A posteriori goal-oriented error estimators based on equilibrated flux and potential reconstructions

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Introduction

Introduction

- Many engineering problems require computing some quantities of interest, which are usually linear functionals applied to the solution of a PDE.
- Error estimations on such functionals are called **goal-oriented error estimations**.
- Such estimations are based on the resolution of an **adjoint problem**, whose solution is used in the estimator definition, and on the use of some **energy-norm error estimators**.
- Goal of this talk :
  - Give an overview of such techniques in different contexts,
  - Provide an upper-bound of the error which can be totally and explicitly computed,
  - Test the behaviour of such estimators on some numerical benchmarks.
- Two models are considered :
  - Reaction-diffusion problems,
  - Eddy-current problems.

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The reaction-diffusion problem

Problem definition

$$\begin{cases} -\mathrm{div}(D\nabla u) + r\,u &= f \quad \text{in } \Omega \in \mathbb{R}^d, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

- $D \in L^\infty(\Omega; \mathbb{R}^{d\times d})$, symmetric matrix-valued function such that

$$D(x)\xi \cdot \xi \gtrsim |\xi|^2,\ \forall\, \xi \in \mathbb{R}^d, \text{ and a.e. } x \in \Omega,$$

- $r \in L^\infty(\Omega)$ supposed to be nonnegative,
- f is supposed to be in $L^2(\Omega)$.

Variational formulation

$$\begin{aligned} B(u,v) &= \int_\Omega (D\nabla u \cdot \nabla v + r\,u\,v)\,dx,\ \forall\, u,v \in H_0^1(\Omega), \\ F(v) &= \int_\Omega f\,v\,dx,\ \forall\, v \in H_0^1(\Omega), \\ B(u,v) &= F(v),\ \forall\, v \in H_0^1(\Omega), \end{aligned}$$

$\Rightarrow$ unique (weak) solution u in $H_0^1(\Omega)$.

Goal-oriented functional and adjoint problem

Output functional

$$q \in L^2(\Omega) \text{ and } Q(v) = \int_\Omega q\, v\, dx, \ \forall\, v \in L^2(\Omega).$$

Question : How to compute an approximation of the value of $Q(u)$?

Adjoint problem

- We now define $u^* \in H_0^1(\Omega)$ solution of the adjoint problem

$$B(v, u^*) = Q(v), \ \forall\, v \in H_0^1(\Omega).$$

- The associated strong formulation is

$$\begin{cases} -\mathrm{div}(D\nabla u^*) + r u^* & = & q & \text{in } \Omega, \\ u^* & = & 0 & \text{on } \partial\Omega. \end{cases}$$

- We clearly have

$$Q(u) = B(u, u^*) = F(u^*).$$

- Since B is here symmetric, we also have:

$$B(u^*, v) = Q(v), \ \forall\, v \in H_0^1(\Omega).$$

Discrete setting

Mesh and discrete spaces

- Let us introduce a triangulation $\mathcal{T}$ of Ω made of polygonal elements T that covers exactly Ω,
- We assume that the mesh is simplicial and matching,
- We introduce the so-called broken Sobolev space

$$H^1(\mathcal{T}) = \{v \in L^2(\Omega) \,|\, v_{|T} \in H^1(T),\ \forall\, T \in \mathcal{T}\}.$$

- We are looking for :
  - $u_h \in V_h \subset H^1(\mathcal{T})$ approximation of u,
  - $u_h^* \in V_h^* \subset H^1(\mathcal{T})$ approximation of u^*.
- Let us recall that

$$H(\mathrm{div}, \Omega) = \{\xi \in L^2(\Omega)^d; \mathrm{div}\xi \in L^2(\Omega)\}.$$

Error estimation

[Mozolevski and Prudhomme, CMAME 2015]
[Mallik, Vohralik and Yousef, JCAM 2020]

Theorem 1

Let $s_h \in H^1_0(\Omega)$, $\theta_h \in H(\mathrm{div}, \Omega)$ and $\theta_h^* \in H(\mathrm{div}, \Omega)$. Then we have :

$$\mathcal{E} = Q(u) - Q(u_h) = Q(u - u_h) = \eta_{QOI} + \mathcal{R},$$

where the estimator η_{QOI} is given by

$$\begin{aligned} \eta_{QOI} &= (q, s_h - u_h)_\Omega + (f - \mathrm{div}\theta_h - r\, u_h, u_h^*)_\Omega \\ &+ (\theta_h + D\nabla s_h, D^{-1}\theta_h^*)_\Omega - (r\, u_h^*, s_h - u_h)_\Omega, \end{aligned}$$

while the remainder term $\mathcal{R}$ is defined by

$$\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 \qquad \text{with}$$

$$\begin{cases} \mathcal{R}_1 = (f - \mathrm{div}\theta_h - r u_h, u^* - u_h^*)_\Omega, \\ \mathcal{R}_2 = -(\theta_h + D\nabla s_h, D^{-1}\theta_h^* + \nabla u^*)_\Omega, \\ \mathcal{R}_3 = (r(u^* - u_h^*), s_h - u_h)_\Omega. \end{cases}$$

Remarks

1. η_{QOI} has three contributions :
   - $(f - \mathrm{div}\theta_h - r\, u_h\,, u_h^*)_\Omega$ represents the data oscillation with respect to the primal problem weighted by the dual approximate solution if $\mathrm{div}\theta_h - r\, u_h$ is equal to the $L^2(\Omega)$ projection of f on the approximation space used to compute u_h,
   - $(\theta_h + D\nabla s_h\,, D^{-1}\theta_h^*)_\Omega$ measures the deviation of $-D\nabla s_h$ from the reconstructed flux θ_h,
   - $(q, s_h - u_h)_\Omega - (r\, u_h^*, s_h - u_h)_\Omega$ measures the deviation of u_h from $H_0^1(\Omega)$.
2. If $V_h \subset H_0^1(\Omega)$, then we can take $s_h = u_h$ and the blue terms vanish.
3. This result occurs whatever the values of

$$s_h \in H_0^1(\Omega),\ \theta_h \in H(\mathrm{div}, \Omega) \text{ and } \theta_h^* \in H(\mathrm{div}, \Omega).$$

$$\Rightarrow |\eta_{QOI}| \text{ and } |\mathcal{R}| \text{ can both be very high...}$$

Potential and Flux reconstructions

[Ern & Vohralik : A unified framework for a posteriori error estimation in elliptic and parabolic problems with application to finite volumes. FVCA6, 2011]

- We assume that a potential reconstruction s_h of u_h is available :

$$s_h \in H_0^1(\Omega) \text{ and } s_h \sim u_h,$$

- We assume that some flux reconstructions θ_h and θ_h^* are available, using respectively (u_h, f) and (u_h^*, q) :
    - $\theta_h \in H(\mathrm{div}, \Omega)$ and $(\mathrm{div}\theta_h + ru_h - f, 1)_T = 0, \ \forall\, T \in \mathcal{T} \Rightarrow \theta_h \sim -D\nabla u_h,$
    - $\theta_h^* \in H(\mathrm{div}, \Omega)$ and $(\mathrm{div}\theta_h^* + ru_h^* - q, 1)_T = 0, \ \forall\, T \in \mathcal{T} \Rightarrow \theta_h^* \sim -D\nabla u_h^*.$

Estimation of the remainder $\mathcal{R}$

Question...

- Once the primal and dual problems have been solved, the value of η_{QOI} **can be computed** (up to oscillation terms).
- Nevertheless, the value of $\mathcal{R}$ **can not be evaluated**, because of the value of u^* in its definition.
- Question :

  Can the value of $\mathcal{R}$ be bounded by known quantities ?

Estimation of the remainder $\mathcal{R}$

Some definitions

- $\forall\, w \in H_0^1(\Omega) \cup V_h$, $\|w\|_h^2 = \|D^{\frac{1}{2}} \nabla_h w\|^2 + \|r^{\frac{1}{2}} w\|^2$,
- $\eta^2 = \sum_{T \in \mathcal{T}} (\eta_{NC,T}^2 + \eta_{R,T}^2 + \eta_{DF,T}^2)$, with

$$\begin{aligned}
\eta_{NC,T} &= \|u_h - s_h\|_{h,T}, \\
\eta_{R,T} &= m_T\,\|f - \mathrm{div}\theta_h + r u_h\|_T, \\
\eta_{DF,T} &= \|D^{-\frac{1}{2}}(\theta_h + D\nabla u_h)\|_T,
\end{aligned}$$

$m_T := \min\{\pi^{-1} h_T \|D^{-\frac{1}{2}}\|_{\infty,T}, \|r^{-\frac{1}{2}}\|_{\infty,T}\}$, when T is convex.

Known results

[Ern & Vohralik : A unified framework for a posteriori error estimation in elliptic and parabolic problems with application to finite volumes. FVCA6, 2011]

$$\|u - u_h\|_h \leq \eta$$

and, similarly,

$$\|u^* - u_h^*\|_h \leq \eta^*$$

Estimation of the remainder $\mathcal{R}$

Theorem 2

With η and η^* as defined before, we have $|\mathcal{R}| \le 4\,\eta\,\eta^*$.

Proof.

We estimate each term of $\mathcal{R}$ separetely:

$$
\begin{aligned}
|\mathcal{R}_1| &= |(f - \mathrm{div}\theta_h - ru_h, u^* - u_h^*)_\Omega| \\
&\le \left| \sum_{T\in\mathcal{T}} \int_T (f - \mathrm{div}\theta_h - ru_h)\left((u^* - u_h^*) - \mathcal{M}_T(u^* - u_h^*)\right)\, dx \right| \\
&\le \sum_{T\in\mathcal{T}} \|f - \mathrm{div}\theta_h - ru_h\|_T\; m_T\; \|u^* - u_h^*\|_{h,T} \le \eta\,\eta^*. \\
|\mathcal{R}_2| &= |(\theta_h + D\nabla s_h, D^{-1}\theta_h^* + \nabla u^*)_\Omega| \\
&\le \|D^{-\frac{1}{2}}(\theta_h + D\nabla s_h)\|\|D^{-\frac{1}{2}}(\theta_h^* + D\nabla u^*)\| \\
&\le \|D^{-\frac{1}{2}}(\theta_h + D\nabla s_h)\|(\|D^{-\frac{1}{2}}(\theta_h^* + D\nabla_h u_h^*)\| + \|D^{\frac{1}{2}}\nabla_h(u^* - u_h^*)\|) \\
&\le 2\,\eta\,\eta^*. \\
|\mathcal{R}_3| &= |(r(u^* - u_h^*), s_h - u_h)_\Omega| \le \|r^{\frac{1}{2}}(u^* - u_h^*)\|\|r^{\frac{1}{2}}(s_h - u_h)\| \le \eta\,\eta^*.
\end{aligned}
$$

Some remarks

1. Thms 1 and 2 $\Rightarrow$
$$|\mathcal{E}| \leq |\eta_{QOI}| + 4\eta\eta^*.$$
Nevertheless, such an estimator can overestimate the error.
2. We can estimate the ratio
$$\frac{|\mathcal{R}|}{|\eta_{QOI}|},$$
by computing $\frac{4\eta\eta*}{|\eta_{QOI}|}$, during a refinement procedure based on the use of η_{QOI} and check if it tends to zero or not.
3. In the positive case, since $\mathcal{E} = \eta_{QOI} + \mathcal{R}$, this means that the ratio $\frac{\mathcal{E}}{\eta_{QOI}}$ tends to one and will validate the asymptotic exactness of the estimator η_{QOI}.
4. In any case, we can use the estimate
$$|\mathcal{E}| \leq |\eta_{QOI}| + 4\eta\eta^*,$$
and then choose as estimator $|\eta_{QOI}| + 4\eta\eta^*$ to implement an adaptive algorithm.

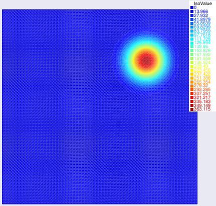
Numerical results

Primal problem : Regular solution

- $d = 2$, $\Omega =]0,1[^2$, $D = I_{\mathbb{R}^2}$ and $r = 0$.
- $u(x,y) = 10^4 x(1-x)y(1-y)e^{-100(\rho(x,y))^2}$, with

$$\rho(x,y) = ((x-0.75)^2 + (y-0.75)^2)^{1/2}.$$

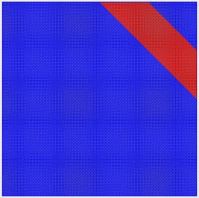
- The right-hand side f is computed accordingly such that $f = -\text{div}(D\nabla u)$.



Dual problem : Regular solution

- $q = 1_\omega$, with

$$\omega = \{(x,y) \in \Omega : 1.5 \leq x + y \leq 1.75\}$$

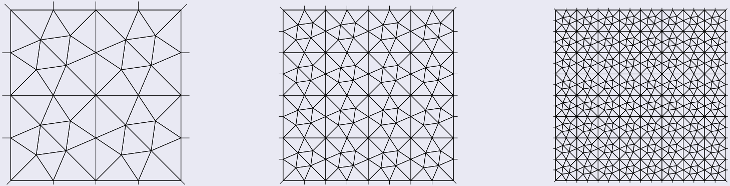


Numerical results

Numerical parameters

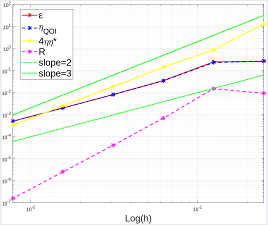
- For u_h : standard conforming $\mathbb{P}_1$ finite elements,
- For θ_h : standard $\mathbb{RT}_1$ finite elements,
- For u_h^* : standard conforming $\mathbb{P}_2$ finite elements,
- For θ_h^* : standard $\mathbb{RT}_2$ finite elements.

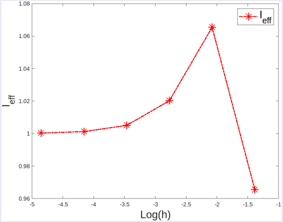
Meshes

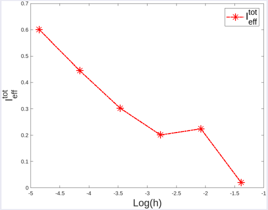


Numerical results

Regular solution







$$I_{eff} = |\mathcal{E}/\eta_{QOI}|$$

$$I_{eff}^{tot} = |\mathcal{E}|/(|\eta_{QOI}| + 4\eta\eta^*)$$

Remarks

If we had chosen :

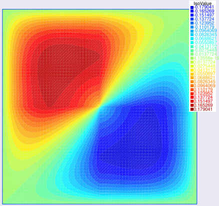
- For u_h : standard conforming $\mathbb{P}_1$ finite elements,
- For θ_h : standard $\mathbb{RT}_1$ finite elements,
- For u_h^* : standard conforming $\mathbb{P}_1$ finite elements,
- For θ_h^* : standard $\mathbb{RT}_1$ finite elements,

then the quantity $\eta\,\eta^*$ is no more superconvergent, even if I_{eff} still tends towards one.

Numerical results

Primal problem : Singular solution

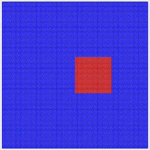
- $d = 2$, $\Omega =]-1,1[^2$ and $r = 0$,
- D is piecewise constant in Ω : $\left|\begin{array}{|c|c|}\hline 1 & a \\ \hline a & 1 \\ \hline\end{array}\right|$, $0 < a < 1$.
- $\alpha = \dfrac{4}{\pi}\text{arctan}(\sqrt{a})$ and $u(x,y) = p(x,y)\,S(x,y)$, where
  - $p(x,y) = (1-x^4)(1-y^4)$ is a truncation function
  - $S(x,y) = \rho^{\alpha} v(\theta)$
- The right-hand side f is computed accordingly.
- For any $\varepsilon > 0$ we have $u \in H^{1+\alpha-\varepsilon}(\Omega)$



Dual problem : Singular solution

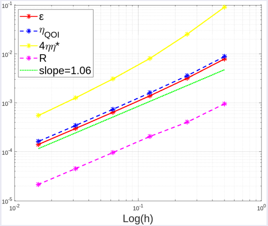
- $q = 1_{\omega}$, with

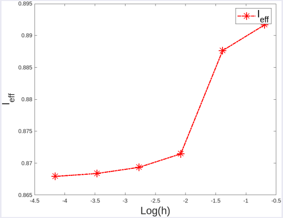
$$\omega = (0, 0.5) \times (-0.25, 0.25).$$

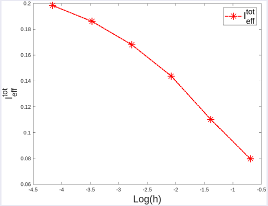


Numerical results

Singular solution with $a = 5$ so that $\alpha \approx 0.53$







$$I_{eff} = |\mathcal{E}/\eta_{QOI}|$$

$$I^{tot}_{eff} = |\mathcal{E}|/(|\eta_{QOI}| + 4\eta\eta^*)$$

Remarks: Singular solution

1. The error, the estimator η_{QOI} and $4\eta\eta^*$ all converge towards zero with order $O(h^{2\alpha})$.
2. I_{eff} remains in the order of unity but is no more close to one.
3. The remainder $\mathcal{R}$ seems to be no more superconvergent.
4. For such problems with singular solutions, an adaptive algorithm should be based on the sum of the estimator $|\eta_{QOI}|$ and of the product $4\,\eta\,\eta^*$,

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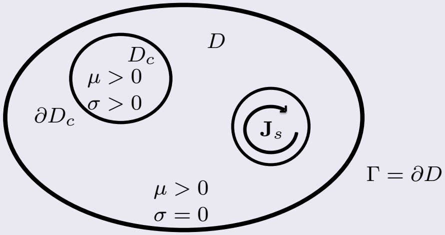
The eddy-current problem

Problem definition



Find the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$ solution of

$$\left\{\begin{array}{rcll} \mathrm{curl}\mathbf{E} &=& -j\omega\mathbf{B} & \text{in } D, \\ \mathrm{curl}\mathbf{H} &=& \mathbf{J}_s + \mathbf{J}_e & \text{in } D, \\ \mathrm{div}\mathbf{B} &=& 0 & \text{in } D, \end{array}\right. \quad \text{with} \left\{\begin{array}{rcll} \mathbf{B} &=& \mu\mathbf{H} & \text{in D}, \\ \mathbf{J}_e &=& \sigma\mathbf{E} & \text{in } D_c. \end{array}\right.$$

Properties and boundary conditions

- $\mathrm{div}\mathbf{J}_e = 0$ in D,
- $\mathbf{J}_e \cdot \mathbf{n} = 0$ on ∂D_c,
- $\mathbf{B} \cdot \mathbf{n} = 0$ on $\Gamma = \partial D$.

The eddy-current problem

Magnetic vector and electric scalar potentials

$$\begin{array}{rcll} \mathbf{B} & = & \mathrm{curl}\mathbf{A} & \text{in } D, \\ \mathbf{E} & = & -j\omega\mathbf{A} - \nabla\varphi & \text{in } D_c. \end{array}$$

Harmonic $\mathbf{A}$-φ formulation

$$\begin{array}{rcll} \mathrm{curl}\left(\mu^{-1}\mathrm{curl}\mathbf{A}\right) + \sigma\left(j\omega\mathbf{A} + \nabla\varphi\right) & = & \mathbf{J}_s & \text{in } D, \\ \mathrm{div}(\sigma(j\omega\mathbf{A} + \nabla\varphi)) & = & 0 & \text{in } D_c, \end{array}$$

with the boundary conditions

$$\begin{array}{rcll} \mathbf{A} \times \mathbf{n} & = & 0 & \text{on } \Gamma, \\ \sigma(j\omega\mathbf{A} + \nabla\varphi) \cdot \mathbf{n} & = & 0 & \text{on } \partial D_c. \end{array}$$

Functional spaces definitions

$$\begin{array}{rcl} H_0(\mathrm{curl}, \mathcal{D}) & = & \left\{ \mathbf{F} \in L^2(\mathcal{D})^3 : \mathrm{curl}\mathbf{F} \in L^2(\mathcal{D})^3, \mathbf{F} \times \mathbf{n} = 0 \text{ on } \partial\mathcal{D} \right\}, \\ \widetilde{X}(\mathcal{D}) & = & \left\{ \mathbf{F} \in H_0(\mathrm{curl}, \mathcal{D}) : (\mathbf{F}, \nabla\xi)_{\mathcal{D}} = 0,\ \forall \xi \in H_0^1(\mathcal{D}) \right\}, \\ \widetilde{H^1}(\mathcal{D}) & = & \left\{ f \in H^1(\mathcal{D}) : (f, 1)_{\mathcal{D}} = 0 \right\}. \end{array}$$

The eddy-current problem

Variational formulation

Find $(\mathbf{A}, \varphi) \in \widetilde{X}(D) \times \widetilde{H^1}(D_c)$ such that

$$B((\mathbf{A}, \varphi), (\mathbf{A}', \varphi')) = (\mathbf{J}_s, \mathbf{A}'), \quad \forall (\mathbf{A}', \varphi') \in \widetilde{X}(D) \times \widetilde{H^1}(D_c),$$

where

$$\begin{aligned} B((\mathbf{A}, \varphi), (\mathbf{A}', \varphi')) &= \left(\mu^{-1} \mathrm{curl} \mathbf{A}, \mathrm{curl} \mathbf{A}'\right)_D \\ &+ j\omega^{-1} \left(\sigma(j\omega \mathbf{A} + \nabla\varphi), (j\omega \mathbf{A}' + \nabla\varphi')\right)_{D_c}, \forall (\mathbf{A}, \varphi), (\mathbf{A}', \varphi') \in \widetilde{X}(D) \times \widetilde{H^1}(D_c). \end{aligned}$$

Well-posedness

[Creusé et al, MMMAS 2012]

Existence and uniqueness of the weak solution $(\mathbf{A}, \varphi)$ since it was shown there that

$$\|(\mathbf{A}', \varphi')\|_B := |B((\mathbf{A}', \varphi'), (\mathbf{A}', \varphi'))|^{\frac{1}{2}}, \forall (\mathbf{A}', \varphi') \in \widetilde{X}(D) \times \widetilde{H^1}(D_c),$$

is a norm on $\widetilde{X}(D) \times \widetilde{H^1}(D_c)$ equivalent to the natural one

$$||(\mathbf{A}, \varphi)||_V = \left(\|\mathbf{A}'\|_D^2 + \|\mu^{-1/2} \mathrm{curl} \mathbf{A}'\|_D^2 + |\varphi'|_{1, D_c}^2\right)^{\frac{1}{2}}, \forall (\mathbf{A}', \varphi') \in \widetilde{X}(D) \times \widetilde{H^1}(D_c).$$

The goal-oriented functional

Definition

We here consider the output functional given by

$$Q(\mathbf{A}) = \int_D \mathbf{q} \cdot \mathrm{curl}\bar{\mathbf{A}}\, dx, \forall \mathbf{A} \in H(\mathrm{curl}, D),$$

where $\mathbf{q} \in L^2(D)^3$ is a given function.

Physical meaning

In many engineering applications, engineers are interested in the computation of the flux through a coil. Indeed, in the case where a coil is included in D, in which a given current $\mathbf{J}_s$ of intensity i is imposed, $\mathbf{N}$ being the unit direction of the coil, it can be shown that the magnetic flux through the surface S of the coil is given by

$$\Phi = \int_S \mathbf{curl A} \cdot \mathbf{n}\, dS,$$

and that it can be evaluated by $\bar{\Phi} = \frac{1}{i} Q(\mathbf{A}) = \frac{1}{i} \int_D \mathbf{q} \cdot \mathbf{curl}\bar{\mathbf{A}}\, dx,$

using $\mathbf{q} = \mathbf{H}_s$ where $\mathbf{curl H}_s = \mathbf{J}_s$, and where as usual $\mathbf{B} = \mathbf{curl A}$.

Adjoint problem

Definition of B^*

$$B^*((\mathbf{A},\varphi),(\mathbf{A}',\varphi')) = \overline{B((\mathbf{A}',\varphi'),(\mathbf{A},\varphi))} \quad \forall(\mathbf{A},\varphi),(\mathbf{A}',\varphi') \in \widetilde{X}(D) \times \widetilde{H^1}(D_c).$$

Adjoint problem

Look for $(\mathbf{A}^*,\varphi^*) \in \widetilde{X}(D) \times \widetilde{H^1}(D_c)$ such that

$$B^*((\mathbf{A}^*,\varphi^*),(\mathbf{A}',\varphi')) = Q(\mathbf{A}'), \quad \forall(\mathbf{A}',\varphi') \in \widetilde{X}(D) \times \widetilde{H^1}(D_c),$$

Strong formulation of the adjoint problem

$$\begin{aligned} \mathrm{curl}\left(\mu^{-1}\mathrm{curl}\mathbf{A}^*\right) - \sigma\left(j\omega\mathbf{A}^* + \nabla\varphi^*\right) &= \mathrm{curl}\mathbf{q} && \text{in } D, \\ \mathrm{div}(\sigma(j\omega\mathbf{A}^* + \nabla\varphi^*)) &= 0 && \text{in } D_c. \end{aligned}$$

Discrete setting

Mesh and discrete spaces

- $H^1(\mathcal{T}) = \{v \in L^2(D) \,|\, v_{|T} \in H^1(T), \forall T \in \mathcal{T}\}$.
- $(\mathbf{A}_h, \varphi_h) \in V_h \subset H^1(\mathcal{T})^3 \times H^1(\mathcal{T}_c)$.
- For $\mathbf{A}_h' \in H^1(\mathcal{T})^3$ and $\varphi_h' \in H^1(\mathcal{T}_c)$, we denote :

$$\begin{array}{rcll} \mathrm{curl}_h \mathbf{A}_h' & = & \mathrm{curl}\mathbf{A}_h' & \text{on } T, \quad \forall T \in \mathcal{T}, \\ \nabla_h \varphi_h' & = & \nabla \varphi_h' & \text{on } T, \quad \forall T \in \mathcal{T}_c. \end{array}$$

- We introduce the discrete counterparts of $\mathbf{B}$ and $\mathbf{E}$ by

$$\begin{array}{rcl} \mathbf{B}_h & = & \mathrm{curl}_h \mathbf{A}_h, \\ \mathbf{E}_h & = & -j\,\omega\,\mathbf{A}_h - \nabla_h \varphi_h. \end{array}$$

Potential and Flux reconstructions

- We assume that
  - a potential reconstruction $(\mathbf{S}_h, \psi_h) \in H_0(\mathrm{curl}, D) \times \widetilde{H^1}(D_c)$ of $(\mathbf{A}_h, \varphi_h)$ is available,
  - some flux reconstructions $\mathbf{H}_h$ and $\mathbf{J}_{e,h}$ are available that belong respectively to $H(\mathrm{curl}, D)$ and $H(\mathrm{div}, D_c)$ and satisfy the following conservation properties :

$$\begin{array}{rcl} (\mathrm{curl}\mathbf{H}_h - \tilde{\mathbf{J}}_{e,h} - \mathbf{J}_s, \mathbf{e})_T & = & 0, \forall T \in \mathcal{T}, \mathbf{e} \in \mathbb{C}^3, \\ \mathrm{div}\mathbf{J}_{e,h} & = & 0 \text{ in } D_c, \\ \mathbf{J}_{e,h} \cdot \mathbf{n} & = & 0 \text{ on } \partial D_c. \end{array}$$

Energy-norm estimator

The energy error

$$\epsilon_{A,\varphi} = \left(\left\| \mu^{-1/2} \mathrm{curl}_h \epsilon_A \right\|^2 + \left\| \omega^{-1/2}\, \sigma^{1/2} (j\, \omega \epsilon_A + \nabla_h \epsilon_\varphi) \right\|_{D_c}^2 \right)^{1/2},$$

The estimators

- Non conforming estimator :

$$\begin{aligned} \eta_{NC} &= \left(\left\| \mu^{-1/2} \mathrm{curl}_h (\mathbf{A}_h - \mathbf{S}_h) \right\|^2 \right. \\ &+ \left. \left\| \omega^{-1/2}\, \sigma^{1/2}\, (j\, \omega (\mathbf{A}_h - \mathbf{S}_h) + \nabla_h (\varphi_h - \psi_h)) \right\|_{D_c}^2 \right)^{1/2}, \end{aligned}$$

- Flux estimator :

$$\eta_{\mathrm{flux}} = \left(\eta_{\mathrm{magn}}^2 + \eta_{\mathrm{elec}}^2 \right)^{1/2}, \text{ with}$$

$$\eta_{\mathrm{magn}} = \left\| \mu^{1/2} (\mathbf{H}_h - \mu^{-1} \mathbf{B}_h) \right\|_D \text{ and } \eta_{\mathrm{elec}} = \left\| (\omega\sigma)^{-1/2} (\mathbf{J}_{e,h} - \sigma \mathbf{E}_h) \right\|_{D_c},$$

- Oscillation estimator (if D is convex)

$$\eta_{\mathcal{O}} = \mu_{\max}^{\frac{1}{2}} \left(\sum_{T \in \mathcal{T}} \pi^{-2} h_T^2 \| \mathbf{J}_s - \mathrm{curl} \mathbf{H}_h + \tilde{\mathbf{J}}_{e,h} \|_T^2 \right)^{\frac{1}{2}}.$$

Energy-norm estimator

Theorem 3

Let us define :

$$\eta = 2\eta_{NC} + \eta_{\text{flux}} + \eta_{\mathcal{O}},$$

Then we have :

$$\epsilon_{A,\varphi} \leq \eta$$

Similarly for the adjoint problem...

- The energy error :

$$\epsilon_{A^*,\varphi^*} = \left(\left\| \mu^{-1/2} \text{curl}_h \epsilon_{A^*} \right\|^2 + \left\| \omega^{-1/2} \sigma^{1/2} (j\,\omega \epsilon_{A^*} + \nabla_h \epsilon_{\varphi^*}) \right\|_{D_c}^2 \right)^{1/2}.$$

- The estimators :

$$\eta^*_{NC},\ \eta^*_{\text{flux}},\ \eta^*_{\mathcal{O}} \text{ and } \eta^* = 2\eta^*_{NC} + \eta^*_{\text{flux}} + \eta^*_{\mathcal{O}},$$

- The estimation :

$$\epsilon_{A^*,\varphi^*} \leq \eta^*.$$

Goal-oriented estimator

Theorem 4

Let $(\mathbf{S}_h, \psi_h) \in H_0(\text{curl}, D) \times \widetilde{H^1}(D_c)$ (resp. $(\mathbf{S}_h^*, \psi_h^*)$) be a potential reconstruction of $(\mathbf{A}_h, \varphi_h)$ (resp. $(\mathbf{A}_h^*, \varphi_h^*)$), then the error on the quantity of interest defined by

$$\mathcal{E} = \sum_{T \in \mathcal{T}} \int_T \mathbf{q} \cdot \text{curl}\overline{(\mathbf{A} - \mathbf{A}_h)}\, dx$$

admits the splitting

$$\mathcal{E} = \eta_{\text{QOI}} + \mathcal{R},$$

where the estimator η_{QOI} is given by

$$\begin{aligned}
\eta_{\text{QOI}} \quad = \quad & \sum_{T \in \mathcal{T}} \int_T \mathbf{q} \cdot \text{curl}\overline{(\mathbf{S}_h - \mathbf{A}_h)}\, dx \\
+ \quad & \int_D \mathbf{S}_h^* \cdot \overline{(\mathbf{J}_s - \text{curl}\mathbf{H}_h + \tilde{\mathbf{J}}_{e,h})}\, dx \\
- \quad & j\omega^{-1} \int_{D_c} \sigma^{-1} \mathbf{J}_{e,h}^* \cdot \overline{\Big(\sigma(j\omega \mathbf{S}_h + \nabla \psi_h) + \mathbf{J}_{e,h}\Big)}\, dx \\
- \quad & \int_D \mathbf{H}_h^* \cdot (\text{curl}\overline{\mathbf{S}_h} - \mu \overline{\mathbf{H}_h})\, dx,
\end{aligned}$$

Goal-oriented estimator

Theorem 4 ctd

while the remainder term $\mathcal{R}$ is defined by

$$\begin{aligned}\mathcal{R} &= \int_D (\mathbf{A}^* - \mathbf{S}_h^*) \cdot \overline{(\mathbf{J}_s - \mathrm{curl}\mathbf{H}_h + \tilde{\mathbf{J}}_{e,h})}\, dx \\ &+ j\omega^{-1} \int_{D_c} (\sigma^{-1}\mathbf{J}_{e,h}^* - \mathbf{E}^*) \cdot \overline{\big(\sigma(j\omega\mathbf{S}_h + \nabla\psi_h) + \mathbf{J}_{e,h}\big)}\, dx \\ &- \int_D (\mu^{-1}\mathrm{curl}\mathbf{A}^* - \mathbf{H}_h^*) \cdot (\mathrm{curl}\overline{\mathbf{S}_h} - \mu\overline{\mathbf{H}_h})\, dx\end{aligned}$$

Theorem 6

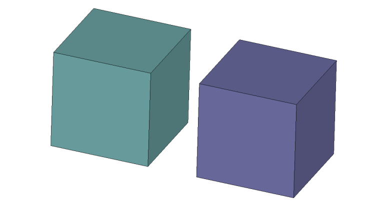
With η (resp. η^*) defined before, we have

$$|\mathcal{R}| \leq 6\eta\eta^*.$$

Numerical results

Primal problem

- $d = 3$,
- $D = [-2, 5] \times [-2, 2] \times [-2, 2]$,
- $D_s = [-1, 1]^3$
- $D_c = [2, 4] \times [-1, 1] \times [-1, 1]$.
- $\mu \equiv 1$ in D, $\sigma \equiv 1$ in D_c and $\omega = 2\pi$.



- The exact solution is given by $\varphi \equiv 0$ and

$$\mathbf{A} = \operatorname{curl} \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix} \text{ with } f(x, y, z) = \begin{cases} (x^2 - 1)^4 (y^2 - 1)^4 (z^2 - 1)^4 & \text{in} \quad D_s, \\ 0 & \text{in} \quad D \backslash D_s. \end{cases}$$

- The value of $\mathbf{J}_s$ is computed accordingly.

Numerical results

Discrete spaces for the primal problem

$$(\mathbf{A}_h, \varphi_h) \in \mathbf{V}_h = \tilde{X}_h \times \tilde{\Theta}_h, \text{ where}$$

$$\begin{aligned}
\tilde{\Theta}_h &= \{\varphi'_h \in \widetilde{H^1}(D_c) \ : \varphi'_{h|T} \in \mathbb{P}_1(T), \forall T \in \mathcal{T} \cap \bar{D}_c\}, \\
\Theta_h^0 &= \{\psi_h \in H_0^1(D) \ : \psi_{h|T} \in \mathbb{P}_1(T), \forall T \in \mathcal{T}\}, \\
X_h &= \{\mathbf{A}'_h \in H_0(\mathrm{curl}, D) \ : \ \mathbf{A}'_{h|T} \in \mathcal{ND}_1(T), \forall T \in \mathcal{T}\}, \\
\tilde{X}_h &= \{\mathbf{A}'_h \in X_h \ : \int_D \mathbf{A}'_h \cdot \nabla \psi_h = 0, \forall \psi_h \in \Theta_h^0\}.
\end{aligned}$$

Dual problem : regular solution

$\mathbf{q} = \mathbf{H}_s = \mathrm{curl}\mathbf{A}$, and we recall that we are interested in

$$\mathcal{E} = \int_D \mathbf{H}_s \cdot \mathrm{curl}\overline{(\mathbf{A} - \mathbf{A}_h)}\, dx.$$

Numerical results

Discrete spaces for the dual problem

$$(\mathbf{A}_h^*, \varphi_h^*) \in \mathbf{V}_h^* = \tilde{X}_h^* \times \tilde{\Theta}_h^*, \text{ where}$$

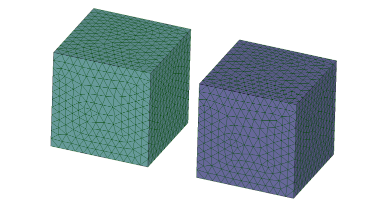
$$\tilde{\Theta}_h^* = \{\varphi_h' \in \widetilde{H^1}(D_c) \ : \varphi'_{h|T} \in \mathbb{P}_2(T), \forall T \in \mathcal{T} \cap \bar{D}_c\},$$

$$\Theta_h^{*,0} = \{\psi_h \in H_0^1(D) \ : \psi_{h|T} \in \mathbb{P}_2(T), \forall T \in \mathcal{T}\},$$

$$X_h^* = \{\mathbf{A}_h' \in H_0(\mathrm{curl}, D) \ : \ \mathbf{A}'_{h|T} \in \mathcal{ND}_2(T), \forall T \in \mathcal{T}\},$$

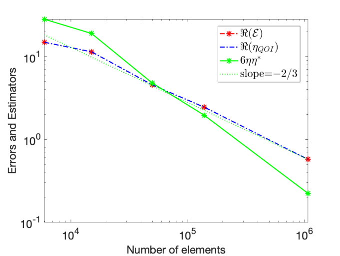
$$\tilde{X}_h^* = \{\mathbf{A}_h' \in X_h^* \ : \int_D \mathbf{A}_h' \cdot \nabla \psi_h = 0, \forall \psi_h \in \Theta_h^{*,0}\}.$$

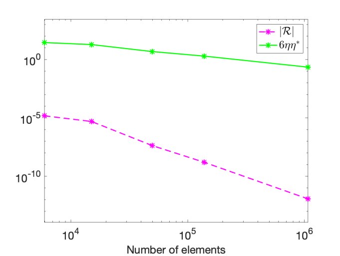
Meshes

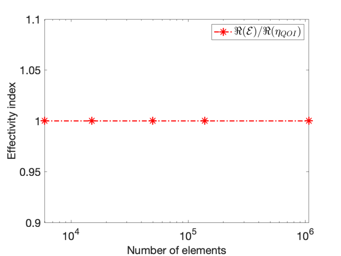


Numerical results

Regular solution, $(\mathbf{A}_h^*, \varphi_h^*) \in \mathbf{V}_h^*$

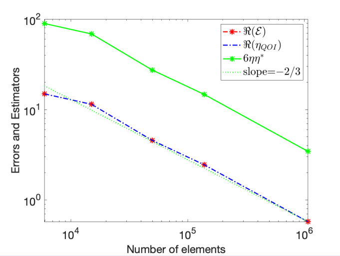


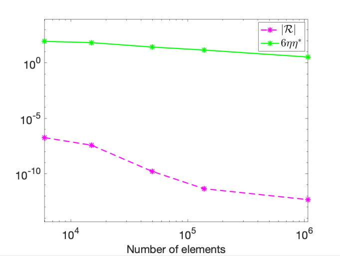


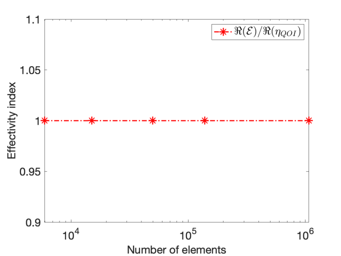


Numerical results

Regular solution $(\mathbf{A}^*, \varphi^*)$







Numerical results

Dual problem : singular solution

$$\mathbf{q} = \begin{pmatrix} \rho_s \\ 0 \\ 0 \end{pmatrix}$$

with

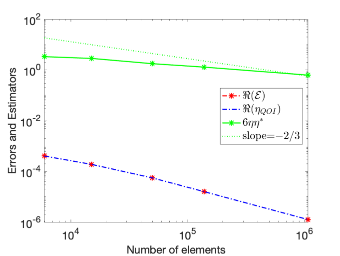
$$\rho_s(x,y,z) = e^{-\frac{(x-3)^2+y^2+z^2}{log(10)/4}}, \forall (x,y,z) \in D,$$

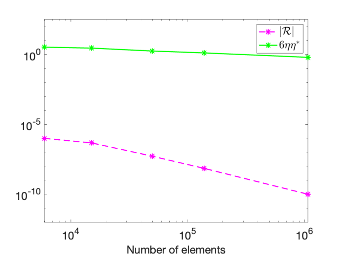
and we recall that we are interested in

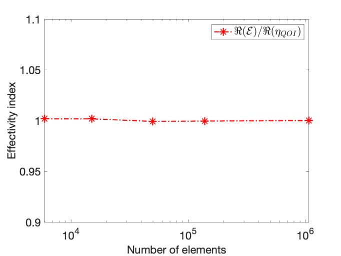
$$\mathcal{E} = \int_D \mathbf{q} \cdot \mathrm{curl}\overline{(\mathbf{A} - \mathbf{A}_h)}\, dx.$$

Numerical results

Singular solution $(\mathbf{A}^*, \varphi^*)$








Thank you