Model order reduction methods: parametric variational inequalities

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Outline of the talk

1. Introduction to model order reduction for parametrized PDEs
2. Reduced-order model for parametric variational inequalities
3. Conclusions and Perspectives

Parametrized Partial Differential Equations

- There are a wide range of contexts where the state of a system of interest can be modeled by means of a set of PDEs.
- This set of equations may depend on some **parameters** $\mu = (\mu_1, \cdots, \mu_p)$ with $p \in \mathbb{N}^*$, the values of which belong to a set of parameter values $\mathcal{D} \subset \mathbb{R}^p$.

For a given value $\mu \in \mathcal{D}$ of the parameters, the solution of the associated PDE system generally is a function $u(\mu)$ which satisfies

$$\mathcal{A}(u(\mu); \mu) = 0,$$

where $\mathcal{A}(\cdot; \mu)$ is a **parametric differential operator**, the values of which depends on μ.

Simple toy example: heat equation

$\Omega \subset \mathbb{R}^d$ and $T > 0$. Find $u(\mu)(t,x)$ ($x \in \Omega$, $t \in [0,T]$) of the parametrized problem

$$\begin{cases} \dfrac{\partial u(\mu)}{\partial t} - \nabla \cdot \Big(\mu \nabla u(\mu)\Big) = f & \text{in } \Omega \times (0,T) \\ -\mu \dfrac{\partial u(\mu)}{\partial n} = h_0 u(\mu), & \text{on } \partial\Omega \times (0,T) \\ u(\mu)(t=0,\cdot) = u_0, & \text{in } \Omega \end{cases}$$

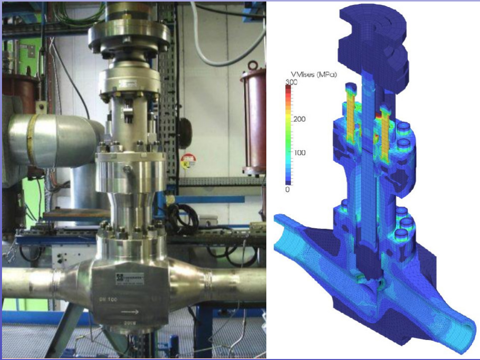
- $\mu \in \mathcal{D} := [\mu_{\min}, \mu_{\max}] \subset \mathbb{R}$: thermal conductivity
- $h_0 \in \mathbb{R}$: thermal exchange coefficient (Robin boundary conditions)
- $f \in L^2(0,T;D)$: source
- $u_0 \in H^1(D)$: initial condition

Example in an industrial context: collaboration with EDF

PhD thesis of **Amina Benaceur** (with Alexandre Ern): Model-order reduction for nonlinear coupled thermo-mechanical problems

Motivation:

- Motivation: Study of flow regulation valve used in nuclear reactor operation.
- Complex simulation: Coupled system with thermal ($\approx$ 4h), mechanical ($\approx$ 0.5j), instationnary nonlinear problems, with contact.



Motivation of model-order reduction methods

- For a given value $\mu \in \mathcal{D}$, the solution $u(\mu)$ is typically computed by means of a simulation code (using for instance finite elements, finite volumes...) which can be **very costly in terms of computational time** for complex systems.
- There is a wide variety of contexts which require the computation of $u(\mu)$ **for a very large number of parameter values μ as quickly as possible!**

Exemples:

- Optimization (design)
- Inverse problems (using experimental data for instance)
- Real time control
- Uncertainty quantification

Naive approaches are doomed to fail in such contexts

Principle of model order reduction

The aim of **model order reduction techniques** is to circumvent this difficulty. Their principle is the following:

- **Offline stage:** Compute $u(\mu)$ with the original costly simulation code for a **small** number of well-chosen values of μ
- Build another model, using the previous computations, which computes (an approximation) $u(\mu)$ for many other values of μ with a **much smaller computational cost than the original simulation code**: this is the **reduced-order model**.
- **Online stage:** Use the reduced-order model instead of the original simulation code to compute (much faster) $u(\mu)$ for a large number of values of μ

Model-order reduction methods

There exists a wide variety of model-order reduction methods.

The most classical approaches are called **linear model-order reduction methods**, among which:

- Proper Orthogonal Decomposition
- Reduced Basis methods
- Proper Generalized Decomposition

These linear approaches work very well typically for parametrized elliptic and parabolic systems.

More recently, **nonlinear model order reduction methods** (for instance based on neural networks or optimal transport) are being developped for othre types of problems for which linear methods do not work.

Linear approximation methods

Solution set:

$$\mathcal{M} := \{u(\mu),\ \mu \in \mathcal{D}\}$$

Assume that $\mathcal{M} \subset \mathcal{V}$ where $\mathcal{V}$ is a Hilbert space (for instance $\mathcal{V} = L^2(\Omega)$ or $\mathcal{V} = H^1(\Omega)$).

A classical method to construct reduced-order models consists in looking for a N-dimensional linear subspace V_N of V (with N small) so that the error

$$\mathrm{dist}^{\mathcal{V}}(\mathcal{M}, V_N) := \sup_{u \in \mathcal{M}} \|u - \Pi_{V_N} u\|_V$$

is as small as possible (here Π_{V_N} denotes the orthogonal projection of $\mathcal{V}$ onto V_N).

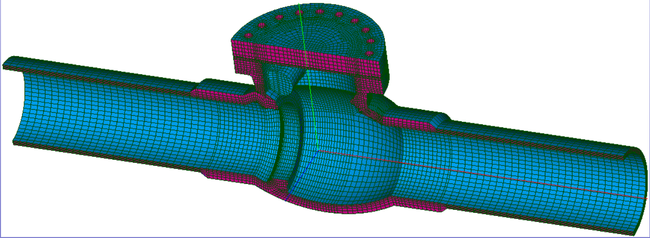
For a fixed value of N, the **best possible approximation error** is given by the **Kolmogorov N-width** of the set $\mathcal{M}$, defined as

$$d_N^{\mathcal{V}}(\mathcal{M}) := \inf_{\substack{V_N \subset \mathcal{V}, \\ \dim V_N = N}} \mathrm{dist}^{\mathcal{V}}(\mathcal{M}, V_N).$$

Example: For parabolic parametrized equations, it can be shown that $\left(d_N^{\mathcal{V}}(\mathcal{M})\right)_{N \in \mathbb{N}}$ decays **exponentially fast** with N [Cohen, DeVore, 2016]

Numerical results on an industrial test case: original model

Instationnary nonlinear heat problem on a flow regulation valve



Original mesh ≈ 51500 nodes 50 times steps

Computational time $\approx 140\ s$

Numerical results on an industrial test case: reduced-order model

Reduced-order model with reduced basis size $N = 25$

Computational time: $\approx 0.1\ s$

Error between the reduced-order model and the original model: below 1% (in $L^2(0,T;H^1(\Omega))$ norm)

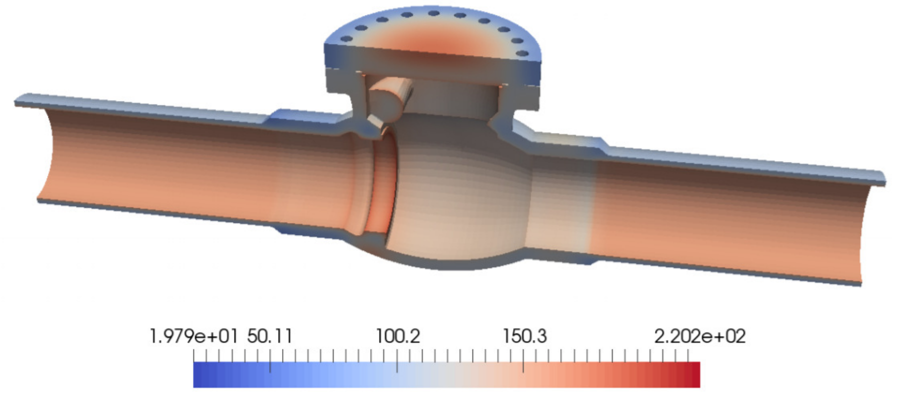


Figure – Solution of reduced-order model

Method 1: Proper Orthogonal Decomposition

Finite training set and snapshots

$$\mathcal{D}_{\mathtt{train}} := \{\mu_p\}_{p\in\{1:P\}} \subset \mathcal{D}, \qquad \{u(\mu_p)\}_{p\in\{1:P\}} \subset \mathcal{V}$$

The POD method selects $V_N \subset \mathcal{V}$ one N-dimensional subspace which minimizes the error

$$\sum_{\mu\in\mathcal{D}_{\mathtt{train}}} \|u(\mu) - \Pi_{V_N} u(\mu)\|_{\mathcal{V}}^2$$

among all N-dimensional subspaces.

In practice, the POD method amounts to computing the Singular Value Decomposition of a large matrix.

Method 2: (Ideal) Greedy algorithms

Greedy algorithms used in reduced basis methods provides a practical way to find a quasi-optimal linear subspace V_N in many situations.

[DeVore et al., 2013]

- Initialization $N = 1$: Choose $\mu_1 \in \mathcal{D}_{\texttt{train}}$ such that

$$\mu_1 \in \operatorname*{argmax}_{\mu \in \mathcal{D}_{\texttt{train}}} \|u(\mu)\|_{\mathcal{V}}.$$

  Let $V_1 := \mathrm{Vect}\{u(\mu_1)\}$.

- Iteration $N \geq 2$: Let

$$\sigma_N = \max_{\mu \in \mathcal{D}_{\texttt{train}}} \|u(\mu) - \Pi_{V_{N-1}} u(\mu)\|_{\mathcal{V}}.$$

  If $\sigma_N < \epsilon$, stop. Otherwise, choose $\mu_N \in \mathcal{D}_{\texttt{train}}$ such that

$$\mu_N \in \operatorname*{argmax}_{\mu \in \mathcal{D}_{\texttt{train}}} \|u(\mu) - \Pi_{V_{N-1}} u(\mu)\|_{\mathcal{V}}.$$

  Let $V_N := \mathrm{Vect}\{u(\mu_1), \cdots, u(\mu_N)\}$.

In practice, **a posteriori error estimators** are used instead of the exact norms in the above greedy algorithm.

Quasi-optimality of greedy algorithms

[DeVore et al., 2013]

Theorem (DeVore, Petrova, Wojtaszczyk,2012)

- *For all* $N \in \mathbb{N}^*$,
$$\sigma_{2N} \leq \sqrt{2}\sqrt{d_N^{\mathcal{V}}(\mathcal{M})}.$$

- *If there exists* $C_0 > 0$ *et* $\alpha > 0$ *such that for all* $n \in \mathbb{N}^*$, $d_N^{\mathcal{V}}(\mathcal{M}) \leq C_0 n^{-\alpha}$, *then there exists* $C > 0$ *such that*
$$\forall N \in \mathbb{N}^*, \quad \sigma_N \leq C N^{-\alpha}.$$

- *If there exists* $C_0 > 0$, $c_0 > 0$ *and* $\alpha > 0$ *such that for all* $N \in \mathbb{N}^*$, $d_N^{\mathcal{V}}(\mathcal{M}) \leq C_0 e^{-c_0 N^\alpha}$, *the there exists* $C > 0$ *and* $c > 0$ *such that*
$$\forall N \in \mathbb{N}^*, \quad \sigma_N \leq C e^{-cN^\alpha}.$$

Model reduction: elliptic problem (1/2)

Generic HF problem

- Minimization problem: $u(\mu) \in \mathcal{V}$, $u(\mu) = \underset{v \in \mathcal{V}}{\text{argmin}}\ \frac{1}{2} a(\mu; v, v) - f(\mu; v)$
- Algebraic representation: $U(\mu) \in \mathbb{R}^{\mathcal{N}}$, $A(\mu)U(\mu) = F(\mu)$

General organization

1. `Offline` phase
2. `Online` phase

`Offline` phase

1. **Training set and snapshots**

$$\mathcal{D}_{\mathtt{train}} := \{\mu_p\}_{p \in \{1:P\}} \subset \mathcal{D}, \qquad \{u(\mu_p)\}_{p \in \{1:P\}} \subset \mathcal{V}$$

2. **Greedy parameter selection drawn by a posteriori error estimator**
3. **Generic sampling and compression by** `POD`

$$V_N := \boldsymbol{Span}\Big(\{\xi_n\}_{n \in \{1:N\}}\Big) = \mathtt{POD}\Big(\{u(\mu_p)\}_{p \in \{1:P\}}; \mathcal{V}, \delta_{\mathtt{POD}} > 0\Big)$$

Model reduction: elliptic problem (2/2)

Online phase

- Reduced problem: $u_N(\mu) \in V_N,\ a(\mu; u_N(\mu), w) = f(\mu; w),\ \forall w \in V_N$
  - Algebraic representation: $U_N(\mu) \in \mathbb{R}^N,\ A_N(\mu)U_N(\mu) = F_N(\mu)$
    - Reduced arrays: $A_N(\mu) := \Xi^\top A(\mu)\Xi, \quad F_N(\mu) := \Xi^\top F(\mu)$, with $\Xi := [\Xi_1 \cdots \Xi_N] \in \mathbb{R}^{\mathcal{N}\times N}$
  - Large-dimensional parameter-dependent matrix $A(\mu) \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}} \longrightarrow$ inefficient Online phase

Affine parametric decomposition

- Empirical Interpolation Method (EIM) [Barrault, Maday, Nguyen & Patera 2004]

$$A(\mu) \approx E^a(\mu) := \sum_{j\in\{1:J^a\}} \alpha_j^a(\mu) A_j, \quad A_j \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}},\ \alpha_j^a(\mu) \in \mathbb{R},\ J^a \in \mathbb{N}^*$$

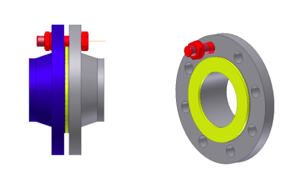
- Efficient Online phase:

$$A_N(\mu) \approx E_N^a(\mu) := \sum_{j\in\{1:J^a\}} \alpha_{\mathrm{on,j}}^a(\mu) A_{N,j}, \quad A_{N,j} := \Xi^\top A_j \Xi \in \mathbb{R}^{N\times N}$$

Outline

Industrial context

Nonlinear computational mechanics



- Structural mechanics problems

  [Johnson 1987], [Kikuchi & Oden 1988], [Wriggers 2006]

- Parameter-dependent high-fidelity (HF) model
  - Example: load, geometry, material properties, ...
- Computationally expensive

Multi-query context

- Parametric study: model calibration, uncertainty quantification, ...
- High Performance Computing (HPC) is often not sufficient
- Reduced Order Model $\longrightarrow$ Reduced Basis Method (RBM)

  [Prud'homme, Rovas, Veroy, Machiels, Maday, Patera & Turinici 2001]
  [Hesthaven, Rozza & Stamm 2016], [Quarteroni, Manzoni & Negri 2016]

Variational inequalities

Optimization problem

- **Energy minimization:** [Glowinski 1984], [Capatina 2014]

$$\exists! u \in \mathcal{K}, \quad u = \underset{v\in\mathcal{K}}{\text{argmin}} \ \frac{1}{2}a(v,v) + \mathcal{F}(v) - f(v)$$

(quadratic term $a(\cdot,\cdot)$, nonlinear energy $\mathcal{F}(\cdot)$, load $f(\cdot)$)

Optimality conditions

- **First kind** ($\mathcal{F} \equiv 0$): Find $u \in \mathcal{K}$ such that

$$a(u, v-u) \geq f(v-u), \quad \forall v \in \mathcal{K}(\mu)$$

- **Second kind**: Find $u(\mu) \in \mathcal{K}(\mu)$ such that

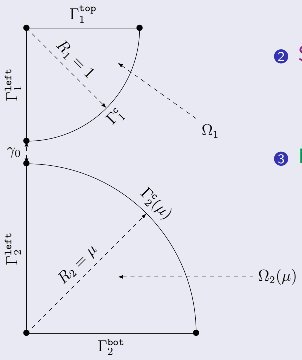
$$a(u, v-u) + \mathcal{F}(v) - \mathcal{F}(u) \geq f(v-u), \quad \forall v \in \mathcal{K}$$

Resolution methods

1. Mixed (primal/dual) $\longrightarrow$ Lagrangian methods
   [Fortin & Glowinski 1983], [Kikuchi & Oden 1988]
2. Primal $\longrightarrow$ Nitsche method (consitent boundary penalty method)
   [Nitsche 1971], [Chouly & Hild 2013], [Chouly, Hild & Renard 2015]

A generic model problem

Hertz contact problem with friction



1. Linear Elasticity: small deformation assumption

$$-\mathrm{div}(\sigma(\boldsymbol{u})) = \boldsymbol{\ell}, \quad \text{in } \Omega, \quad \sigma(\boldsymbol{v}) := \mathbb{C} : \varepsilon(\boldsymbol{v})$$

$$a(\boldsymbol{u}, \boldsymbol{v}) := \int_\Omega \sigma(\boldsymbol{u}) : \varepsilon(\boldsymbol{v})\, d\Omega,\ f(\boldsymbol{v}) := \int_\Omega \boldsymbol{\ell} \cdot \boldsymbol{v}\, d\Omega$$

2. Signorini contact conditions:

$$u_{\boldsymbol{n}} \leq d^{\boldsymbol{n}},\ \sigma_{\boldsymbol{nn}}(\boldsymbol{u}) \leq 0,\ \sigma_{\boldsymbol{nn}}(\boldsymbol{u})(u_{\boldsymbol{n}} - d^{\boldsymbol{n}}) = 0,\ \text{on } \Gamma^{\mathrm{c}}$$

$$\mathcal{K}^{\boldsymbol{n}} := \{v \in \mathcal{V} \ \mid u_{\boldsymbol{n}} \leq d^{\boldsymbol{n}}, \quad \text{on } \Gamma^{\mathrm{c}}\}$$

3. Friction conditions:
   - Tresca:

$$\begin{cases} \|\boldsymbol{\sigma}_{\boldsymbol{n\tau}}(\boldsymbol{u})\| \leq s, & \text{if } \boldsymbol{u}_{\boldsymbol{\tau}} = \mathbf{0} \\ \boldsymbol{\sigma}_{\boldsymbol{n\tau}}(\boldsymbol{u}) = -s\dfrac{\boldsymbol{u}_{\boldsymbol{\tau}}}{\|\boldsymbol{u}_{\boldsymbol{\tau}}\|}, & \text{otherwise} \end{cases}$$

$$\mathcal{F}(\boldsymbol{v}) := \int_{\Gamma^{\mathrm{c}}} s\|\boldsymbol{v}_{\boldsymbol{\tau}}\|\, d\Gamma$$

   - Coulomb: (depends from above setting)

$$s \longrightarrow \nu_{\mathcal{F}} |\sigma_{\boldsymbol{nn}}(\boldsymbol{u})|$$

$$\mathcal{F}(\boldsymbol{w}, \boldsymbol{v}) := \int_{\Gamma^{\mathrm{c}}} \nu_{\mathcal{F}} |\sigma_{\boldsymbol{nn}}(\boldsymbol{w})| \|\boldsymbol{v}_{\boldsymbol{\tau}}\|\, d\Gamma$$

Outline

State of the art

[Gerner & Veroy 2012], [Haasdonk, Salomon & Wohlmuth 2012], [Balajewicz, Amsallem & Farhat 2016], [Fauque, Ramière & Ryckelynck 2018], [Benaceur, Ern & Ehrlacher 2020]

Contributions

1. **Stable model reduction for mixed formulation**
   - Theoretical result on inf-sup stability
   - Projected Greedy Algorithm (PGA)
2. **Contact problem in mixed formulation**
   - modified Cone Projected Greedy (mCPG) algorithm
3. **Friction problem in mixed formulation**
   - Greedy Collocation Nodes Selection (GCNS) algorithm
4. **Nitsche's method for contact and friction problems**
   - Challenging nonlinearities in Nitsche's tangent matrices

1 + 2 $\longrightarrow$ M2AN

Parameter-dependent HF model

- Well-posed parametric constrained minimization problem:

$$u(\mu) = \operatorname*{argmin}_{v \in \mathcal{K}(\mu)} \frac{1}{2} a(\mu; v, v) - f(\mu; v), \quad \forall \mu \in \mathcal{D} \subset \mathbb{R}^m$$

- Admissible set

$$\mathcal{K}(\mu) := \left\{ v \in \mathcal{V} \;\middle|\; b(\mu; v, \eta) \leq g(\mu; \eta), \quad \forall \eta \in \mathcal{W}^+ \right\} \neq \emptyset$$

Mathematical setting

- $\mathcal{V}, \mathcal{W} \longrightarrow$ finite-dimensional HF spaces
- $\mathcal{W}^+ \subset \mathcal{W} \longrightarrow$ positive cone
- $a(\mu; \cdot, \cdot), b(\mu; \cdot, \cdot), f(\mu; \cdot), g(\mu; \cdot)$: bounded (bi)linear forms
- $a(\mu; \cdot, \cdot)$: symmetric and uniformly coercive

Mixed formulation

- Lagrangian:

$$\mathcal{L}(\mu; v, \eta) := \frac{1}{2} a(\mu; v, v) - f(\mu; v) + b(\mu; v, \eta) - g(\mu; \eta)$$

- Saddle-point problem:

$$(u(\mu), \lambda(\mu)) = \quad \text{arg} \min_{v \in \mathcal{V}} \max_{\eta \in \mathcal{W}^+} \quad \mathcal{L}(\mu; v, \eta)$$

- Key property: uniform inf-sup condition

$$\exists \beta_0 > 0, \quad \beta_{\mathrm{HF}}(\mu) := \inf_{\eta \in \mathcal{W}^+} \sup_{v \in \mathcal{V}} \frac{b(\mu; v, \eta)}{\|u\|_{\mathcal{V}} \|\eta\|_{\mathcal{W}}} \geq \beta_0$$

- Critical point of Lagrangian $(u(\mu), \lambda(\mu)) \in \mathcal{V} \times \mathcal{W}^+$ such that

$$\begin{cases} a(\mu; u(\mu), v) + b(\mu; v, \lambda(\mu)) = f(\mu; v), & \forall v \in \mathcal{V} \\ b(\mu; u(\mu), \eta - \lambda(\mu)) \leq g(\mu; \eta - \lambda(\mu)), & \forall \eta \in \mathcal{W}^+ \end{cases}$$

Decorrelated model reduction

1. Training set and snapshots

$$\mathcal{D}_{\texttt{train}} := \{\mu_p\}_{p\in\{1:P\}} \subset \mathcal{D}, \quad \{(u(\mu_p), \lambda(\mu_p))\}_{p\in\{1:P\}} \subset \mathcal{V} \times \mathcal{W}^+$$

2. Compression of reduced spaces
   - Proper Orthogonal Decomposition (POD)

$$V_N := \boldsymbol{Span}\Big(\{\xi_n\}_{n\in\{1:N\}}\Big) = \texttt{POD}\Big(\{u(\mu_p)\}_{p\in\{1:P\}}; \mathcal{V}, \delta_{\texttt{POD}} > 0\Big)$$

   - Angle Greedy or Cone Projected Greedy (CPG) algorithms

$$W_R^+ := \boldsymbol{Span}^+\Big(\{\upsilon_r\}_{r\in\{1:R\}}\Big) = \texttt{CPG}\Big(\{\lambda(\mu_p)\}_{p\in\{1:P\}}; \mathcal{W}, \delta_{\texttt{mCPG}} > 0\Big)$$

   [Burkovska, Haasdonk, Salomon & Wohlmuth 2015], [Benaceur, Ern & Ehrlacher 2020]

3. Reduced problem

$$(u_N(\mu), \lambda_R(\mu)) = \underset{v\in V_N}{\text{arg min}} \max_{\eta\in W_R^+} \mathcal{L}(\mu; v, \eta)$$

Main issue

We cannot guarantee that

$$\beta^{\text{dec}}(\mu) := \inf_{\eta\in W_R^+} \sup_{v\in V_N} \frac{b(\mu; v, \eta)}{\|v\|_{\mathcal{V}}\|\eta\|_{\mathcal{W}}} > 0$$

Enhancement by supremizers

State of the art

- Key assumption: $b(\cdot,\cdot)$ parameter-independent, represented by operator $\mathcal{B} : \mathcal{W}^+ \to \mathcal{V}$
  [Rozza & Veroy 2007], [Gerner & Veroy 2012], [Haasdonk, Salomon & Wohlmuth 2012]
  - Completion of $\{v_n\}_{n\in\{1:N\}}$ with `supremizers`
  - Enriched reduced primal space

$$V_{N,R} := V_N + S_R, \quad S_R := \boldsymbol{Span}\Big(\{\mathcal{B}v_r\}_{r\in\{:R\}}\Big)$$

  - Inf-sup stability

$$\inf_{\eta\in W_R^+} \sup_{v\in V_{N,R}} \frac{b(v,\eta)}{\|v\|_{\mathcal{V}}\|\eta\|_{\mathcal{W}}} \geq \beta_{\mathrm{HF}} \geq \beta_0 > 0$$

Parameter-dependent context

- Enriched reduced primal space

$$V_{N,R}^{\mathrm{on}}(\mu) := V_N + S_R(\mu), \quad S_R(\mu) := \boldsymbol{Span}\Big(\{\mathcal{B}(\mu)v_r\}_{r\in\{:R\}}\Big)$$

- Inf-sup stability

$$\beta^{\mathrm{on}}(\mu) := \inf_{\eta\in W_R^+} \sup_{v\in V_{N,R}^{\mathrm{on}}(\mu)} \frac{b(\mu;v,\eta)}{\|v\|_{\mathcal{V}}\|\eta\|_{\mathcal{W}}} \geq \beta_{\mathrm{HF}}(\mu) \geq \beta_0 > 0$$

- Construction must be performed online $\longrightarrow$ computationally inefficient

Our contribution

Goal

- Compute a primal space offline ensuring inf-sup stability
- Approximate $S_R := \underset{\mu\in\mathcal{D}}{+} S_R(\mu)$ by parameter-independent low-dimensional subspace

Approximation of S_R

- Let S be any finite-dimensional subspace of S_R. To measure how well S represents S_R, let

$$\sigma_S(\mu) := \|(\mathbb{I}^{\mathcal{V}} - \Pi^{\mathcal{V}}_{V_N+S})|_{S_R(\mu)}\|_{\mathcal{L}(\mathcal{V})}$$

- Basic properties: For all $\mu \in \mathcal{D}$,
  1. $(\sigma_S(\mu))_S \searrow$ if $S \nearrow$
  2. $\sigma_{S_R}(\mu) = 0$

Main result

Let $c_{\mathrm{HF}}(\mu)$ be the boundedness constant of $b(\mu;\cdot,\cdot)$.

If $\sigma_S(\mu) < \frac{\beta^{\mathrm{on}}(\mu)}{c_{\mathrm{HF}}(\mu)}$, then $\inf\limits_{\eta\in W_R^+} \sup\limits_{v\in V_N+S} \frac{b(\mu;v,\eta)}{\|v\|_{\mathcal{V}}\|\eta\|_{\mathcal{W}}} \geq \beta_S^*(\mu) := \frac{\beta^{\mathrm{on}}(\mu)-c_{\mathrm{HF}}(\mu)\sigma_S(\mu)}{1+\sigma_S(\mu)} > 0$.

Projected Greedy Algorithm (PGA)

- Build $S_R^{\mathrm{red}} := \mathtt{PGA}\left(\mathcal{D}_{\mathtt{train}}, V_N, \delta_{\mathtt{PGA}}\right) \subset S_R$ such that $\max\limits_{\mu \in \mathcal{D}_{\mathtt{train}}} \sigma_{S_R^{\mathrm{red}}}(\mu) \leq \delta_{\mathtt{PGA}}$
- Primal reduced space $V_{N,R}^{\mathrm{off}} := V_N + S_R^{\mathrm{red}}$
- Inf-sup stability:

$$\beta^{\mathrm{off}}(\mu) := \inf_{\eta \in W_R^+} \sup_{v \in V_{N,R}^{\mathrm{off}}} \frac{b(\mu; v, \eta)}{\|v\|_{\mathcal{V}} \|\eta\|_{\mathcal{W}}} > 0, \forall \mu \in \mathcal{D}_{\mathtt{train}}$$

Notice that nothing is asserted out of $\mathcal{D}_{\mathtt{train}}$.

Some details

while $e_n > \delta$ **do**

$S^n := S^{n-1} + \boldsymbol{Span}\{v_{n-1}\}$

$\mu_n \in \underset{\mu \in \mathcal{D}_{\mathtt{train}}}{\mathtt{argmax}} \; \sigma_{S^n}(\mu)$

$v_n^{(1)} := \underset{\substack{v \in S_R(\mu_n) \\ \|v\|_{\mathcal{V}} \leq 1}}{\mathtt{argmax}} \left\| \left(\mathbb{I}^{\mathcal{V}} - \Pi^{\mathcal{V}}_{V_N + S^n} \right)(v) \right\|_{\mathcal{V}}$

$v_n := (\mathbb{I}^{\mathcal{V}} - \Pi^{\mathcal{V}}_{V_N + S^n})(v_n^{(1)})$

$e_{n+1} := \sigma_{S^n}(\mu_n)$

$n = n + 1$

end while

Main points

- Monotonicity:

$$\left(\max_{\mu \in \mathcal{D}_{\mathtt{train}}} \sigma_{S^n}(\mu) \right)_{n \geq 0} \searrow$$

- Recall:

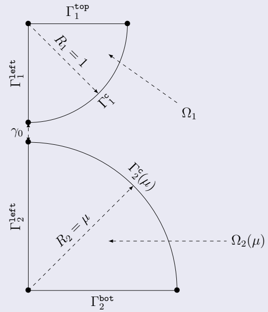
$$\sigma_{S_R}(\mu) = 0, \forall \mu \in \mathcal{D}$$

- Finite termination: $\exists n_0 \leq P \times R$,

$$\max_{\mu \in \mathcal{D}_{\mathtt{train}}} \sigma_{S^n}(\mu) = 0, \; \forall n \geq n_0$$

Hertz contact between two half-disks

- Parametric domain:
  1. $\mathcal{D} := [0.7,\ 1.3]$(m)
  2. $\mathcal{D}_{\mathrm{train}}$ uniform discretization
- Geometric load:
  1. $\gamma_0 = 0.001$m
  2. $d = 0.09$m $(\leq 10\%\ \max(R_1, R_2))$
- Potential contact manifold: $\theta \in [-\frac{5\pi}{8}, -\frac{3\pi}{8}]$
- Finite Element discretization:
  1. $\Omega(\mu)$ are μ-dependent
  2. Geometric mapping: $h(\mu) : \widehat{\Omega} \to \Omega(\mu)$
  3. displacement: $\mathbb{P}_1$
  4. Lagrange multiplier: $\mathbb{P}_0$ using LAC method [Abbas, Drouet & Hild 2018]



Contact problem

Setting

- $\Omega(\mu) \subset \mathbb{R}^d$; $\Gamma(\mu) = \Gamma^{\mathsf{D}}(\mu) \cup \Gamma^{\mathsf{N}}(\mu) \cup \Gamma^{\mathsf{c}}(\mu)$
- $\boldsymbol{u}(\mu) : \Omega(\mu) \to \mathbb{R}^d$; $\boldsymbol{\ell}(\mu) : \Omega(\mu) \to \mathbb{R}^d$
- $\boldsymbol{n}(\mu) \in \mathbb{R}^d$; $\boldsymbol{\tau}(\mu) \in \mathbb{R}^{d\times(d-1)}$
- $\sigma(\boldsymbol{v}) := \mathbb{C} : \varepsilon(\boldsymbol{v})$;
  $\varepsilon(\boldsymbol{v}) := \frac{1}{2}\left(\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^\top\right)$
- $\boldsymbol{v} = v_{\boldsymbol{n}}\boldsymbol{n} + \boldsymbol{\tau}\boldsymbol{v}_{\boldsymbol{\tau}}$
- $\sigma(\boldsymbol{v})\boldsymbol{n} = \sigma_{\boldsymbol{nn}}(\boldsymbol{v})\boldsymbol{n} + \boldsymbol{\tau}\boldsymbol{\sigma}_{\boldsymbol{n\tau}}(\boldsymbol{v})$

Linear Elasticity

$$
\begin{aligned}
-\mathrm{div}(\sigma(\boldsymbol{u}(\mu))) = \boldsymbol{\ell}(\mu), &\quad \text{in } \Omega(\mu)\\
\boldsymbol{u}(\mu) = \boldsymbol{0}, &\quad \text{on } \Gamma^{\mathsf{D}}(\mu)\\
\sigma(\boldsymbol{u}(\mu))\boldsymbol{n} = \boldsymbol{0}, &\quad \text{on } \Gamma^{\mathsf{N}}(\mu)
\end{aligned}
$$

Signorini contact conditions on $\Gamma^{\mathsf{c}}(\mu)$

$$
\begin{aligned}
u_{\boldsymbol{n}}(\mu) &\le d^{\boldsymbol{n}}(\mu)\\
\sigma_{\boldsymbol{nn}}(\boldsymbol{u}(\mu)) &\le 0\\
\sigma_{\boldsymbol{nn}}(\boldsymbol{u}(\mu))\left(u_{\boldsymbol{n}}(\mu) - d^{\boldsymbol{n}}(\mu)\right) &= 0
\end{aligned}
$$

HF model

HF spaces

- $\boldsymbol{\mathcal{V}}(\mu) \subset \Big\{ \boldsymbol{v} \in H^1(\Omega(\mu); \mathbb{R}^d) \mid \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma^{\mathsf{D}}(\mu) \Big\}$; $\mathcal{W}(\mu) \subset L^2(\Gamma^{\mathsf{c}}(\mu); \mathbb{R})$
- Positive cone $\mathcal{W}^{\boldsymbol{n}}(\mu) := \Big\{ \eta^{\boldsymbol{n}} \in \mathcal{W}(\mu) \mid \eta^{\boldsymbol{n}} \geq 0 \Big\} \subset L^2(\Gamma^{\mathsf{c}}(\mu); \mathbb{R}_+)$

Variational formulation

- Variational inequality: For all $\mu \in \mathcal{D}$, find $\boldsymbol{u}(\mu) \in \boldsymbol{\mathcal{K}}^{\boldsymbol{n}}(\mu)$ such that

$$a(\mu; \boldsymbol{u}(\mu), \boldsymbol{v} - \boldsymbol{u}(\mu)) \geq f(\mu; \boldsymbol{v} - \boldsymbol{u}(\mu)), \quad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{K}}(\mu)$$

- Admissible set:

$$\boldsymbol{\mathcal{K}}^{\boldsymbol{n}}(\mu) := \Big\{ \boldsymbol{v} \in \boldsymbol{\mathcal{V}}(\mu) \;\Big|\; b^{\boldsymbol{n}}(\mu; \boldsymbol{v}, \eta^{\boldsymbol{n}}) \leq g^{\boldsymbol{n}}(\mu; \eta^{\boldsymbol{n}}), \quad \forall \eta^{\boldsymbol{n}} \in \mathcal{W}^{\boldsymbol{n}}(\mu) \Big\}$$

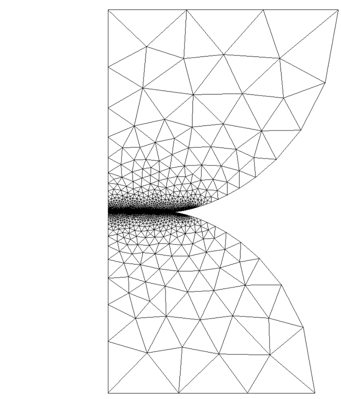
Bilinear Forms

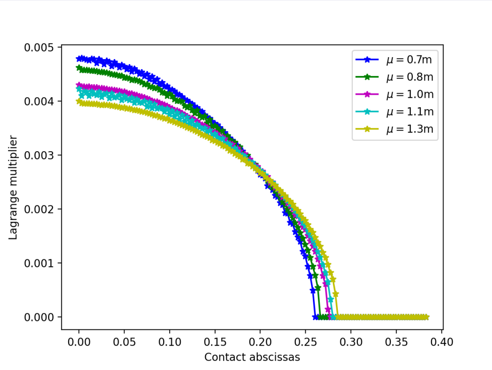
- $a(\mu; \boldsymbol{u}, \boldsymbol{v}) := \int_{\Omega(\mu)} \sigma(\boldsymbol{u}) : \varepsilon(\boldsymbol{v}) \, d\Omega(\mu)$
- $b^{\boldsymbol{n}}(\mu; \boldsymbol{v}, \eta^{\boldsymbol{n}}) := \int_{\Gamma^{\mathsf{c}}} v_{\boldsymbol{n}} \eta^{\boldsymbol{n}} \, d\Gamma$

Linear Forms

- $f(\mu; \boldsymbol{v}) := \int_{\Omega(\mu)} \boldsymbol{\ell}(\mu) \cdot \boldsymbol{v} \, d\Omega(\mu)$
- $g^{\boldsymbol{n}}(\mu; \eta^{\boldsymbol{n}}) := \int_{\Gamma^{\mathsf{c}}} d^{\boldsymbol{n}}(\mu) \eta^{\boldsymbol{n}} \, d\Gamma$

HF solutions





$u(\mu)$, $\mu = 0.9$m

$\lambda^n(\mu)$, $\mu \in \left\{0.7, 0.8, 1.0, 1.1, 1.3\right\}$(m).

Computed with `FreeFem++` and `Python`.

Projected Greedy Algorithm (1/2)

- $\beta_{S^n}(\mu) := \inf_{\eta^{\boldsymbol{n}} \in W_R^+} \sup_{v \in V_N + S^n} \frac{b(\mu; v, \eta^{\boldsymbol{n}})}{\|v\|_{\mathcal{V}} \|\eta^{\boldsymbol{n}}\|_{\mathcal{W}}}$
- $\sigma_{S^n}(\mu) := \|(\mathbb{I}^{\mathcal{V}} - \Pi^{\mathcal{V}}_{V_N + S^n})|_{S_R(\mu)}\|_{\mathcal{L}(\mathcal{V})}$
- $\beta^n_{\min} := \min_{\mu \in \mathcal{D}_{\texttt{train}}} \beta_{S^n}(\mu)$
- $\sigma^n_{\max} := \max_{\mu \in \mathcal{D}_{\texttt{train}}} \sigma_{S^n}(\mu)$

- $\sigma^n_{\max}$

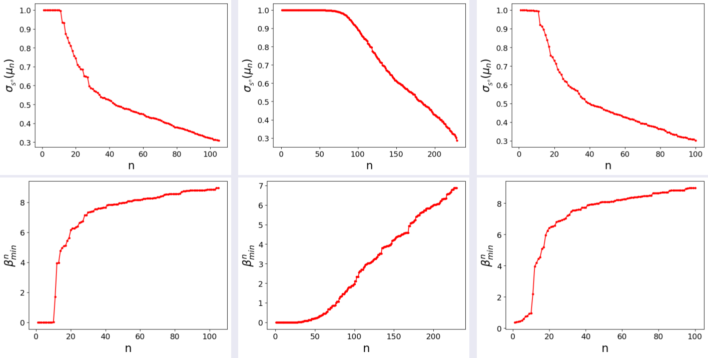
$\searrow$

Convergence

- $\beta^n_{\min}$

$\nearrow$

Effective



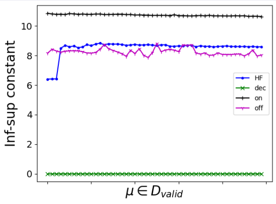
$(N, R) = (3, 12)$ $(N, R) = (46, 70)$ $(N, R) = (46, 12)$

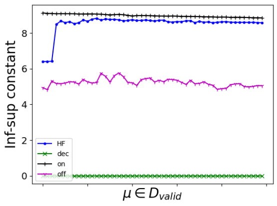
Projected Greedy Algorithm (2/2)

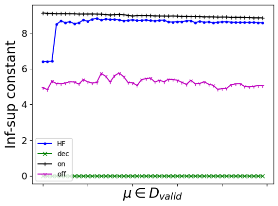
Inf-sup constants

- $\beta_{\mathrm{HF}}(\mu) := \inf\limits_{\eta^{\boldsymbol{n}} \in \mathcal{W}^+} \sup\limits_{v \in \mathcal{V}} \dfrac{b(\mu; v, \eta^{\boldsymbol{n}})}{\|u\|_{\mathcal{V}} \|\eta^{\boldsymbol{n}}\|_{\mathcal{W}}}$
- $\beta^{\mathrm{dec}}(\mu) := \inf\limits_{\eta^{\boldsymbol{n}} \in W_R^+} \sup\limits_{v \in V_N} \dfrac{b(\mu; v, \eta^{\boldsymbol{n}})}{\|v\|_{\mathcal{V}} \|\eta^{\boldsymbol{n}}\|_{\mathcal{W}}}$
- $\beta^{\mathrm{on}}(\mu) := \inf\limits_{\eta^{\boldsymbol{n}} \in W_R^+} \sup\limits_{v \in V_{N,R}^{\mathrm{on}}(\mu)} \dfrac{b(\mu; v, \eta^{\boldsymbol{n}})}{\|v\|_{\mathcal{V}} \|\eta^{\boldsymbol{n}}\|_{\mathcal{W}}}$
- $\beta^{\mathrm{off}}(\mu) := \inf\limits_{\eta^{\boldsymbol{n}} \in W_R^+} \sup\limits_{v \in V_{N,R}^{\mathrm{off}}} \dfrac{b(\mu; v, \eta^{\boldsymbol{n}})}{\|v\|_{\mathcal{V}} \|\eta^{\boldsymbol{n}}\|_{\mathcal{W}}}$

Validation: (uniform distributed) random points in $\mathcal{D}$







Main conclusions

- We guarantee

$$\beta^{\mathrm{off}}(\mu) > 0 \ \forall \mu \in \mathcal{D}_{\mathtt{train}}$$

- We obtain

$$\beta^{\mathrm{off}}(\mu) > 0 \ \forall \mu \in \mathcal{D}_{\mathtt{valid}}$$

modified Cone Projected Greedy (mCPG) algorithm (1/4)

In practice, the CPG algorithm is applied to the family $\{\theta_q\}_{q\in\{1:Q\}} = \{\lambda(\mu_p)\}_{p\in\{1:P\}}$.

CPG algorithm

[Benaceur, Ern & Ehrlacher 2020]

1. **Input:** HF positive cone $\mathcal{W}^+$; Family $\{\theta_q\}_{q\in\{1:Q\}} \subset \mathcal{W}^+$; Tolerance $\delta_{\mathtt{CPG}} > 0$
2. **Iteration** $r \geq 1$**:** Select $q_r \in \{1:Q\}$ such that

$$q_r \in \operatorname*{argmax}_{q\in\{1:Q\}} \left\| \left(\mathbb{I}^{\mathcal{W}} - \Pi^{\mathcal{W}}_{W^+_{r-1}} \right) (\theta_q) \right\|_{\mathcal{W}}$$

   and define $W_r^+ = \boldsymbol{Span}^+ \{\theta_{q_1}, \cdots, \theta_{q_r}\}$.

3. **Output:** Subset $\{\theta_{q_r}\}_{r\in\{1:R\}} \subset \{\theta_q\}_{q\in\{1:Q\}}$ s.t. $W_R^+ := \boldsymbol{Span}^+ \left(\{\theta_{q_r}\}_{r\in\{1:R\}} \right)$ satisfies

$$W_R^+ \subset W^+ := \boldsymbol{Span}^+ \left(\{\theta_q\}_{q\in\{1:Q\}} \right) \text{ and } e_{\mathtt{CPG}}(R) := \frac{\max_{q\in\{1:Q\}} \left\| \left(\mathbb{I}^{\mathcal{W}} - \Pi^{\mathcal{W}}_{W^+_R} \right) (\theta_q) \right\|_{\mathcal{W}}}{\max_{q\in\{1:Q\}} \|\theta_q\|_{\mathcal{W}}} \leq \delta_{\mathtt{CPG}}$$

mCPG algorithm (2/4)

mCPG algorithm

1. **Input:** HF positive cone $\mathcal{W}^+$; Family $\{\theta_q\}_{q\in\{1:Q\}} \subset \mathcal{W}^+$; Tolerance $\delta_{\mathtt{mCPG}} > 0$
2. **Iteration** $r \geq 1$**:** Select $q_r \in \{1:Q\}$ such that

$$q_r \in \underset{q\in\{1:Q\}}{\operatorname{argmax}} \left\| \left(\mathbb{I}^{\mathcal{W}} - \Pi^{\mathcal{W}}_{W^+_{r-1}} \right)(\theta_q) \right\|_{\mathcal{W}}.$$

Define $\nu_r = \theta_{q_r} - \widetilde{\theta}_{r-1}$ where

$$\widetilde{\theta}_{r-1} \in \underset{\widetilde{\theta}\in W^+_{r-1},\, \theta_{q_r} - \widetilde{\theta}\in\mathcal{W}^+}{\operatorname{argmin}} \left\| \theta_{q_r} - \widetilde{\theta} \right\|_{\mathcal{W}}$$

and define $W^+_r = \boldsymbol{Span}^+\{\nu_1, \cdots, \nu_r\}$.

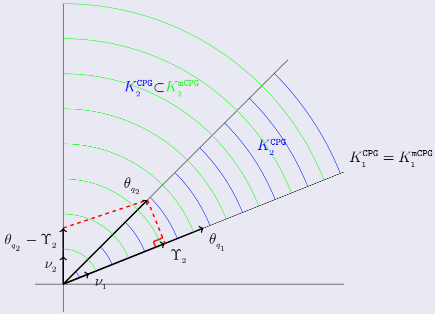
3. **Output:** Subset $\{\nu_r\}_{r\in\{1:R\}}$ s.t. $W^+_R := \boldsymbol{Span}^+\left(\{\nu_r\}_{r\in\{1:R\}}\right)$ satisfies

$$W^+_R \subset W^+ := \boldsymbol{Span}^+\left(\{\theta_q\}_{q\in\{1:Q\}}\right) \text{ and } e_{\mathtt{mCPG}}(R) := \frac{\max_{q\in\{1:Q\}} \left\| \left(\mathbb{I}^{\mathcal{W}} - \Pi^{\mathcal{W}}_{W^+_R} \right)(\theta_q) \right\|_{\mathcal{W}}}{\max_{q\in\{1:Q\}} \|\theta_q\|_{\mathcal{W}}} \leq \delta_{\mathtt{mCPG}}$$

mCPG algorithm (3/4)

mCPG algorithm

- Main advantage: avoid ill-conditioning issues (Gram-Schmidt not available on cones)
- Progressive construction of a cone with wider aperture than CPG
- Enlarged Nonnegative Greedy (ENG) algorithm:
  [Bakhta, Boiveau, Maday & Mula 2020]

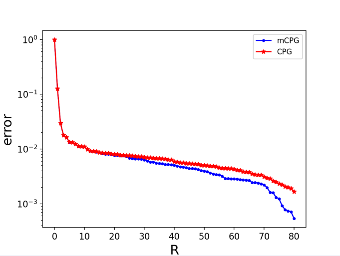


mCPG algorithm (4/4)

Comparison of CPG and mCPG

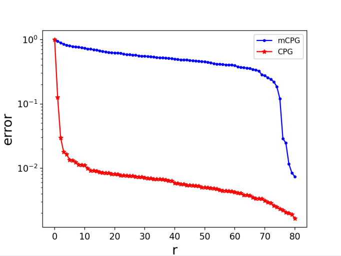
Projection error on cone

$$e_{\mathrm{proj}}(R) := \frac{\max\limits_{q\in\{1:Q\}} ||(\mathbb{I}^{\mathcal{W}} - \Pi^{\mathcal{W}}_{W_R^+})(\theta_q)||_{\mathcal{W}}}{\max\limits_{q\in\{1:Q\}} \|\theta_q\|_{\mathcal{W}}}$$



Measure of cone aperture

$$e_{\mathrm{orth}}(r) := \|(\mathbb{I}^{\mathcal{W}} - \Pi^{\mathcal{W}}_{W_R^+})(\nu_r)\|_{\mathcal{W}}$$



Main conclusions

- Accuracy $\nearrow$
- Cone aperture $\nearrow$

Extension 1: Coulomb contact problem

Strong formulation

$$\begin{cases} \|\boldsymbol{\sigma}_{\boldsymbol{n\tau}}(\boldsymbol{u}(\mu))\| \leq \nu_{\mathcal{F}}|\sigma_{\boldsymbol{nn}}(\boldsymbol{u}(\mu))|, & \text{if } \boldsymbol{u}_{\boldsymbol{\tau}}(\mu) = \mathbf{0} \\ \boldsymbol{\sigma}_{\boldsymbol{n\tau}}(\boldsymbol{u}(\mu)) = -\nu_{\mathcal{F}}|\sigma_{\boldsymbol{nn}}(\boldsymbol{u}(\mu))|\dfrac{\boldsymbol{u}_{\boldsymbol{\tau}}(\mu)}{\|\boldsymbol{u}_{\boldsymbol{\tau}}(\mu)\|}, & \text{otherwise} \end{cases}$$

Weak formulation

For all $\mu \in \mathcal{D}$, find $\boldsymbol{u}(\mu) \in \boldsymbol{\mathcal{K}}(\mu)$ such that

$$a(\mu; \boldsymbol{u}(\mu), \boldsymbol{v} - \boldsymbol{u}(\mu)) + \mathcal{F}(\mu; \boldsymbol{u}(\mu), \boldsymbol{v}) - \mathcal{F}(\mu; \boldsymbol{u}(\mu), \boldsymbol{u}(\mu)) \geq f(\mu; \boldsymbol{v} - \boldsymbol{u}(\mu)), \quad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{K}}(\mu)$$

Extension 2: Nitsche's method (frictionless contact problems)

State of art

[Nitsche 1971], [Chouly & Hild 2013], [Chouly 2014], [Chouly, Hild & Renard 2015]
[Mlika, Renard & Chouly 2017], [Chouly, Ern & Pignet 2020], [Chouly, Hild, Lleras & Renard 2022]

Alart–Curnier reformulation

$$\sigma_{\boldsymbol{nn}}(\boldsymbol{u}(\mu)) = \Big[\sigma_{\boldsymbol{nn}}(\boldsymbol{u}(\mu)) - \gamma\Big(u_{\boldsymbol{n}}(\mu) - d^{\boldsymbol{n}}(\mu)\Big)\Big]_-$$

Nitsche's energy

$$J^{\mathrm{Nitsche}}(\mu;\boldsymbol{v}) := \mathcal{J}(\mu;\boldsymbol{v}) - \frac{1}{2}\int_{\Gamma^{\mathrm{c}}(\mu)} \frac{1}{\gamma}|\sigma_{\boldsymbol{nn}}(\boldsymbol{v})|^2 d\Gamma(\mu) + \frac{1}{2}\int_{\Gamma^{\mathrm{c}}(\mu)} \frac{1}{\gamma}\Big[\sigma_{\boldsymbol{nn}}(\boldsymbol{v}) - \gamma\Big(v_{\boldsymbol{n}} - d^{\boldsymbol{n}}(\mu)\Big)\Big]_-^2 d\Gamma(\mu)$$

Variational formulation

Well-posed parametric unconstrained minimization problem:

$$\boldsymbol{u}(\mu) = \underset{\boldsymbol{v}\in\boldsymbol{\mathcal{V}}(\mu)}{\mathrm{argmin}}\ J^{\mathrm{Nitsche}}(\mu;\boldsymbol{v})$$

Outline

Conclusions

Reduced-order models can yield tremendous gains in terms of computational costs for parametric studies.
How to choose a good method? Depends on the nature of the problem at hand!

Reduced-order model for parametrized variational inequalities

- Inf-sup stability in the framework of a mixed formulation and parameter-dependent constraints
- An effective and stable RBM for the frictional contact problem in the mixed formulation
- An effective RBM for contact problems formulated with Nitsche's method
- Applications: Mechanical contact problem with friction, hydromechanical coupling [Plassart 2011]

Perspectives

Other works on applications of model order reduction methods

- Parametrized non-symmetric eigenvalue problems: collaboration with CEA for applications in neutronics
- Non-parametrized geometrical variability: collaboration with SAFRANTech for applications in aircraft engine design (based on neural networks)

Perspective of collaboration with IFPEN (Guillaume Enchéry)

- Model order reduction of flow simulations in cracked porous media : linear and nonlinear model order reduction methods
- Preliminary work on nonlinear methods (based on optimal transport techniques) [Battisti, Blickhan, Ehrlacher, Enchéry, Lombardi & Mula, 2022]

Coulomb contact problem

Strong formulation

$$\begin{cases} \|\boldsymbol{\sigma}_{\boldsymbol{n\tau}}(\boldsymbol{u}(\mu))\| \leq \nu_{\mathcal{F}}|\sigma_{\boldsymbol{nn}}(\boldsymbol{u}(\mu))|, & \text{if } \boldsymbol{u}_{\boldsymbol{\tau}}(\mu) = \mathbf{0} \\ \boldsymbol{\sigma}_{\boldsymbol{n\tau}}(\boldsymbol{u}(\mu)) = -\nu_{\mathcal{F}}|\sigma_{\boldsymbol{nn}}(\boldsymbol{u}(\mu))|\dfrac{\boldsymbol{u}_{\boldsymbol{\tau}}(\mu)}{\|\boldsymbol{u}_{\boldsymbol{\tau}}(\mu)\|}, & \text{otherwise} \end{cases}$$

Weak formulation

For all $\mu \in \mathcal{D}$, find $\boldsymbol{u}(\mu) \in \boldsymbol{\mathcal{K}}(\mu)$ such that

$$a(\mu; \boldsymbol{u}(\mu), \boldsymbol{v} - \boldsymbol{u}(\mu)) + \mathcal{F}(\mu; \boldsymbol{u}(\mu), \boldsymbol{v}) - \mathcal{F}(\mu; \boldsymbol{u}(\mu), \boldsymbol{u}(\mu)) \geq f(\mu; \boldsymbol{v} - \boldsymbol{u}(\mu)), \quad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{K}}(\mu)$$

HF model

Mixed formulation

- Space triplet $\boldsymbol{\mathcal{V}}(\mu) \times \mathcal{W}^{\boldsymbol{n}}(\mu) \times \boldsymbol{\mathcal{X}}^{\boldsymbol{\tau}}(\mu; \nu_{\mathcal{F}}\lambda^{\boldsymbol{n}}(\mu))$
- solution denoted $(\boldsymbol{u}(\mu), \lambda^{\boldsymbol{n}}(\mu), \boldsymbol{\lambda}^{\boldsymbol{\tau}}(\mu))$
- Uzawa algorithm

$$\begin{cases} a(\mu; \boldsymbol{u}_{k+1}(\mu); \boldsymbol{v}) + b^{\boldsymbol{n}}(\mu; \boldsymbol{v}, \lambda_k^{\boldsymbol{n}}(\mu)) + b^{\boldsymbol{\tau}}(\mu; \boldsymbol{v}, \boldsymbol{\lambda}_k^{\boldsymbol{\tau}}(\mu)) = f(\mu; \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{V}}(\mu) \\ \lambda_{k+1}^{\boldsymbol{n}}(\mu) = \Pi_{\mathcal{W}^{\boldsymbol{n}}(\mu)}^{\mathcal{W}(\mu)}(\lambda_k^{\boldsymbol{n}}(\mu) - \rho(u_{k+1,\boldsymbol{n}}(\mu) - d^{\boldsymbol{n}}(\mu))) \\ \boldsymbol{\lambda}_{k+1}^{\boldsymbol{\tau}}(\mu) = \Pi_{\boldsymbol{\mathcal{X}}^{\boldsymbol{\tau}}(\mu; \nu_{\mathcal{F}}\lambda_{k+1}^{\boldsymbol{n}}(\mu))}^{\boldsymbol{\mathcal{X}}(\mu)}(\boldsymbol{\lambda}_k^{\boldsymbol{\tau}}(\mu) - \rho \boldsymbol{u}_{k+1,\boldsymbol{\tau}}(\mu)) \end{cases}$$

Tangential constraints

- Tangential space:

$$\boldsymbol{\mathcal{X}}(\mu) \subset L^2(\Gamma^{\mathrm{c}}(\mu); \mathbb{R}^{d-1})$$

- Admissible tangential stresses:

$$\boldsymbol{\mathcal{X}}^{\boldsymbol{\tau}}(\mu; \eta^{\boldsymbol{n}}) := \left\{ \boldsymbol{\theta}^{\boldsymbol{\tau}} \in \boldsymbol{\mathcal{X}}(\mu) \mid \|\boldsymbol{\theta}^{\boldsymbol{\tau}}\| \leq \eta^{\boldsymbol{n}} \text{ a.e on } \Gamma^{\mathrm{c}}(\mu) \right\}$$

- Tangential contact operator:

$$b^{\boldsymbol{\tau}}(\mu; \boldsymbol{v}, \boldsymbol{\theta}^{\boldsymbol{\tau}}) := \langle \boldsymbol{v}_{\boldsymbol{\tau}}, \boldsymbol{\theta}^{\boldsymbol{\tau}} \rangle_{\Gamma^{\mathrm{c}}(\mu)}$$

- Friction functional:

$$\mathcal{F}(\mu; \boldsymbol{w}, \boldsymbol{v}) := \langle -\nu_{\mathcal{F}} |\sigma_{\boldsymbol{nn}}(\boldsymbol{w})|, \|\boldsymbol{v}_{\boldsymbol{\tau}}\| \rangle_{\Gamma^{\mathrm{c}}(\mu)}$$

Algebraic formulation

Collocation method

- Admissible tangential stresses:

$$\mathcal{X}^{\boldsymbol{\tau}}_{\mathcal{C}(\mu)}(\mu;\eta^{\boldsymbol{n}}) := \left\{\boldsymbol{\theta}^{\boldsymbol{\tau}} \in \mathcal{X}(\mu) \;\middle|\; \|\boldsymbol{\theta}^{\boldsymbol{\tau}}(\boldsymbol{c}_s(\mu))\| \le \eta^{\boldsymbol{n}}(\boldsymbol{c}_s(\mu)),\ \forall s \in \{1{:}\mathcal{S}_0\}\right\}.$$

Uzawa

Find $\left(\boldsymbol{U}_{k+1}(\mu), \Lambda^{\boldsymbol{n}}_{k+1}(\mu), \boldsymbol{\Lambda}^{\boldsymbol{\tau}}_{k+1}(\mu)\right) \in \mathbb{R}^{\mathcal{N}} \times \mathbb{R}^{\mathcal{R}}_{+} \times \boldsymbol{X}^{\boldsymbol{\tau}}_{\mathcal{C}(\mu)}(\mu;\nu_{\mathcal{F}}\Lambda^{\boldsymbol{n}}_{k+1}(\mu))$ such that

$$\begin{cases} A(\mu)\boldsymbol{U}_{k+1}(\mu) + B^{\boldsymbol{n}}(\mu)^\top \Lambda^{\boldsymbol{n}}_{k+1}(\mu) + B^{\boldsymbol{\tau}}(\mu)^\top \boldsymbol{\Lambda}^{\boldsymbol{\tau}}_{k+1}(\mu) = F(\mu) \\ \Lambda^{\boldsymbol{n}}_{k+1}(\mu) = \Pi_{\mathbb{R}^{\mathcal{R}}_{+}}(\Lambda^{\boldsymbol{n}}_{k+1}(\mu) - \rho(B^{\boldsymbol{n}}(\mu)\boldsymbol{U}_{k+1}(\mu) - G^{\boldsymbol{n}}(\mu))) \\ \boldsymbol{\Lambda}^{\boldsymbol{\tau}}_{k+1}(\mu) = \Pi_{\boldsymbol{X}^{\boldsymbol{\tau}}_{\mathcal{C}(\mu)}(\mu;\nu_{\mathcal{F}}\Lambda^{\boldsymbol{n}}_{k+1}(\mu))}(\boldsymbol{\Lambda}^{\boldsymbol{\tau}}_{k+1}(\mu) - \rho B^{\boldsymbol{\tau}}(\mu)\boldsymbol{U}_{k+1}(\mu)) \end{cases}$$

Discretization

- $\mathbb{P}_1$ shape functions
- $\mathcal{S} = (d-1) \times \mathcal{R}$
- $\mathcal{S}_0 = \mathcal{R}$
- $\boldsymbol{X}^{\boldsymbol{\tau}}_{\mathcal{C}(\mu)}(\mu;\nu_{\mathcal{F}}\Lambda^{\boldsymbol{n}}_{k+1}(\mu)) := \left\{\boldsymbol{\Theta}^{\boldsymbol{\tau}} \in \mathbb{R}^{(d-1)\times\mathcal{R}} \;\middle|\; \|\boldsymbol{\Theta}^{\boldsymbol{\tau},r}\| \le \nu_{\mathcal{F}}\Lambda^{\boldsymbol{n},r}_{k+1}(\mu),\ \forall r \in \{1{:}\mathcal{R}\}\right\}$

Plain RBM

Setting

- Geometric mapping: $h(\mu) : \widehat{\Omega} \to \Omega(\mu)$ such that $\widehat{\Gamma}^{c} := h_c^{-1}(\mu)(\Gamma^{c}(\mu))$
- Reference mesh: μ-independent $\mathcal{N}$, $\mathcal{R}$ and $\mathcal{S}$
- Collocations nodes: $\widehat{\boldsymbol{\mathcal{C}}} := \{\widehat{\boldsymbol{c}}_s\}_{s\in\{1:\mathcal{S}_0^{\mathrm{HF}}\}} \subset \widehat{\Gamma}^{c}$ such that $\boldsymbol{\mathcal{C}}(\mu) := h(\mu)(\widehat{\boldsymbol{\mathcal{C}}})$

Preliminaries

1. RB spaces
   - POD $\longrightarrow$ $\boldsymbol{V}_N := \boldsymbol{Span}\Big(\{\boldsymbol{\xi}_n\}_{n\in\{1:N\}}\Big)$, $\boldsymbol{X}_S^{\boldsymbol{\tau}} := \boldsymbol{Span}\Big(\{\boldsymbol{\upsilon}_s^{\boldsymbol{\tau}}\}_{s\in\{1:S\}}\Big)$
   - mCPG $\longrightarrow$ $W_R^{\boldsymbol{n}} := \boldsymbol{Span}^{+}\Big(\{\upsilon_r^{\boldsymbol{n}}\}_{r\in\{1:R\}}\Big)$
2. Stabilization using PGA algorithm

$$b(\mu;\boldsymbol{v},\boldsymbol{\eta}) := b^{\boldsymbol{n}}(\mu;\boldsymbol{v},\eta^{\boldsymbol{n}}) + b^{\boldsymbol{\tau}}(\mu;\boldsymbol{v},\boldsymbol{\eta}^{\boldsymbol{\tau}}),\quad \forall \boldsymbol{v}\in\boldsymbol{\mathcal{V}}(\mu),\ \forall\boldsymbol{\eta} := (\eta^{\boldsymbol{n}},\boldsymbol{\eta}^{\boldsymbol{\tau}})\in\boldsymbol{\mathcal{Y}}(\mu)$$

3. Dependence of the RB problem on the HF dimensions $\longrightarrow$ EIM
4. Large collocation sets

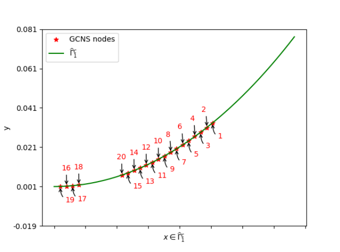
$$\boldsymbol{X}^{\boldsymbol{\tau}}_{S,\boldsymbol{\mathcal{C}}^{\mathrm{HF}}(\mu)}(\mu;\nu_{\mathcal{F}}\Lambda^{\boldsymbol{n}}_{R,k+1}(\mu)) := \Big\{\boldsymbol{\Theta}\in\mathbb{R}^S \mid \|(Q^{\boldsymbol{\tau}}\boldsymbol{\Theta})^r\| \le \nu_{\mathcal{F}}\Big(Q^{\boldsymbol{n}}\Lambda^{\boldsymbol{n}}_{R,k+1}(\mu)\Big)^r,\ \forall r\in\{1:\mathcal{R}\}\Big\}$$

Greedy Collocation Node Selection (GCNS)

1. **Input:** Training set $\mathcal{D}_{\mathtt{train}}$; HF collocation nodes $\widehat{\boldsymbol{\mathcal{C}}}$; RB spaces $\left(\boldsymbol{V}_N, W_R^{\boldsymbol{n}}, \boldsymbol{X}_S^{\boldsymbol{\tau}}\right)$
2. **Output:** Reduced set of collocation nodes $\widehat{C}(q) \subset \widehat{\boldsymbol{\mathcal{C}}}$

Iteration $q \geq 1$

1. $\mu_q \in \underset{\mu \in \mathcal{D}_{\mathtt{train}}}{\mathtt{argmax}} \left\| \left[\|\boldsymbol{\lambda}_S^{\boldsymbol{\tau},q}(\mu)\| - \nu_{\mathcal{F}} \lambda_R^{\boldsymbol{n},q}(\mu) \right]_+ \right\|_{\ell^\infty(\boldsymbol{\mathcal{C}}(\mu))}$
2. $\overline{\mathcal{Q}}(q) := \underset{s \in \{1:\mathcal{S}_0\}}{\mathtt{argmax}} \left[\left(\|\boldsymbol{\lambda}_S^{\boldsymbol{\tau},q}(\mu_q)\| - \nu_{\mathcal{F}} \lambda_R^{\boldsymbol{n},q}(\mu_q) \right) (h(\mu_q)(\widehat{\boldsymbol{c}}_s)) \right]_+$
3. $e_{\mathtt{GCNS}}(q) := \underset{\mu \in \mathcal{D}_{\mathtt{train}}}{\max} \frac{\|\boldsymbol{\lambda}^{\boldsymbol{\tau}}(\mu) - \boldsymbol{\lambda}_S^{\boldsymbol{\tau},q}(\mu)\|_{\Gamma^c(\mu)}}{\|\boldsymbol{\lambda}^{\boldsymbol{\tau}}(\mu)\|_{\Gamma^c(\mu)}}$
4. $\widehat{\boldsymbol{C}}(q+1) := \widehat{\boldsymbol{C}}(q) \cup \{\widehat{\boldsymbol{c}}_s\}_{s \in \overline{\mathcal{Q}}(q)}$; $\mathcal{Q}(q+1) := \mathcal{Q}(q) \cup \overline{\mathcal{Q}}(q)$

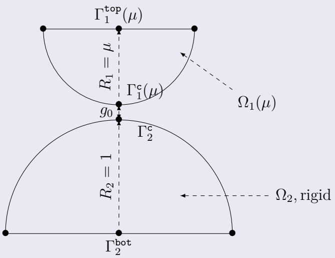


Main points

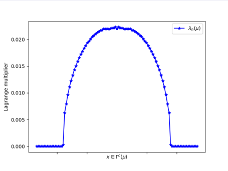
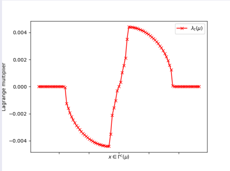
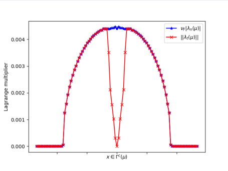
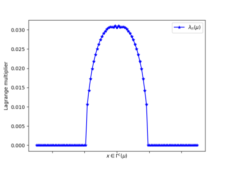
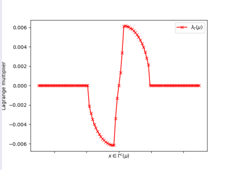
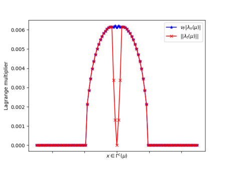
- Initialization $\longrightarrow$ no friction constraints ($\widehat{C}(0) := \emptyset$)
- Stopping criterion $\longrightarrow$ $e_{\mathtt{GCNS}}(q) > e_{\mathtt{GCNS}}(q-1)$

Hertz contact between two half-disks

- Parametric domain:
  1. $\mathcal{D} := [0.7,\ 1.3](\mathrm{m})$
  2. $\mathcal{D}_{\mathrm{train}}$ uniform discretization
- Geometric load:
  1. $g_0 = 0.001\mathrm{m}$
  2. $d = 0.09\mathrm{m}$ $(\leq 10\%\ \max(R_1))$
- Friction coefficient:
  1. $\nu_{\mathcal{F}} := 0.2$
- Potential contact manifold: $\theta \in [-\frac{5\pi}{8}, -\frac{3\pi}{8}]$
- Finite Element discretization:
  1. $\Omega(\mu)$ are μ-dependent
  2. Geometric mapping: $h(\mu) : \widehat{\Omega} \to \Omega(\mu)$
  3. displacement: $\mathbb{P}_2$
  4. Lagrange multipliers: $\mathbb{P}_1$



HF solutions

	$\lambda^{\boldsymbol{n}}(\mu)$	$\boldsymbol{\lambda}^{\boldsymbol{\tau}}(\mu)$	$\nu_{\mathcal{F}}\lvert\lambda^{\boldsymbol{n}}(\mu)\rvert$ and $\lVert\boldsymbol{\lambda}^{\boldsymbol{\tau}}(\mu)\rVert$
• $\mu = 0.7$m			
• $\mu = 1.3$m			

Projected Greedy Algorithm (PGA)

Reduced bases

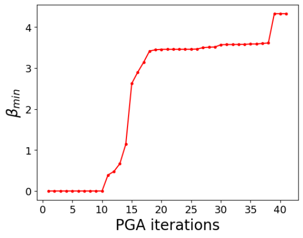
- Plain bases
  - primal $\longrightarrow N = 30$
  - dual $\longrightarrow R = S = 20$
- PGA algorithm
  - Tolerance:

$$\delta_{\mathtt{PGA}} := \sup_{\mu \in \mathcal{D}_{\mathtt{train}}} \frac{\beta_{\mathrm{HF}}(\mu)}{c_{\mathrm{HF}}(\mu)} = 0.072$$

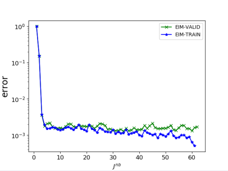
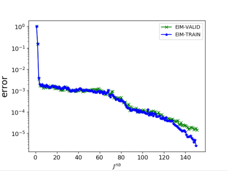
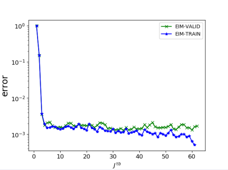
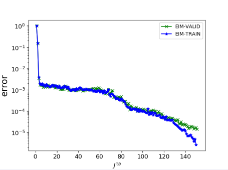
  - Enriched primal basis:

$$\dim(\mathrm{V}^{\mathrm{off}}_{\mathrm{N,R}}) = 70$$

Inf-sup stability: $\beta^n_{\min} := \min_{\mu \in \mathcal{D}_{\mathtt{train}}} \beta_{S^n}(\mu)$



EIM errors

	$\lvert\mathcal{D}_{\mathtt{train}}\rvert = 61$	$\lvert\mathcal{D}_{\mathtt{train}}\rvert = 150$
• $B^{\boldsymbol{n}}(\mu)$		
• $B^{\boldsymbol{\tau}}(\mu)$		

Greedy Collocation Node Selection (GCNS) algorithm

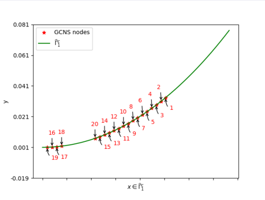
- Tangential constraint error

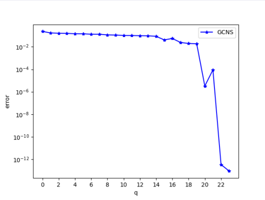
$$e^{\boldsymbol{n\tau}}_{\mathtt{GCNS}}(q) := \max_{\mu \in \mathcal{D}_{\mathtt{train}}} \left\| \left[\|\boldsymbol{\lambda}^{\boldsymbol{\tau},q}_{S}(\mu)\| - \nu_{\mathcal{F}} \lambda^{\boldsymbol{n},q}_{R}(\mu) \right]_{+} \right\|_{\ell^\infty(\boldsymbol{\mathcal{C}}(\mu))}$$

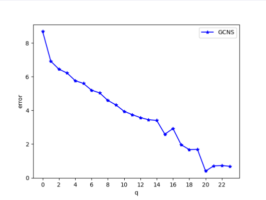
- RB approximation error on $\boldsymbol{\lambda}^{\boldsymbol{\tau}}(\mu)$

$$e_{\mathtt{GCNS}}(q) := \max_{\mu \in \mathcal{D}_{\mathtt{train}}} \frac{\|\boldsymbol{\lambda}^{\boldsymbol{\tau}}(\mu) - \boldsymbol{\lambda}^{\boldsymbol{\tau},q}_{S}(\mu)\|_{\Gamma^c(\mu)}}{\|\boldsymbol{\lambda}^{\boldsymbol{\tau}}(\mu)\|_{\Gamma^c(\mu)}}$$

$\widehat{\boldsymbol{\mathcal{C}}}(q) := \{\widehat{\boldsymbol{c}}_{s_j}\}_{j \in \{1:S_q\}}$ $\quad$ $e^{\boldsymbol{n\tau}}_{\mathtt{GCNS}}(q)$ $\quad$ $e_{\mathtt{GCNS}}(q)$



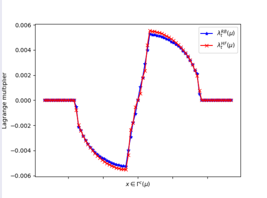


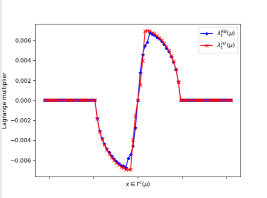


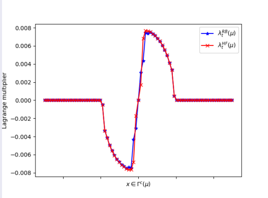
- At convergence $\longrightarrow$ $q = 20, \quad S_q = 40$

RB solutions

$\mu = 0.7\text{m}$	$\mu = 1\text{m}$	$\mu = 1.3\text{m}$







$\boldsymbol{\lambda}^{\boldsymbol{\tau}}(\mu)$ and $\boldsymbol{\lambda}_S^{\boldsymbol{\tau},q}(\mu)$

Comparison of the computational cost between the HF model and the RB model

(N, R, S)	$(30, 20, 20)$
HF times(s)	450
RB time(s)	35

Frictionless contact problem

1. Linear Elasticity: small deformation assumption

$$-\mathrm{div}(\sigma(\boldsymbol{u}(\mu))) = \boldsymbol{\ell}(\mu), \quad \text{in } \Omega(\mu), \quad \sigma(\boldsymbol{v}) := \mathbb{C} : \varepsilon(\boldsymbol{v})$$

$$a(\mu; \boldsymbol{u}, \boldsymbol{v}) := \int_{\Omega(\mu)} \sigma(\boldsymbol{u}) : \varepsilon(\boldsymbol{v})\, d\Omega(\mu), \quad f(\mu; \boldsymbol{v}) := \int_{\Omega(\mu)} \boldsymbol{\ell}(\mu) \cdot \boldsymbol{v}\, d\Omega(\mu)$$

2. Signorini contact conditions:

$$u_{\boldsymbol{n}}(\mu) \leq d^{\boldsymbol{n}}(\mu), \; \sigma_{\boldsymbol{nn}}(\boldsymbol{u}(\mu)) \leq 0, \; \sigma_{\boldsymbol{nn}}(\boldsymbol{u}(\mu))(u_{\boldsymbol{n}}(\mu) - d^{\boldsymbol{n}}(\mu)) = 0, \; \text{on } \Gamma^{\mathrm{c}}(\mu)$$

$$\mathcal{K}^{\boldsymbol{n}}(\mu) := \{v \in \mathcal{V}(\mu) \;\; | \; u_{\boldsymbol{n}}(\mu) \leq d^{\boldsymbol{n}}(\mu), \quad \text{on } \Gamma^{\mathrm{c}}(\mu)\}$$

Variational formulation

Well-posed parametric constrained minimization problem:

$$u(\mu) = \underset{v \in \mathcal{K}(\mu)}{\mathrm{argmin}}\; \mathcal{J}(\mu; \boldsymbol{v}) := \frac{1}{2} a(\mu; v, v) - f(\mu; v), \quad \forall \mu \in \mathcal{D} \subset \mathbb{R}^m$$

Nitsche's method (1/2)

State of art

[Nitsche 1971], [Chouly & Hild 2013], [Chouly 2014], [Chouly, Hild & Renard 2015]
[Mlika, Renard & Chouly 2017], [Chouly, Ern & Pignet 2020], [Chouly, Hild, Lleras & Renard 2022]

Alart–Curnier reformulation

$$\sigma_{\boldsymbol{nn}}(\boldsymbol{u}(\mu)) = \Big[\sigma_{\boldsymbol{nn}}(\boldsymbol{u}(\mu)) - \gamma\Big(u_{\boldsymbol{n}}(\mu) - d^{\boldsymbol{n}}(\mu)\Big)\Big]_-$$

Nitsche's energy

$$\begin{aligned} J^{\text{Nitsche}}(\mu;\boldsymbol{v}) := \ \mathcal{J}(\mu;\boldsymbol{v}) & - \frac{1}{2}\int_{\Gamma^c(\mu)} \frac{1}{\gamma}|\sigma_{\boldsymbol{nn}}(\boldsymbol{v})|^2 d\Gamma(\mu) \\ & + \frac{1}{2}\int_{\Gamma^c(\mu)} \frac{1}{\gamma}\Big[\sigma_{\boldsymbol{nn}}(\boldsymbol{v}) - \gamma\Big(v_{\boldsymbol{n}} - d^{\boldsymbol{n}}(\mu)\Big)\Big]_-^2 d\Gamma(\mu) \end{aligned}$$

Variational formulation

Well-posed parametric unconstrained minimization problem:

$$\boldsymbol{u}(\mu) = \underset{\boldsymbol{v}\in\boldsymbol{\mathcal{V}}(\mu)}{\text{argmin}} \ J^{\text{Nitsche}}(\mu;\boldsymbol{v})$$

Nitsche's method (2/2)

Optimality conditions

$$a_\gamma^{\boldsymbol{n}}(\mu;\boldsymbol{u}(\mu),\boldsymbol{v}) + \int_{\Gamma^c(\mu)} \frac{1}{\gamma} \left[P^{\boldsymbol{n}}_{\gamma,d^{\boldsymbol{n}}}(\mu;\boldsymbol{u}(\mu))\right]_- P^{\boldsymbol{n}}_{\gamma,0}(\mu;\boldsymbol{v})\, d\Gamma(\mu) = f(\mu;\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{V}}(\mu)$$

Notations

- Bilinear form

$$a_\gamma^{\boldsymbol{n}}(\mu;\boldsymbol{u},\boldsymbol{v}) := a(\mu;\boldsymbol{u},\boldsymbol{v}) - \int_{\Gamma^c(\mu)} \frac{1}{\gamma}\sigma_{\boldsymbol{nn}}(\boldsymbol{u})\sigma_{\boldsymbol{nn}}(\boldsymbol{v})\, d\Gamma(\mu)$$

- Operators
  - $P^{\boldsymbol{n}}_{\gamma,d^{\boldsymbol{n}}}(\mu;\boldsymbol{v}) := \sigma_{\boldsymbol{nn}}(\boldsymbol{v}) - \gamma(v_{\boldsymbol{n}} - d^{\boldsymbol{n}}(\mu))$
  - $P^{\boldsymbol{n}}_{\gamma,0}(\mu;\boldsymbol{v}) := \sigma_{\boldsymbol{nn}}(\boldsymbol{v}) - \gamma v_{\boldsymbol{n}}$

Resolution method

- Nonlinear problem $\longrightarrow$ iterative method

$$\left[P^{\boldsymbol{n}}_{\gamma,d^{\boldsymbol{n}}}(\mu;\boldsymbol{u}_k(\mu)+\boldsymbol{\delta u}_k(\mu))\right]_- \approx \left[P^{\boldsymbol{n}}_{\gamma,d^{\boldsymbol{n}}}(\mu;\boldsymbol{u}_k(\mu))\right]_- + H(-P^{\boldsymbol{n}}_{\gamma,d^{\boldsymbol{n}}}(\mu;\boldsymbol{u}_k(\mu)))P^{\boldsymbol{n}}_{\gamma,0}(\mu;\boldsymbol{\delta u}_k(\mu))$$

HF model

Iterative algorithm

- $\boldsymbol{u}_{k+1}(\mu) = \boldsymbol{u}_k(\mu) + \boldsymbol{\delta u}_k(\mu)$
- Sequence of problems: For all $k \geq 0$, find $\boldsymbol{\delta u}_k(\mu) \in \boldsymbol{\mathcal{V}}(\mu)$ such that

$$a_\gamma^{\boldsymbol{n}}(\mu; \boldsymbol{\delta u}_k(\mu), \boldsymbol{v}) + b_\gamma^{\boldsymbol{n}}(\mu; \boldsymbol{u}_k(\mu); \boldsymbol{\delta u}_k(\mu), \boldsymbol{v}) = -r_\gamma^{\boldsymbol{n}}(\mu; \boldsymbol{u}_k(\mu); \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{V}}(\mu)$$

Algebraic formulation

For all $k \geq 0$, find $\boldsymbol{\Delta U}_k(\mu) \in \mathbb{R}^{\mathcal{N}}$ such that

$$A_\gamma^{\boldsymbol{n}}(\mu)\boldsymbol{\Delta U}_k(\mu) + B_\gamma^{\boldsymbol{n}}(\mu, \boldsymbol{u}_k(\mu))\boldsymbol{\Delta U}_k(\mu) = -R_\gamma^{\boldsymbol{n}}(\mu, \boldsymbol{u}_k(\mu))$$

Notations

- Tangent matrix $B_\gamma^{\boldsymbol{n}}(\mu, \boldsymbol{w})$

$$b_\gamma^{\boldsymbol{n}}(\mu; \boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) := \int_{\Gamma^c(\mu)} \frac{1}{\gamma} H(-P_{\gamma,d\boldsymbol{n}}^{\boldsymbol{n}}(\mu; \boldsymbol{w})) P_{\gamma,0}^{\boldsymbol{n}}(\mu; \boldsymbol{u}) P_{\gamma,0}^{\boldsymbol{n}}(\mu; \boldsymbol{v})\, d\Gamma(\mu)$$

- Residual vector $R_\gamma^{\boldsymbol{n}}(\mu, \boldsymbol{w})$

$$r_\gamma^{\boldsymbol{n}}(\mu; \boldsymbol{w}; \boldsymbol{v}) := a_\gamma^{\boldsymbol{n}}(\mu; \boldsymbol{w}, \boldsymbol{v}) + \theta_\gamma^{\boldsymbol{n}}(\mu; \boldsymbol{w}; \boldsymbol{v}) - f(\mu; \boldsymbol{v}),$$

$$\theta_\gamma^{\boldsymbol{n}}(\mu; \boldsymbol{w}, \boldsymbol{v}) := \int_{\Gamma^c(\mu)} \frac{1}{\gamma} \left[P_{\gamma,d\boldsymbol{n}}^{\boldsymbol{n}}(\mu; \boldsymbol{w})\right]_- P_{\gamma,0}^{\boldsymbol{n}}(\mu; \boldsymbol{v})\, d\Gamma(\mu)$$

RB model

Plain approach

- Setting: geometric mapping $h(\mu) : \widehat{\Omega} \to \Omega(\mu)$ such that μ-independent $\mathcal{N}$
- RB space: POD $\longrightarrow \boldsymbol{V_N} := \boldsymbol{Span}\Big(\{\boldsymbol{\xi}_n\}_{n\in\{1:N\}}\Big)$
- Reduced problem: For all $k \geq 0$, find $\boldsymbol{\Delta U}_{N,k}(\mu) \in \mathbb{R}^N$ such that

$$A^{\boldsymbol{n}}_{\gamma,N}(\mu)\boldsymbol{\Delta U}_{N,k}(\mu) + B^{\boldsymbol{n}}_{\gamma,N}(\mu,k)\boldsymbol{\Delta U}_{N,k}(\mu) = -R^{\boldsymbol{n}}_{\gamma,N}(\mu,k)$$

- Large-dimensional "parameter/iteration"-dependent arrays: $\Xi := [\Xi_1 \cdots \Xi_N] \in \mathbb{R}^{\mathcal{N}\times N}$
  - $B^{\boldsymbol{n}}_{\gamma,N}(\mu,k) := \Xi^\top B^{\boldsymbol{n}}_{\gamma}(\mu,\boldsymbol{u}_{N,k}(\mu))\ \Xi$
  - $R^{\boldsymbol{n}}_{\gamma,N}(\mu,k) := \Xi^\top R^{\boldsymbol{n}}_{\gamma}(\mu,\boldsymbol{u}_{N,k}(\mu))$

Computationally efficient approach

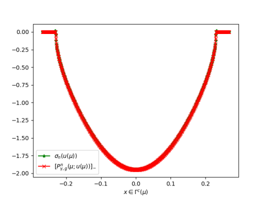
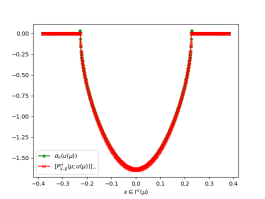
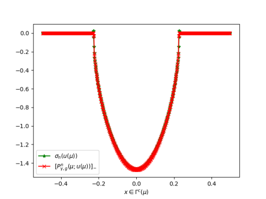
- Affine decomposition
  - Offline phase $\longrightarrow$ training set

$$B^{\boldsymbol{n}}_{\gamma}(\mu,\boldsymbol{u}_k(\mu)) \approx E^{b^{\boldsymbol{n}}}(\mu,k) := \sum_{s\in\{1:S^{b^{\boldsymbol{n}}}\}} \alpha^{b^{\boldsymbol{n}}}_s(\mu,k) B^{\boldsymbol{n}}_{\gamma,s}, \quad B^{\boldsymbol{n}}_{\gamma,s} \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}},\ \alpha^{b^{\boldsymbol{n}}}_s(\mu,k) \in \mathbb{R}$$

  - Online phase $\longrightarrow$ validation set

$$B^{\boldsymbol{n}}_{\gamma,N}(\mu,k) \approx E^{b^{\boldsymbol{n}}}_N(\mu,k) := \sum_{s\in\{1:S^{b^{\boldsymbol{n}}}\}} \alpha^{b^{\boldsymbol{n}}}_{N,s}(\mu,k) B^{\boldsymbol{n}}_{\gamma,N,s}, \quad B^{\boldsymbol{n}}_{\gamma,N,s} := \Xi^\top B^{\boldsymbol{n}}_{\gamma,s}\Xi \in \mathbb{R}^{N\times N}$$

HF solutions

$\mu = 0.7\text{m}$	$\mu = 1\text{m}$	$\mu = 1.3\text{m}$
		

$\boldsymbol{\sigma_{nn}}(\boldsymbol{u}(\mu))$ and $\left[P^{\boldsymbol{n}}_{\gamma,d\boldsymbol{n}}(\mu;\boldsymbol{u}(\mu))\right]_-$

Relative error on the Alart–Curnier reformulation of Signorini's contact conditions

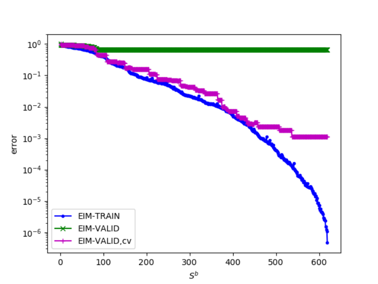
μ(m)	0.7			1			1.3		
h(mm)	5	2.5	1.25	5	2.5	1.25	5	2.5	1.25
$e^{\boldsymbol{n}}_{\mathrm{AC}}(\%)$	1	0.52	0.28	1.45	0.72	0.33	1.17	1.1	0.43

- $e^{\boldsymbol{n}}_{\mathrm{AC}}(\mu) := \dfrac{\left\| \boldsymbol{\sigma_{nn}}(\boldsymbol{u}(\mu)) - \left[P^{\boldsymbol{n}}_{\gamma,d\boldsymbol{n}}(\mu;\boldsymbol{u}(\mu))\right]_- \right\|_{\ell^2(\Gamma^c(\mu))}}{\|\boldsymbol{\sigma_{nn}}(\boldsymbol{u}(\mu))\|_{\ell^2(\Gamma^c(\mu))}}$
- Convergence of order 1

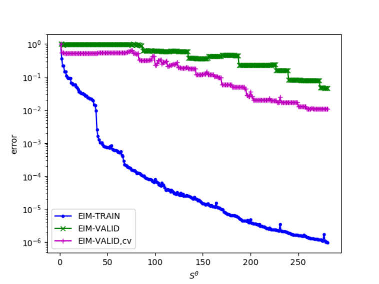
EIM errors

- $e^{b^{\boldsymbol{n}}}_{\mathtt{EIM}}(S^{b^{\boldsymbol{n}}}, \mathcal{D}_*) := \dfrac{\max\limits_{\mu\in\mathcal{D}_*} \max\limits_{k\in\{1:k^{\mathrm{cv}}(\mu)\}} \|B^{\boldsymbol{n}}_\gamma(\mu,\boldsymbol{u}_k(\mu)) - E^{b^{\boldsymbol{n}}}(\mu,k)\|_{\ell^\infty(ij)}}{\max\limits_{\mu\in\mathcal{D}_*} \max\limits_{k\in\{1:k^{\mathrm{cv}}(\mu)\}} \|B^{\boldsymbol{n}}_\gamma(\mu,\boldsymbol{u}_k(\mu))\|_{\ell^\infty(ij)}}$, $\mathcal{D}_* = \mathcal{D}_{\mathtt{train}}$ or $\mathcal{D}_{\mathtt{valid}}$
- $e^{b^{\boldsymbol{n}},\mathrm{cv}}_{\mathtt{EIM}}(S^{b^{\boldsymbol{n}}}) := \dfrac{\max\limits_{\mu\in\mathcal{D}_{\mathtt{valid}}} \|B^{\boldsymbol{n}}_\gamma(\mu,\boldsymbol{u}_{k^{\mathrm{cv}}(\mu)}(\mu)) - E^{b^{\boldsymbol{n}}}(\mu,k^{\mathrm{cv}}(\mu))\|_{\ell^\infty(ij)}}{\max\limits_{\mu\in\mathcal{D}_{\mathtt{valid}}} \|B^{\boldsymbol{n}}_\gamma(\mu,\boldsymbol{u}_{k^{\mathrm{cv}}(\mu)}(\mu))\|_{\ell^\infty(ij)}}$

$B^{\boldsymbol{n}}_\gamma(\mu, \boldsymbol{u}_k(\mu))$



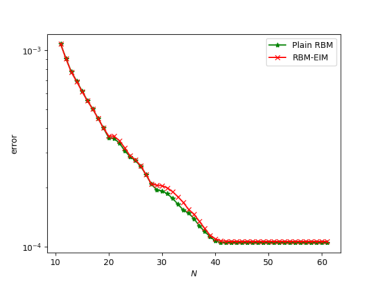
$\Theta^{\boldsymbol{n}}_\gamma(\mu, \boldsymbol{u}_{\mathrm{cv}}(\mu))$



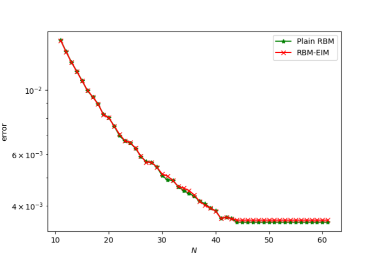
RB approximation errors

- $e_N^{\boldsymbol{u}}(\mu) := \dfrac{\|\boldsymbol{u}(\mu)-\boldsymbol{u}_N(\mu)\|_{\boldsymbol{\mathcal{V}}(\mu)}}{\|\boldsymbol{u}(\mu)\|_{\boldsymbol{\mathcal{V}}(\mu)}}$
- $e_N^{\boldsymbol{nn}}(\mu) := \dfrac{\|\sigma_{\boldsymbol{nn}}(\boldsymbol{u}(\mu))-\sigma_{\boldsymbol{nn}}(\boldsymbol{u}_N(\mu))\|_{\ell^2(\Gamma^c(\mu))}}{\|\sigma_{\boldsymbol{nn}}(\boldsymbol{u}(\mu))\|_{\ell^2(\Gamma^c(\mu))}}$
- $e_{N,\max}^{\boldsymbol{u}} := \max\limits_{\mu\in\mathcal{D}_{\mathtt{valid}}} e_N^{\boldsymbol{u}}(\mu)$
- $e_{N,\max}^{\boldsymbol{nn}} := \max\limits_{\mu\in\mathcal{D}_{\mathtt{valid}}} e_N^{\boldsymbol{nn}}(\mu)$

$\boldsymbol{u}(\mu)$



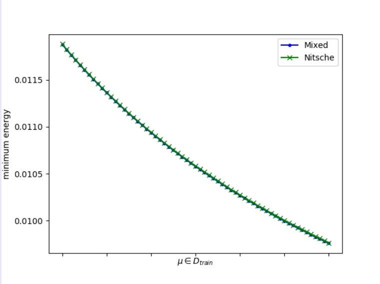
$\sigma_{\boldsymbol{nn}}(\boldsymbol{u}(\mu))$



Plain RBM vs RBM-EIM

Comparison with the mixed formulation (1/2)

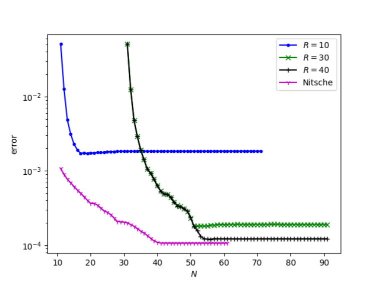
HF energy $\mathcal{J}(\mu; \boldsymbol{u}(\mu)) := \frac{1}{2} a(\mu; \boldsymbol{u}(\mu), \boldsymbol{u}(\mu)) - f(\mu; \boldsymbol{u}(\mu))$, $\mu \in \mathcal{D}_{\texttt{train}}$

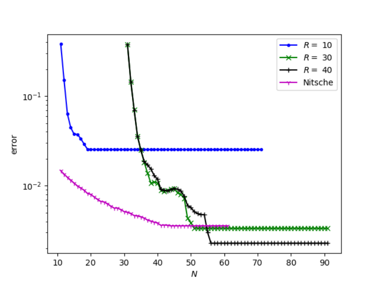


Comparison with the mixed formulation (2/2)

- $e^{\boldsymbol{u}}_{N,R}(\mu) := \dfrac{\|\boldsymbol{u}(\mu) - \boldsymbol{u}_N(\mu)\|_{\boldsymbol{\mathcal{V}}(\mu)}}{\|\boldsymbol{u}(\mu)\|_{\boldsymbol{\mathcal{V}}(\mu)}}$
- $e^{\lambda}_{N,R}(\mu) := \dfrac{\|\lambda(\mu) - \lambda_R(\mu)\|_{\ell^2(\Gamma^c(\mu))}}{\|\lambda(\mu)\|_{\ell^2(\Gamma^c(\mu))}}$
- $e^{\boldsymbol{u}}_{N,R,\max} := \max\limits_{\mu \in \mathcal{D}_{\mathtt{valid}}} e^{\boldsymbol{u}}_{N,R}(\mu)$
- $e^{\lambda}_{N,R,\max} := \max\limits_{\mu \in \mathcal{D}_{\mathtt{valid}}} e^{\lambda}_{N,R}(\mu)$

$e^{\boldsymbol{u}}_{N,R,\max}$ and $e^{\boldsymbol{u}}_{N,\max}$ — $e^{\lambda}_{N,R,\max}$ and $e^{nn}_{N,\max}$





$R = 10$, $R = 30$, $R = 40$