Polynomial-degree-robust a posteriori error estimates for virtual element methods

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Motivation

In this talk, we would like to approximately solve the Poisson problem

$$\left\{ \begin{array}{rcll} -\Delta u & = & f & \text{in } \Omega \subset \mathbb{R}^2, \\ u & = & 0 & \text{on } \partial\Omega, \end{array} \right.$$

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Specifically, we would like to associated with vertex $\boldsymbol{a} \in \mathcal{V}_h$ an a number $\eta_{\boldsymbol{a}}$ s.t.

Reliability and efficiency

$$\|u - u_h\|_\Omega^2 \lesssim \sum_{\boldsymbol{a} \in \mathcal{V}_h} \eta_{\boldsymbol{a}}^2, \qquad \eta_{\boldsymbol{a}} \lesssim \|u - u_h\|_{\omega^{\boldsymbol{a}}} \quad \forall \boldsymbol{a} \in \mathcal{V}_h,$$

with (ideally) constants only depending on the geometry (shape-regularity) of $\mathcal{T}_h$.

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with (ideally) constants only depending on the geometry (shape-regularity) of $\mathcal{T}_h$.

In particular, the constants are independent of p and the choice of stabilization.

Disclaimer!

I will make to (harmless) simplification throughout the talk:
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Due to time constraints, I will only talk about <u>some</u> aspects of the problem.
I will mainly focus on p-robustness, not on robustness w.r.t. the stabilization.

Outline

1. What are the challenges associated with VEM?
2. The approach for “standard” non-conforming methods
3. A modified approach suitable for VEM

What are the challenges associated with VEM?

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What is VEM anyway?

VEM in a nutshell

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$$V_h := \left\{ w_h \in H^1_0(\Omega) \;\middle|\; \begin{array}{rcll} \Delta w_h|_K & \in & \mathcal{P}_{p-2}(K) & \forall K \in \mathcal{T}_h \\ w_h|_F & \in & \mathcal{P}_p(F) & \forall F \in \mathcal{F}_h \end{array} \right\}.$$

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The VEM discrete problem is to find $u_h \in V_h$ such that

$$(\nabla_h(\Pi^{\nabla} u_h), \nabla_h(\Pi^{\nabla} v_h))_\Omega + s_h(u_h - \Pi^{\nabla} u_h, v_h - \Pi^{\nabla} v_h) = (f, v_h)_\Omega, \quad \forall v_h \in V_h$$

for a suitable stabilization form s_h computable through the dofs.

What are the challenges associated with VEM?

What is the “standard” setting?

The setting for "standard" non-conforming method

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(a) for all $\boldsymbol{a} \in \mathcal{V}_h$, $(\boldsymbol{\nabla}_h u_h, \boldsymbol{\nabla} \psi^{\boldsymbol{a}})_\Omega = (f, \psi^{\boldsymbol{a}})_\Omega$, with $\psi^{\boldsymbol{a}}$ the hat function of $\boldsymbol{a}$.
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Lagrange and Crouzeix–Raviart elements of arbitrary order satisfy these assumptions.

We will see that (a) is crucial for localizing computations.

The condition in (b) is important to employ the broken Poincaré inequality

$$\|w\|_U \lesssim h_U^{-1} \|\boldsymbol{\nabla}_h w\|_U$$

for all $w \in H^1(\mathcal{T}_h)$ with $([\![w]\!], 1)_F = 0$ and $(w, 1)_U = 0$.

What are the challenges associated with VEM?

How does VEM fail to enter the framework?

Non-polynomial solution

A first problem is that $u_h \notin \mathcal{P}_p(\mathcal{T}_h)$ for VEM.
Perhaps more importantly, we only know the dofs of u_h not its actual values.
Hence, $\|\nabla(u - u_h)\|_\Omega$ is not a desirable error measure.

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This problem can be remedied by considering $\Pi^{\boldsymbol{\nabla}} u_h$ as the "solution".

Indeed, then we have $\Pi^{\boldsymbol{\nabla}} u_h \in \mathcal{P}_p(\mathcal{T}_h)$ and we can use

$$\|\boldsymbol{\nabla}_h(u - \Pi^{\boldsymbol{\nabla}} u_h)\|_\Omega$$

as an error measure.

Virtual partition of unity

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Nevertheless, the VEM space does contain a partition of unity, given by

$$\Delta\psi^{\boldsymbol{a}}|_K = 0 \ \forall K \in \mathcal{T}_h, \quad \psi^{\boldsymbol{a}}|_F \in \mathcal{P}_1(F) \ \forall F \in \mathcal{F}_h, \quad \psi^{\boldsymbol{a}}(\boldsymbol{b}) = \delta_{\boldsymbol{a},\boldsymbol{b}} \ \forall \boldsymbol{b} \in \mathcal{V}_h.$$

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for all $\boldsymbol{a} \in \mathcal{V}_h$.

However, unless $\mathcal{T}_h$ contains simplices, these $\psi^{\boldsymbol{a}}$ are “virtual”.

Lack of Galerkin orthogonality

In order to follow the standard framework, we would need

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Unfortunately, due to the stabilization form, we only have

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This can be remedied by post-processing the solution and constructing $\boldsymbol{\mathcal{G}}_h$ such that

$$(\boldsymbol{\mathcal{G}}_h, \boldsymbol{\nabla}\psi^{\boldsymbol{a}}) = (f, \psi^{\boldsymbol{a}})_\Omega$$

and then measure the error with

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This notion of generalized gradient has been previously used in the past:

A. Ern and M. Vohralík, *SIAM J. Numer. Anal.*, 2015.

D.A. Di Pietro, J. Droniou, and G. Manzini, *J. Comput. Phys.*, 2018.

What are the challenges associated with VEM?

What is in this talk?

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This leads to in a modified framework, providing to p-robust estimates.

The approach for “standard” non-conforming methods

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We want to estimate the error in the norm

$$\|\boldsymbol{\nabla} u - \boldsymbol{\nabla}_h u_h\|_\Omega = \|\boldsymbol{\nabla}_h (u - u_h)\|_\Omega.$$

The approach for “standard” non-conforming methods

Prager–Synge identity

Error splitting

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The Euler-Lagrange equations defining $s^\star \in H^1_0(\Omega)$ are

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$$(\boldsymbol{\nabla} s^\star, \boldsymbol{\nabla} v)_\Omega = (\boldsymbol{\nabla}_h u_h, \boldsymbol{\nabla} v)_\Omega \qquad \forall v \in H^1_0(\Omega).$$

In particular, we have the Pythagorean identity

$$\|\boldsymbol{\nabla}_h(u - u_h)\|_\Omega^2 = \|\boldsymbol{\nabla}_h(u_h - s^\star)\|_\Omega^2 + \|\boldsymbol{\nabla}(u - s^\star)\|_\Omega^2$$

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We thus split the error as “distance to $H^1_0(\Omega)$” + “something else”.

What is the second term?

Since $u - s^\star \in H^1_0(\Omega)$, we have

$$\|\nabla(u - s^\star)\|_\Omega = \sup_{\substack{v \in H^1_0(\Omega) \\ \|\nabla v\|_\Omega = 1}} (\nabla(u - s^\star), \nabla v)_\Omega.$$

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Recall that, whenever $v \in H^1_0(\Omega)$, we do have

$$(\nabla u, \nabla v)_\Omega = (f, v)_\Omega, \qquad (\nabla s^\star, \nabla v)_\Omega = (\nabla_h u_h, \nabla v)_\Omega;$$

and therefore

$$\|\nabla(u - s^\star)\|_\Omega = \sup_{\substack{v \in H^1_0(\Omega) \\ \|\nabla v\|_\Omega = 1}} \left\{(f, v)_\Omega - (\nabla_h u_h, \nabla v)_\Omega\right\}.$$

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This is a standard Hilbert space result.

Recall that, whenever $v \in H_0^1(\Omega)$, we do have

$$(\nabla u, \nabla v)_\Omega = (f, v)_\Omega, \qquad (\nabla s^\star, \nabla v)_\Omega = (\nabla_h u_h, \nabla v)_\Omega;$$

and therefore

$$\|\nabla(u - s^\star)\|_\Omega = \sup_{\substack{v \in H_0^1(\Omega) \\ \|\nabla v\|_\Omega = 1}} \{(f, v)_\Omega - (\nabla_h u_h, \nabla v)_\Omega\}.$$

In other words,

$$\|\nabla(u - s^\star)\|_\Omega = \sup_{\substack{v \in H_0^1(\Omega) \\ \|\nabla v\|_\Omega = 1}} \langle f + \nabla \cdot (\nabla_h u_h), v\rangle_\Omega = \|f + \nabla \cdot \nabla_h u_h\|_{H^{-1}(\Omega)},$$

so that this term measures the PDE residual.

Reformulating as a minimization problem

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and with a bit of extra work, we can show that equality holds.

The Prager-Synge identity

Putting together the pieces, we have shown that

Prager-Synge identity

$$\|\nabla_h(u - u_h)\|_\Omega^2 = \min_{s \in H_0^1(\Omega)} \|\nabla_h(u_h - s)\|_\Omega^2 + \min_{\substack{\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla \cdot \boldsymbol{\sigma} = f}} \|\boldsymbol{\sigma} + \nabla_h u_h\|_\Omega^2.$$

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The equation $-\Delta u = f$ means (a) $u \in H_0^1(\Omega)$ and (b) $\boldsymbol{\nabla}\cdot(-\boldsymbol{\nabla} u) = f$.
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Since we "just" want an upper bound, we can input any admissible field s and $\boldsymbol{\sigma}$.
Constructing a "potential" s and an equilibrated flux "$\boldsymbol{\sigma}$" makes an estimator.

Of course, to have a good estimator, these need to be close to $\boldsymbol{\nabla}_h u_h$.

The approach for "standard" non-conforming methods

Practical reconstructions

Idealized reconstructions

We have shown earlier that

$$\|\nabla_h(u - u_h)\|_\Omega^2 = \min_{s \in H_0^1(\Omega)} \|\nabla_h(u_h - s)\|_\Omega^2 + \min_{\substack{\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla \cdot \boldsymbol{\sigma} = f}} \|\boldsymbol{\sigma} + \nabla_h u_h\|_\Omega^2.$$

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A natural idea to obtain a guaranteed error bound is simply to say that

$$\|\boldsymbol{\nabla}_h(u-u_h)\|_\Omega^2 \leq \min_{s_h \in H^1_0(\Omega) \cap \mathcal{P}_p(\mathcal{T}_h)} \|\boldsymbol{\nabla}_h(u_h - s)\|_\Omega^2 + \min_{\substack{\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div},\Omega) \cap \boldsymbol{RT}_p(\mathcal{T}_h) \\ \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = f}} \|\boldsymbol{\sigma} + \boldsymbol{\nabla}_h u_h\|_\Omega^2.$$

where the second minimization problem is well-posed since we assumed $f \in \mathcal{P}_{p-1}(\mathcal{T}_h)$.

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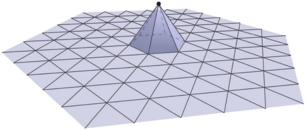
This approach is "feasible": It does lead to a guaranteed upper bound.

However, it is expensive and it is not clear that it leads to localized lower bound.

Localization with the hat functions

As we consider a simplicial mesh $\mathcal{T}_h$ here, the "hat functions" $\{\psi^{\boldsymbol{a}}\}_{\boldsymbol{a}\in\mathcal{V}_h}$ form a partition of unity.



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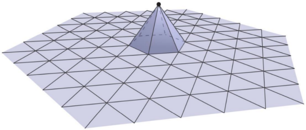

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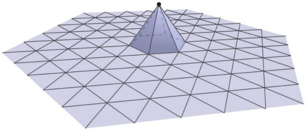

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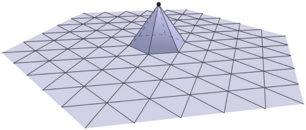

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We use this partition of unity to localize the potential and flux reconstructions.

Potential reconstruction

We focus on the term

$$\min_{s \in H_0^1(\Omega)} \|\nabla_h (s - u_h)\|_\Omega$$

and provide an element $s_h \in H_0^1(\Omega) \cap \mathcal{P}_{p+1}(\mathcal{T}_h)$ close to u_h from local computations.

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Observe that $s_h \in H_0^1(\Omega)$ should mimic u_h on Ω.

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Observe that $s_h \in H^1_0(\Omega)$ should mimic u_h on Ω. The decomposition

$$u_h = \sum_{\mathbf{a} \in \mathcal{V}_h} \psi^{\mathbf{a}} u_h$$

motivates to build $s_h^{\mathbf{a}} \in H^1_0(\omega^{\mathbf{a}})$ close to $\psi^{\mathbf{a}} u_h$, and then set

$$s_h = \sum_{\mathbf{a} \in \mathcal{V}_h} s_h^{\mathbf{a}}.$$

Potential reconstruction (continued)

We solve for each $\boldsymbol{a} \in \mathcal{V}_h$ the problem

Localized potential reconstruction

$$s_h^{\boldsymbol{a}} := \arg \min_{w_h \in H_0^1(\omega^{\boldsymbol{a}}) \cap \mathcal{P}_{p+1}(\mathcal{T}_h^{\boldsymbol{a}})} \|\boldsymbol{\nabla}_h(\psi^{\boldsymbol{a}} u_h - s_h^{\boldsymbol{a}})\|_{\omega^{\boldsymbol{a}}}.$$

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This leads to a set of uncoupled small finite element problem each involving few dofs.

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Note that the values of the $\psi^{\boldsymbol{a}}$ are required to assemble the right-hand sides.

Flux reconstruction

We follow a similar strategy to build $\boldsymbol{\sigma}_h \in \boldsymbol{H}(\mathrm{div},\Omega) \cap \boldsymbol{RT}_{p+1}(\mathcal{T}_h)$. For each $\boldsymbol{a} \in \mathcal{V}_h$,

Localized flux reconstruction

$$\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{\xi}_h \in \boldsymbol{H}_0(\mathrm{div},\omega^{\boldsymbol{a}}) \cap \boldsymbol{RT}_{p+1}(\mathcal{T}_h^{\boldsymbol{a}}) \\ \boldsymbol{\nabla}\cdot\boldsymbol{\xi}_h = \psi^{\boldsymbol{a}} f - \boldsymbol{\nabla}\psi^{\boldsymbol{a}}\cdot\boldsymbol{\nabla}_h u_h}} \|\boldsymbol{\xi}_h + \psi^{\boldsymbol{a}} \boldsymbol{\nabla}_h u_h\|_{\omega^{\boldsymbol{a}}}.$$

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Crucially the Stokes' compatibility condition is satisfied due to Galerkin orthogonality:

$$(\psi^{\boldsymbol{a}} f - \boldsymbol{\nabla} \psi^{\boldsymbol{a}} \cdot \boldsymbol{\nabla}_h u_h, 1)_{\omega^{\boldsymbol{a}}} = (\boldsymbol{\nabla}_h u_h, \boldsymbol{\nabla} \psi^{\boldsymbol{a}})_\Omega - (f, \psi^{\boldsymbol{a}})_\Omega = 0.$$

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After summation over $\boldsymbol{a} \in \mathcal{V}_h$, we have $\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_h = f$. We control the second term with

$$\min_{\substack{\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div},\Omega) \\ \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = f}} \|\boldsymbol{\sigma} + \boldsymbol{\nabla}_h u_h\|_\Omega \leq \|\boldsymbol{\sigma}_h + \boldsymbol{\nabla}_h u_h\|_\Omega.$$

Summary

We solve the local problems

Localized potential reconstruction

$$s_h^{a} := \arg \min_{w_h \in H_0^1(\omega^{a}) \cap \mathcal{P}_{p+1}(\mathcal{T}_h^{a})} \|\nabla_h(\psi^{a} u_h - s_h^{a})\|_{\omega^{a}}.$$

and

Localized flux reconstruction

$$\sigma_h^{a} := \arg \min_{\substack{\xi_h \in H_0(\mathrm{div}, \omega^{a}) \cap RT_{p+1}(\mathcal{T}_h^{a}) \\ \nabla \cdot \xi_h = \psi^{a} f - \nabla \psi^{a} \cdot \nabla_h u_h}} \|\xi_h + \psi^{a} \nabla_h u_h\|_{\omega^{a}}.$$

for each $a \in \mathcal{V}_h$.

After summing up the contributions, we have

Guaranteed upper bound

$$\|\nabla_h(u - u_h)\|_{\Omega}^2 \leq \|\nabla_h(u_h - s_h)\|_{\Omega}^2 + \|\sigma_h + \nabla u_h\|_{\Omega}^2.$$

The approach for “standard” non-conforming methods

Efficiency

Discrete stable minimization

For all $\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{P}}_p(\mathcal{T}_h^{\boldsymbol{a}})$, we have

Unconstrained H^1 minimization

$$\min_{w_h \in H^1_0(\omega^{\boldsymbol{a}}) \cap \mathcal{P}_{p+1}(\mathcal{T}_h^{\boldsymbol{a}})} \|\boldsymbol{\tau}_h - \boldsymbol{\nabla} w_h\|_{\omega^{\boldsymbol{a}}} \lesssim \min_{w \in H^1_0(\omega^{\boldsymbol{a}})} \|\boldsymbol{\tau}_h - \boldsymbol{\nabla} w\|_{\omega^{\boldsymbol{a}}}$$

with a constant independent of p.

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Similarly, for all $\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{P}}_{p+1}(\mathcal{T}_h^{\boldsymbol{a}})$ and $q_h \in \mathcal{P}_{p+1}(\mathcal{T}_h^{\boldsymbol{a}})$ with $(q_h, 1)_{\omega^{\boldsymbol{a}}} = 0$, we have

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$$\min_{\substack{\boldsymbol{\xi}_h \in \boldsymbol{H}_0(\mathrm{div}, \omega^{\boldsymbol{a}}) \cap \boldsymbol{RT}_{p+1}(\mathcal{T}_h^{\boldsymbol{a}}) \\ \boldsymbol{\nabla} \cdot \boldsymbol{\xi}_h = q_h}} \|\boldsymbol{\xi}_h + \boldsymbol{\tau}_h\|_{\omega^{\boldsymbol{a}}} \lesssim \min_{\substack{\boldsymbol{\xi} \in \boldsymbol{H}_0(\mathrm{div}, \omega^{\boldsymbol{a}}) \\ \boldsymbol{\nabla} \cdot \boldsymbol{\xi} = q_h}} \|\boldsymbol{\xi} + \boldsymbol{\tau}_h\|_{\omega^{\boldsymbol{a}}}$$

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T. Chaumont-Frelet and M. Vohralík, *arXiv*, 2023.

Application to the localized reconstructions

Using the discrete minimization, we have

$$\begin{aligned}
\|\nabla_h(\psi^{a} u_h - s_h^{a})\|_{\omega^{a}} &= \min_{w_h \in H_0^1(\omega^{a}) \cap \mathcal{P}_{p+1}(\mathcal{T}_h^{a})} \|\nabla_h(\psi^{a} u_h) - \nabla w_h\|_{\omega^{a}} \\
&\lesssim \min_{w \in H_0^1(\omega^{a})} \|\nabla_h(\psi^{a} u_h) - \nabla w\|_{\omega^{a}} \\
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by picking $w = \psi^{a} u$.

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$$\|\boldsymbol{\nabla}_h(\psi^{\boldsymbol{a}}(u_h - s_h^{\boldsymbol{a}}))\|_{\omega^{\boldsymbol{a}}} \lesssim \|\boldsymbol{\nabla}_h(u - u_h)\|_{\omega^{\boldsymbol{a}}}$$

using Galerkin orthogonality and a broken Poincaré inequality.

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A similar argument applies to $\|\boldsymbol{\sigma}_h^{a} + \psi^{a}\nabla_h u_h\|_{\omega^{a}}$.

Efficiency

We have established earlier that

Guaranteed upper bound

$$\|\nabla(u - u_h)\|_\Omega^2 \leq \|\nabla_h(u_h - s_h)\|_\Omega^2 + \|\boldsymbol{\sigma}_h + \nabla_h u_h\|_\Omega^2.$$

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$$\|\nabla_h(u_h - s_h)\|_K^2 + \|\boldsymbol{\sigma}_h + \nabla_h u_h\|_K^2 \lesssim \|\nabla(u - u_h)\|_{\widetilde{K}}^2,$$

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holds, even locally, up a constant independent of p.

In particular, the overestimation in the upper bound cannot be too large.

The approach for “standard” non-conforming methods

Summary

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The error bounds

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$$\|\boldsymbol{\nabla}(u - u_h)\|_\Omega^2 \leq \|\boldsymbol{\nabla}_h(u_h - s_h)\|_\Omega^2 + \|\boldsymbol{\sigma}_h + \boldsymbol{\nabla}_h u_h\|_\Omega^2$$

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It is crucial that the $\psi^{\boldsymbol{a}}$ are computable and polynomial.

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can be obtained by solving local uncoupled finite element problems.

The partition of the unity by the hat function $\psi^{\boldsymbol{a}}$ plays a important role.
It is crucial that the $\psi^{\boldsymbol{a}}$ are computable and polynomial.

Unfortunately, the VEM partition in unity is virtual.

A modified approach suitable for VEM

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Key ideas

Virtual partition of unity

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The expression of the $\psi^{\boldsymbol{a}}$ must be available to compute $s_h^{\boldsymbol{a}}$ and $\boldsymbol{\sigma}_h^{\boldsymbol{a}}$.
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Here, we want to extend the approach to a situation where
a) a partition of unity $\psi^{\boldsymbol{a}}$ exists
b) the $\psi^{\boldsymbol{a}}$ satisfy the natural scaling $|\psi^{\boldsymbol{a}}| \lesssim 1$ and $|\boldsymbol{\nabla}\psi^{\boldsymbol{a}}| \lesssim h_{\omega^{\boldsymbol{a}}}^{-1}$.
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c) the $\psi^{\boldsymbol{a}}$ need not be computable.

In other words, the $\psi^{\boldsymbol{a}}$ will appear in the analysis, but not in the algorithms.

Broken Prager–Synger inequality

Earlier, we used the

Prager–Synge identity

$$\|\nabla_h(u-u_h)\|_\Omega^2 = \min_{s\in H^1_0(\Omega)} \|\nabla_h(u_h-s)\|_\Omega^2 + \min_{\substack{\boldsymbol{\sigma}\in \boldsymbol{H}(\mathrm{div},\Omega)\\ \nabla\cdot\boldsymbol{\sigma}=f}} \|\boldsymbol{\sigma}+\nabla_h u_h\|_\Omega^2,$$

with suitable s and $\boldsymbol{\sigma}$ constructed through local problems explicitly involving $\psi^{\boldsymbol{a}}$.

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$$\|\nabla_h(u - u_h)\|_\Omega^2 \lesssim \sum_{a \in \mathcal{V}_h} \left(\min_{s^a \in H^1(\omega^a)} \|\nabla_h(u_h - s^a)\|_{\omega^a}^2 + \min_{\substack{\sigma \in \boldsymbol{H}(\mathrm{div},\omega^a) \\ \nabla \cdot \sigma = f}} \|\sigma^a + \nabla_h u_h\|_{\omega^a}^2 \right)$$

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where s^a and σ^a may be computed without explicitly knowing ψ^a.

The scaling properties of ψ^a will appear in $\lesssim$.

A modified approach suitable for VEM

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Our goal is to derive an inequality of the form

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$$\|\boldsymbol{\nabla}_h(u - u_h)\|_{\Omega}^2 \lesssim \sum_{\boldsymbol{a} \in \mathcal{V}_h} \left(\min_{s^{\boldsymbol{a}} \in H^1(\omega^{\boldsymbol{a}})} \|\boldsymbol{\nabla}_h(u_h - s^{\boldsymbol{a}})\|_{\omega^{\boldsymbol{a}}}^2 + \min_{\substack{\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \omega^{\boldsymbol{a}}) \\ \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = f}} \|\boldsymbol{\sigma}^{\boldsymbol{a}} + \boldsymbol{\nabla}_h u_h\|_{\omega^{\boldsymbol{a}}}^2 \right)$$

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More specifically, it is in fact possible to show that

$$\min_{s\in H_0^1(\Omega)}\|\boldsymbol{\nabla}(u_h-s)\|_{\omega^{\boldsymbol{a}}}^2 \lesssim \min_{s^{\boldsymbol{a}}\in H^1(\omega^{\boldsymbol{a}})}\|\boldsymbol{\nabla}_h(u_h-s^{\boldsymbol{a}})\|_{\omega^{\boldsymbol{a}}}^2$$

and

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I am going to detail how the first inequality is obtained.

Potential reconstruction

We want to upper bound

$$\min_{s \in H^1_0(\Omega)} \|\nabla_h(u_h - s)\|_\Omega.$$

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We will do so by using functions $s \in H_0^1(\Omega)$ of a specific form. Namely,

$$s^\star := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \psi^{\boldsymbol{a}} s^{\boldsymbol{a}} \in H_0^1(\Omega).$$

where, for each $\boldsymbol{a} \in \mathcal{V}_h$, $s^{\boldsymbol{a}} \in H^1(\omega^{\boldsymbol{a}})$ satisfies $(s^{\boldsymbol{a}}, 1)_{\omega^{\boldsymbol{a}}} = (u_h, 1)_{\omega^{\boldsymbol{a}}}$.

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Our first task is to show that

$$\|\boldsymbol{\nabla}_h (u_h - s^\star)\|^2_{\Omega} \leq (d+1) \sum_{\boldsymbol{a} \in \mathcal{V}_h} \|\boldsymbol{\nabla}_h (\psi^{\boldsymbol{a}} (u_h - s^{\boldsymbol{a}}))\|^2_{\omega^{\boldsymbol{a}}}.$$

Potential reconstruction (continued)

To do so, we fix an element $K \in \mathcal{T}_h$. Due to the limited support of the $\psi^{\boldsymbol{a}}$, we have

$$\|\boldsymbol{\nabla}_h(u - s^\star)\|_K = \|\boldsymbol{\nabla}_h(u_h - \sum_{\boldsymbol{a} \in \mathcal{V}_h(K)} \psi^{\boldsymbol{a}} s^{\boldsymbol{a}})\|_K = \|\sum_{\boldsymbol{a} \in \mathcal{V}_h(K)} \boldsymbol{\nabla}_h(\psi^{\boldsymbol{a}}(u_h - s^{\boldsymbol{a}}))\|_K.$$

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Since each K as $d+1$ vertices, the triangle and Cauchy-Schwarz inequality gives:

$$\|\boldsymbol{\nabla}_h(u_h - \sum_{\boldsymbol{a} \in \mathcal{V}_h} \psi^{\boldsymbol{a}} s^{\boldsymbol{a}})\|_K^2 \leq (d+1) \sum_{\boldsymbol{a} \in \mathcal{V}_h(K)} \|\boldsymbol{\nabla}_h(\psi^{\boldsymbol{a}}(u_h - s^{\boldsymbol{a}}))\|_K^2$$

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After summation over the K, we obtain

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and we have sucessfully loacalized the norm.

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and we have sucessfully loacalized the norm.

The next step is to remove the hat function.

Removing the hat function

Consider a vertex $\boldsymbol{a} \in \mathcal{V}_h$. Then, due to assumptions on $\psi^{\boldsymbol{a}}$, we have

$$\|\boldsymbol{\nabla}_h(\psi^{\boldsymbol{a}}(u_h - s^{\boldsymbol{a}}))\|_{\omega^{\boldsymbol{a}}} \lesssim \|\boldsymbol{\nabla}_h(u_h - s^{\boldsymbol{a}})\|_{\omega^{\boldsymbol{a}}} + h_{\omega^{\boldsymbol{a}}}^{-1}\|u_h - s^{\boldsymbol{a}}\|_{\omega^{\boldsymbol{a}}}.$$

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Since $(u_h - s^{\boldsymbol{a}}, 1)_{\omega^{\boldsymbol{a}}} = 0$, the broken Poincaré inequality controls the second term, and

$$\|\boldsymbol{\nabla}_h(\psi^{\boldsymbol{a}}(u_h - s^{\boldsymbol{a}}))\|_{\omega^{\boldsymbol{a}}} \lesssim \|\boldsymbol{\nabla}_h(u_h - s^{\boldsymbol{a}})\|_{\omega^{\boldsymbol{a}}}$$

for all $s^{\boldsymbol{a}} \in H^1(\omega^{\boldsymbol{a}})$ with $(s^{\boldsymbol{a}}, 1)_{\omega^{\boldsymbol{a}}} = (u_h, 1)_{\omega^{\boldsymbol{a}}}$.

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for all $s^a \in H^1(\omega^a)$ with $(s^a, 1)_{\omega^a} = (u_h, 1)_{\omega^a}$.

After summation over $a \in \mathcal{V}_h$, we have

$$\min_{s \in H^1_0(\Omega)} \|\nabla_h(u_h - s)\|^2_\Omega \leq \|\nabla_h(u_h - s^\star)\|^2_\Omega \lesssim \sum_{a \in \mathcal{V}_h} \|\nabla_h(u_h - s^a)\|^2_{\omega^a}$$

for all $s^a \in H^1(\omega^a)$ with $(s^a, 1)_{\omega^a} = (u_h, 1)_{\omega^a}$.

Broken localization of the potential reconstruction

We have just established

$$\min_{s \in H^1_0(\Omega)} \|\nabla_h (u_h - s)\|^2_\Omega \lesssim \sum_{a \in \mathcal{V}_h} \|\nabla_h (u_h - s^a)\|^2_{\omega^a}$$

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for all $s^{\boldsymbol{a}} \in H^1(\omega^{\boldsymbol{a}})$ with $(s^{\boldsymbol{a}}, 1)_{\omega^{\boldsymbol{a}}} = (u_h, 1)_{\omega^{\boldsymbol{a}}}$.

As can be seen, only the gradients of the $s^{\boldsymbol{a}}$ matter in the last bound.
We can freely shift them by a constant to remove the mean-value constraint.

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We can freely shift them by a constant to remove the mean-value constraint.

After minimizing, we obtain

Broken localization of the potential

$$\min_{s \in H^1_0(\Omega)} \|\nabla_h(u_h - s)\|^2_\Omega \lesssim \sum_{a \in \mathcal{V}_h} \min_{s^a \in H^1(\omega^a)} \|\nabla_h(u_h - s^a)\|^2_{\omega^a}$$

where $\lesssim$ depends on the broken Poincaré constants and the scaling of ψ^a.

Broken localization of the flux

The proof of the equilibrated flux term is slightly more involved, but essentially uses the same ideas. It is used in another context in

T. Chaumont-Frelet, A. Ern and M. Vohralík, *Math. Comp.*, 2022.

We can rigorously show that

Broken localization of the potential

$$\min_{\substack{\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div},\Omega) \\ \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = f}} \|\boldsymbol{\sigma} + \boldsymbol{\nabla}_h u_h\|_{\Omega}^2 \lesssim \sum_{\boldsymbol{a} \in \mathcal{V}_h} \min_{\substack{\boldsymbol{\sigma}^{\boldsymbol{a}} \in \boldsymbol{H}(\mathrm{div},\omega^{\boldsymbol{a}}) \\ \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}^{\boldsymbol{a}} = f}} \|\boldsymbol{\sigma}^{\boldsymbol{a}} + \boldsymbol{\nabla}_h u_h\|_{\omega^{\boldsymbol{a}}}^2$$

where $\lesssim$ depends on the Poincaré constants and the scaling of $\psi^{\boldsymbol{a}}$.

A broken Prager–Synge inequality

Combining the two estimates we commented earlier, we have

A broken Prager–Synge inequality

$$\|\nabla_h(u - u_h)\|_\Omega^2 \lesssim \sum_{a \in \mathcal{V}_h} \left(\min_{s^a \in H^1(\omega^a)} \|\nabla_h(u_h - s^a)\|_{\omega^a}^2 + \min_{\substack{\sigma^a \in H(\mathrm{div}, \omega^a) \\ \nabla \cdot \sigma^a = f}} \|\sigma^a + \nabla_h u_h\|_{\omega^a}^2 \right)$$

The hidden constant only depends on
a) the Poincaré constant of the patch ω^a
b) the scaling of the ψ^a,
i.e., only on geometrical property of $\mathcal{T}_h$.

A modified approach suitable for VEM

Practical construction and efficiency

Practical construction

The localization has been performed at the continuous level:

A broken Prager–Synge inequality

$$\|\boldsymbol{\nabla}_h(u-u_h)\|_\Omega^2 \lesssim \sum_{\boldsymbol{a}\in\mathcal{V}_h}\left(\min_{s^{\boldsymbol{a}}\in H^1(\omega^{\boldsymbol{a}})}\|\boldsymbol{\nabla}_h(u_h-s^{\boldsymbol{a}})\|_{\omega^{\boldsymbol{a}}}^2+\min_{\substack{\boldsymbol{\sigma}^{\boldsymbol{a}}\in\boldsymbol{H}(\mathrm{div},\omega^{\boldsymbol{a}})\\ \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^{\boldsymbol{a}}=f}}\|\boldsymbol{\sigma}^{\boldsymbol{a}}+\boldsymbol{\nabla}_h u_h\|_{\omega^{\boldsymbol{a}}}^2\right)$$

Practical construction

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$$\|\nabla_h(u - u_h)\|_\Omega^2 \lesssim \sum_{a \in \mathcal{V}_h} \left(\min_{s^a \in H^1(\omega^a)} \|\nabla_h(u_h - s^a)\|_{\omega^a}^2 + \min_{\substack{\sigma^a \in H(\mathrm{div}, \omega^a) \\ \nabla \cdot \sigma^a = f}} \|\sigma^a + \nabla_h u_h\|_{\omega^a}^2 \right)$$

To obtain a practical estimator, we simply use the discretized version

Computable upper-bound

$$\|\nabla_h(u - u_h)\|_\Omega^2 \lesssim \sum_{a \in \mathcal{V}_h} \eta_a^2$$

where

$$\eta_a^2 = \min_{s_h^a \in H^1(\omega^a) \cap \mathcal{P}_p(\mathcal{T}_h^a)} \|\nabla_h(u_h - s_h^a)\|_{\omega^a}^2 + \min_{\substack{\sigma_h^a \in H(\mathrm{div}, \omega^a) \cap RT_p(\mathcal{T}_h^a) \\ \nabla \cdot \sigma_h^a = f}} \|\sigma_h^a + \nabla_h u_h\|_{\omega^a}^2.$$

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For instance, we have

$$\min_{s_h^{\boldsymbol{a}} \in H^1(\omega^{\boldsymbol{a}}) \cap \mathcal{P}_p(\mathcal{T}_h)} \|\boldsymbol{\nabla}_h(u_h - s_h^{\boldsymbol{a}})\|_{\omega^{\boldsymbol{a}}} \lesssim \min_{s^{\boldsymbol{a}} \in H^1(\omega^{\boldsymbol{a}})} \|\boldsymbol{\nabla}_h(u_h - s^{\boldsymbol{a}})\|_{\omega^{\boldsymbol{a}}} \leq \|\boldsymbol{\nabla}_h(u - u_h)\|_{\omega^{\boldsymbol{a}}}$$

for all $\boldsymbol{a} \in \mathcal{T}_h$.

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A similar analysis of the flux term shows that

Efficiency

$$\eta_{a} \lesssim \|\nabla_h(u - u_h)\|_{\omega^{a}}$$

with a constant independent of p.

Concluding remarks

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The framework introduced in

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can be extended to situation where the partition of unity is virtual.

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Reliability and efficiency

$$\|\boldsymbol{\nabla} u - \boldsymbol{\mathcal{G}}_h\|_{\Omega}^2 \lesssim \sum_{\boldsymbol{a} \in \mathcal{T}_h} \eta_{\boldsymbol{a}}^2, \qquad \eta_{\boldsymbol{a}} \lesssim \|\boldsymbol{\nabla} u - \boldsymbol{\mathcal{G}}_h\|_{\omega^{\boldsymbol{a}}}$$

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T. Chaumont-Frelet, J. Gedicke and L. Mascotto, *arXiv*, next Monday.