. Polyhedral multigrid methods ⋆ **PhD2** (Montpellier & Nice)
- WP3. Software development and proof-of-concept applications
  - T5. Implementation in ParaSkel++ ⋆ **Thoma Zoto** (Lille), PhD1/2, PdA/B
  - T6. PoC on T.E.A.M. benchmarks ⋆ PdB

Outline

The whys

The wherefores

Groundwork

Trivial topology

F. Chave, D. A. Di Pietro, and SL

A discrete Weber inequality on three-dimensional hybrid spaces with application to the HHO approximation of magnetostatics

M3AS, 2022

Nontrivial topology

SL and S. Pitassi

Discrete Weber inequalities and related Maxwell compactness for hybrid spaces over polyhedral partitions of domains with general topology

Found. Comput. Math., 2024

Let $\mathcal{D} \subset \mathbb{R}^3$ be an open, bounded, connected, Lipschitz polyhedral domain.

Recall the Betti numbers:

- $\beta_0(\mathcal{D})$: number of connected components of $\mathcal{D}$ (here, $\beta_0(\mathcal{D}) = 1$);
- $\beta_1(\mathcal{D})$: number of tunnels crossing through $\mathcal{D}$;
- $\beta_2(\mathcal{D})$: number of voids encapsulated into $\mathcal{D}$.

Magnetostatics

Given a current density $\boldsymbol{j} : \mathcal{D} \to \mathbb{R}^3$ satisfying $\operatorname{div} \boldsymbol{j} = 0$ in $\mathcal{D}$ and $\boldsymbol{j}\cdot\boldsymbol{n} = 0$ on $\partial\mathcal{D}$, find the magnetic field $\boldsymbol{h} : \mathcal{D} \to \mathbb{R}^3$ such that

$$\begin{cases} \mathbf{curl}\, \boldsymbol{h} = \boldsymbol{j} & \text{in } \mathcal{D}, \\ \operatorname{div} \boldsymbol{b} = 0 & \text{in } \mathcal{D}, \\ \boldsymbol{h}\times\boldsymbol{n} = \boldsymbol{0} & \text{on } \partial\mathcal{D}, \end{cases} \tag{$\mathfrak{P}_{\boldsymbol{\tau}}$}$$

with constitutive law $\boldsymbol{b} = \mu\boldsymbol{h}$, where $\mu \in \mathbb{R}^\star_+$ is the magnetic permeability.

Variational formulation [Kikuchi; 89]

Given $\boldsymbol{j} \in \boldsymbol{H}_0(\mathrm{div}^0; \mathcal{D})$, find $(\boldsymbol{h}, p) \in \boldsymbol{H}_\mathbf{0}(\mathbf{curl}; \mathcal{D}) \times H_0^1(\mathcal{D})$ such that

$$\begin{cases} \int_\mathcal{D} \mathbf{curl}\,\boldsymbol{h} \cdot \mathbf{curl}\,\boldsymbol{v} + \mu \int_\mathcal{D} \boldsymbol{v} \cdot \mathbf{grad}\,p = \int_\mathcal{D} \boldsymbol{j} \cdot \mathbf{curl}\,\boldsymbol{v} & \forall \boldsymbol{v} \in \boldsymbol{H}_\mathbf{0}(\mathbf{curl}; \mathcal{D}), \\ -\mu \int_\mathcal{D} \boldsymbol{h} \cdot \mathbf{grad}\,q = 0 & \forall q \in H_0^1(\mathcal{D}). \end{cases} \qquad (\mathrm{P}_{\boldsymbol{\tau}})$$

Remark that $p \equiv 0$ (test with $\boldsymbol{v} = \mathbf{grad}\,p \in \mathbf{grad}\left(H_0^1(\mathcal{D})\right) \subset \boldsymbol{H}_\mathbf{0}(\mathbf{curl}^\mathbf{0}; \mathcal{D})$).

Weak-strong equivalence

The problems $(\mathfrak{P}_{\boldsymbol{\tau}})$ and $(\mathrm{P}_{\boldsymbol{\tau}})$ are equivalent in the following sense:

- if $\boldsymbol{h}$ solves $(\mathfrak{P}_{\boldsymbol{\tau}})$, then $\boldsymbol{h}$ solves $(\mathrm{P}_{\boldsymbol{\tau}})$;
- reciprocally, assume that $\beta_1(\mathcal{D}) = 0$; then, if $\boldsymbol{h}$ solves $(\mathrm{P}_{\boldsymbol{\tau}})$, $\boldsymbol{h}$ solves $(\mathfrak{P}_{\boldsymbol{\tau}})$.

Well-posedness

Assume that $\beta_2(\mathcal{D}) = 0$. Then, ($\mathsf{P}_{\boldsymbol{\tau}}$) is well-posed.

Well-posedness for ($\mathsf{P}_{\boldsymbol{\tau}}$) hinges on the first Weber inequality.

Weber inequalities

- are named after Christian Weber [Weber; 80];
- are generalizations of the Poincaré inequality to the case of vector fields belonging to $\boldsymbol{H}(\mathbf{curl};\mathcal{D}) \cap \boldsymbol{H}(\mathrm{div};\mathcal{D}) \supset \boldsymbol{H}^1(\mathcal{D})$, and featuring on $\partial\mathcal{D}$ either vanishing tangential trace (first) or vanishing normal trace (second);
- their statement is strongly topology-dependent.

First Weber inequality

Assume that $\beta_2(\mathcal{D}) = 0$. Then, for all $\boldsymbol{v} \in \boldsymbol{H}_{\mathbf{0}}(\mathbf{curl};\mathcal{D}) \cap \boldsymbol{H}(\mathrm{div}^0;\mathcal{D})$,

$$\|\boldsymbol{v}\|_{0,\mathcal{D}} \lesssim \|\,\mathbf{curl}\,\boldsymbol{v}\|_{0,\mathcal{D}}.$$

Let $(\mathcal{T}_h, \mathcal{F}_h)$ be a polyhedral mesh of $\mathcal{D} \subset \mathbb{R}^3$, and $\ell \in \mathbb{N}$ a given polynomial degree.

Cell polynomial decomposition

For $T \in \mathcal{T}_h$, let $\boldsymbol{x}_T$ be some point inside T such that T contains a ball centered at $\boldsymbol{x}_T$ of radius comparable to h_T. There holds

$$\boldsymbol{\mathcal{P}}^\ell(T) = \boldsymbol{\mathcal{G}}^\ell(T) \oplus \boldsymbol{\mathcal{P}}^{\ell-1}(T) \times (\boldsymbol{x} - \boldsymbol{x}_T),$$

where $\boldsymbol{\mathcal{G}}^\ell(T) := \mathbf{grad}\left(\mathcal{P}^{\ell+1}(T)\right)$, and the polynomial space $\boldsymbol{\mathcal{P}}^{\ell-1}(T) \times (\boldsymbol{x} - \boldsymbol{x}_T)$ is the so-called Koszul complement.

Face polynomial decomposition

For $F \in \mathcal{F}_h$, let $\boldsymbol{x}_F$ be some point inside F such that F contains a disk centered at $\boldsymbol{x}_F$ of radius comparable to h_F. There holds

$$\boldsymbol{\mathcal{P}}^\ell(F) = \boldsymbol{\mathcal{R}}^\ell(F) \oplus \mathcal{P}^{\ell-1}(F)(\boldsymbol{x} - \boldsymbol{x}_F),$$

where $\boldsymbol{\mathcal{R}}^\ell(F) := \left(\mathbf{grad}_F\left(\mathcal{P}^{\ell+1}(F)\right)\right)^\perp$, with $\boldsymbol{z}^\perp$ the rotation of angle $-\frac{\pi}{2}$ of $\boldsymbol{z}$ in the oriented hyperplane H_F, and $\mathcal{P}^{\ell-1}(F)(\boldsymbol{x} - \boldsymbol{x}_F)$ is the Koszul complement.

For all $T \in \mathcal{T}_h$ and $F \in \mathcal{F}_T$, $\boldsymbol{\mathcal{G}}^\ell(T)_{|F} \times \boldsymbol{n}_F = \boldsymbol{\mathcal{R}}^\ell(F)$.

$\boldsymbol{H}(\mathbf{curl})$-like hybrid space

For $\ell \in \mathbb{N}$, we consider the hybrid space

$$\underline{\boldsymbol{X}}_h^\ell := \left\{ \underline{\boldsymbol{v}}_h := ((\boldsymbol{v}_T)_{T\in\mathcal{T}_h}, (\boldsymbol{v}_{F,\boldsymbol{\tau}})_{F\in\mathcal{F}_h}) \ : \ \begin{array}{ll} \boldsymbol{v}_T \in \boldsymbol{\mathcal{P}}^\ell(T) & \forall T \in \mathcal{T}_h \\ \boldsymbol{v}_{F,\boldsymbol{\tau}} \in \boldsymbol{\mathcal{R}}^\ell(F) & \forall F \in \mathcal{F}_h \end{array} \right\},$$

endowed with the semi-norm

$$|\underline{\boldsymbol{v}}_h|^2_{\mathbf{curl},h} := \sum_{T\in\mathcal{T}_h} \left(\|\mathbf{curl}\,\boldsymbol{v}_T\|^2_{0,T} + \sum_{F\in\mathcal{F}_T} h_F^{-1} \|\boldsymbol{\pi}^\ell_{\boldsymbol{\mathcal{R}},F}(\boldsymbol{v}_{T|F}\times\boldsymbol{n}_F) - \boldsymbol{v}_{F,\boldsymbol{\tau}}\|^2_{0,F} \right).$$

For $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}_h^\ell$, we let $\boldsymbol{v}_h \in \boldsymbol{\mathcal{P}}^\ell(\mathcal{T}_h)$ be such that $\boldsymbol{v}_{h|T} := \boldsymbol{v}_T$ for all $T \in \mathcal{T}_h$.

Is $|\cdot|_{\mathbf{curl},h}$ a norm on a div-free subset of $\underline{\boldsymbol{X}}_{h,\mathbf{0}}^\ell := \left\{ \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}_h^\ell \mid \boldsymbol{v}_{F,\boldsymbol{\tau}} \equiv \mathbf{0} \ \forall F \in \mathcal{F}_h^\partial \right\}$?

First hybrid Weber inequality [Chave, Di Pietro, SL; 22]

Assume that $\beta_2(\mathcal{D}) = 0$. Then, for any $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}_{h,\mathbf{0}}^\ell$ such that $\int_{\mathcal{D}} \boldsymbol{v}_h \cdot \mathbf{grad}\, q = 0$ for all $q \in H^1_0(\mathcal{D})$, the following holds true:

$$\|\boldsymbol{v}_h\|_{0,\mathcal{D}} \lesssim |\underline{\boldsymbol{v}}_h|_{\mathbf{curl},h}.$$

Let $k \in \mathbb{N}^\star$ be a given polynomial degree. Define

$$A_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) := \int_{\mathcal{D}} \mathbf{curl}_h \boldsymbol{u}_h \cdot \mathbf{curl}_h \boldsymbol{v}_h + \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} h_F^{-1} \int_F [\boldsymbol{\pi}^k_{\mathcal{R},F}(\boldsymbol{u}_{T|F} \times \boldsymbol{n}_F) - \boldsymbol{u}_{F,\boldsymbol{\tau}}] \cdot [\boldsymbol{\pi}^k_{\mathcal{R},F}(\boldsymbol{v}_{T|F} \times \boldsymbol{n}_F) - \boldsymbol{v}_{F,\boldsymbol{\tau}}],$$

$$B_h(\underline{\boldsymbol{u}}_h, \underline{q}_h) := \int_{\mathcal{D}} \boldsymbol{u}_h \cdot \boldsymbol{G}_h^k(\underline{q}_h),$$

$$N_h(\underline{r}_h, \underline{q}_h) := \int_{\mathcal{D}} r_h q_h + \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} h_F \int_F r_F q_F.$$

Discrete problem

Find $(\underline{\boldsymbol{h}}_h, \underline{p}_h) \in \underline{\boldsymbol{X}}^k_{h,\mathbf{0}} \times \underline{Y}^k_{h,0}$ such that

$$\begin{cases} A_h(\underline{\boldsymbol{h}}_h, \underline{\boldsymbol{v}}_h) + \mu B_h(\underline{\boldsymbol{v}}_h, \underline{p}_h) = \int_{\mathcal{D}} \boldsymbol{j} \cdot \mathbf{curl}_h \boldsymbol{v}_h & \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{X}}^k_{h,\mathbf{0}}, \\ -\mu B_h(\underline{\boldsymbol{h}}_h, \underline{q}_h) + N_h(\underline{p}_h, \underline{q}_h) = 0 & \forall \underline{q}_h \in \underline{Y}^k_{h,0}. \end{cases}$$

Assume that $\beta_2(\mathcal{D}) = 0$. Then, the discrete problem has a unique solution satisfying

$$\left(|\underline{\boldsymbol{h}}_h|^2_{\mathbf{curl},h} + \|\underline{p}_h\|^2_{0,h}\right)^{1/2} \leq \|\boldsymbol{j}\|_{0,\mathcal{D}},$$

where $\|\underline{q}_h\|^2_{0,h} := N_h(\underline{q}_h, \underline{q}_h)$.

Energy-error estimate [Chave, Di Pietro, SL; 22]

Assume that $\beta_1(\mathcal{D}) = 0$ and $\beta_2(\mathcal{D}) = 0$. Assume also that $\boldsymbol{h} \in \boldsymbol{H}^{k+1}(\mathcal{T}_h)$. Then, the following holds true:

$$\left(|\underline{\boldsymbol{h}}_h - \underline{\boldsymbol{I}}_h^k(\boldsymbol{h})|^2_{\mathbf{curl},h} + \|\underline{p}_h\|^2_{0,h}\right)^{1/2} \lesssim \left(\sum_{T\in\mathcal{T}_h} h_T^{2k} |\boldsymbol{h}|^2_{k+1,T}\right)^{1/2}.$$

- $\rightsquigarrow$ convergence of order $k \geq 1$ of $\|\mathbf{curl}_h \boldsymbol{h}_h - \mathbf{curl}\,\boldsymbol{h}\|_{0,\mathcal{D}}$
- $\rightsquigarrow$ observed convergence of order $k+1$ of $\|\boldsymbol{h}_h - \boldsymbol{h}\|_{0,\mathcal{D}}$ for $\mathcal{D}$ convex
- $\rightsquigarrow$ in practice, local elimination of all (magnetic and pressure) cell unknowns
- $\rightsquigarrow$ in the matching tetrahedral case, N_h can be removed

Academic test-case: $\mathcal{D} := (0,1)^3$, with $\mu = 1$ and exact solution

$$\boldsymbol{h}(x,y,z) = \big(\cos(\pi y)\cos(\pi z), \cos(\pi x)\cos(\pi z), \cos(\pi x)\cos(\pi y)\big)$$

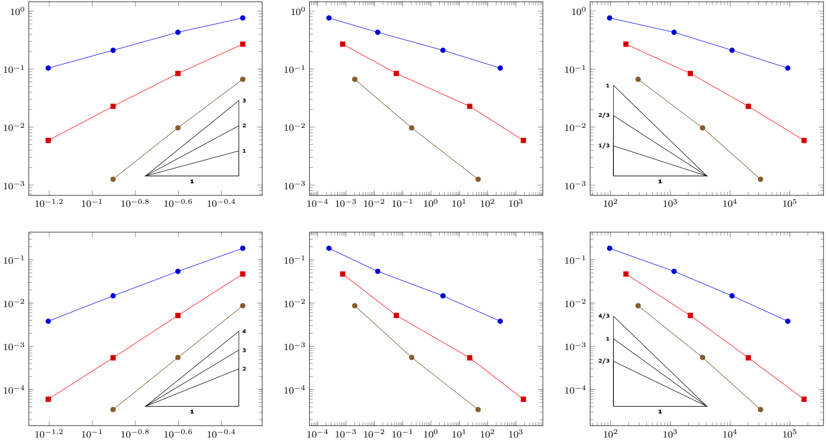


Figure: Relative energy-error (*top row*) and L^2-error (*bottom row*) vs. meshsize h (*left*), solution time in s (*center*), and #dof (*right*) on cubic meshes for $k \in \{1, 2, 3\}$.

THANK YOU