New approach to Data-Driven Variational Multiscale Reduced Order Model (D2-VMS-ROM)

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Education and Work Experience

• (05/21–05/22) — University of Seville, Seville, Spain
  • working with Professor Tomás Chacón on the modeling of the subscales.

• (08/16–05/21) — Virginia Tech, Blacksburg, USA, Ph.D. Mathematics
  • Advisor: Professor Traian Iliescu
  • Dissertation: Numerical Analysis for Data-Driven Reduced Order Model Closures

• (08/16–12/18) — Virginia Tech, Blacksburg, USA, M.S. Mathematics
  • Advisor: Professor Traian Iliescu
  • Thesis: Commutation Error in Reduced Order Modeling
Research Interest

• Applied Mathematics,

• Turbulent Flows,

• Reduced Order Modeling (ROM),

• Closure Modeling,

• Data-Driven Modeling,

• Variational Multiscale Methods,

• Numerical Analysis.
Collaborators

- Guillaume Enchéry, IFPEN
- Angelo Iollo, INRIA & University of Bordeaux
- Tommaso Taddei, INRIA
- Tomás Chacón, University of Seville
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- Gianluigi Rozza, SISSA
- Zhu Wang, University of South Carolina
- Omer San, Oklahoma State University
- Leo Rebholz, Clemson University
• Introduction

• Reduced Order Model (ROM)

• Proper Orthogonal Decomposition (POD)

• Galerkin ROM (G-ROM)

• ROM Closure Problem

• Data-Driven Variational Multiscale ROM (D2-VMS-ROM)
  • Variational Multiscale Framework
  • Data-Driven Approach
    • Coefficient-based D2-VMS-ROM (C-D2-VMS-ROM)
    • Residual-based D2-VMS-ROM (R-D2-VMS-ROM)

• Conclusions
Full Order Models (FOMs), e.g., FE, FV, FD,...
(i) dimension: \( N = \mathcal{O}(10^6) \) 😞 (e.g., FE mass matrix, \( M_h \in \mathbb{R}^{N \times N} \))
(ii) computational cost 😞
(iii) accuracy 😞

Reduced Order Models (ROMs)
(i) dimension: \( r = \mathcal{O}(10) \) 😞 (e.g., ROM mass matrix, \( M_r = C^T M_h C \in \mathbb{R}^{r \times r} \))
(ii) computational cost 😞
(iii) accuracy
   (iiiia) laminar flows: G-ROM 😞
   (iiiib) turbulent flows: G-ROM 😞 G-ROM + closure term 😞
Question: What kind of setting is good for the ROM?
Proper Orthogonal Decomposition (POD)

- one of the most popular ROM techniques,
- seeks a low-dimensional basis \( \{\varphi_j\}_{j=1}^r \),
  - data for the parabolic PDE:
    \[
    \dot{u} = f(u),
    \] (1)
  - collect the snapshots \( \{u_h^1, \ldots, u_h^M\} \) from FE solutions,
  - solve the minimization problem:
    \[
    \min_{(\varphi_i, \varphi_j) \mathcal{H} = \delta_{ij}} \frac{1}{M} \sum_{j=1}^{M} \left\| u_h(t_j) - \sum_{i=1}^{d} \left( u_h(t_j), \varphi_i(x) \right)_{\mathcal{H}} \varphi_i(x) \right\|^2,
    \] (2)
  - solve the eigenvalue problem:
    \[
    Kv_i = \lambda_i v_i , \quad K = \frac{1}{M} Y^T M_h Y
    \] (3)
    \[
    \lambda_1 \geq \lambda_2 \geq \ldots \lambda_d \geq \lambda_{d+1} = \ldots = \lambda_M = 0
    \] (4)
    \[
    (C_r)_i = \frac{1}{\sqrt{M}} \frac{1}{\sqrt{\lambda_i}} Y v_i , \quad i = 1, \ldots, r.
    \] (5)
We derive the G-ROM framework on the incompressible Navier-Stokes equations:

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} - Re^{-1} \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p &= 0, \\
\nabla \cdot \mathbf{u} &= 0,
\end{align*}
\]

(6) (7)

• the most straightforward ROM,

• build the ROM solution, i.e., \( \mathbf{u}_r \) as (\( \mathbf{u}_d \) is truth solution)

\[
\mathbf{u}_r = \sum_{i=1}^{r} (a_r)_i \varphi_i.
\]

(8)

• find the weak form of (6)-(7) by inserting \( \mathbf{u}_r \) into \( \mathbf{u} \) in (6)-(7)

\[
\text{G-ROM: } \dot{a}_r = A a_r + a_r^T B a_r,
\]

(9)

• \( a \) is the unknown,

• the matrix \( A_{im} = -Re^{-1} (\nabla \varphi_m, \nabla \varphi_i) \),

• the tensor \( B_{imn} = - (\varphi_m \cdot \nabla \varphi_n, \varphi_i) \) 1 \( \leq i, m, n \leq r \).
ROM Closure Problem

Structure-dominated Regime

• the rate is fast,
• resolved regime,
• closure term is not needed, e.g., G-ROM

Convection-dominated Regime

• the rate is not fast,
• under-resolved regime,
• to increase the numerical accuracy
  • $r \uparrow \infty$
  • add a closure $\ominus$

\[
(\dot{u}_r, \varphi_i) = (f(u_r), \varphi_i) + \text{Closure term.}
\]  

Closure term:

• models the interaction between the resolved and unresolved ROM modes,
• is not a closed term,
• is modeled with $r$-dimensional operators in an offline stage.
Question: How do we model the closure term?

- Functional ROM closures
  - physical insight,
  - e.g., eddy viscosity.
- Structural ROM closures
  - mathematical arguments,
  - e.g., approximate deconvolution, parameterized manifolds, data-driven.

Data-Driven Framework (D2):

- ansatz construction
  \[
  \text{Closure term}(a_d, a_r) \approx \text{Ansatz}(a_r),
  \]  
  \( (11) \)

- least-squares problem
  \[
  \min_{D2 \text{ operators}} \sum_{j=1}^{M} \left\| \text{Closure term}((a_d)_j^{\text{FOM}}, (a_r)_j^{\text{FOM}}) - \text{Ansatz}((a_r)_j^{\text{FOM}}) \right\|^2
  \]  
  \( (12) \)

D2-VMS-ROM:

\[
(\dot{u}_r, \varphi_i) = (f(u_r), \varphi_i) + \text{Ansatz}(a_r)
\]  
(13)
Coefficient-based D2-VMS-ROM


2S-D2-VMS-ROM

- using hierarchical structure of ROM space and basis
- resolved ROM scales \( X_1 := \text{span}\{\varphi_1, \ldots, \varphi_r\} \)
- unresolved ROM scales \( X_2 := \text{span}\{\varphi_{r+1}, \ldots, \varphi_d\} \)
- two-scale decomposition
  - \( u_r \in X_1 \) resolved ROM component of \( u \)
  - \( u' \in X_2 \) unresolved ROM component of \( u \)

\[
\begin{align*}
  u_d &= \sum_{j=1}^d a_j \varphi_j = \sum_{j=1}^r (a_r)_j \varphi_j + \sum_{j=r+1}^d (a')_j \varphi_j = u_r + u'
\end{align*}
\]

- VMS-ROM closure term

\[
(\ddot{u}_r, \varphi_i) = (f(u_r), \varphi_i) + \left[\left(f(u_d), \varphi_i\right) - \left(f(u_r), \varphi_i\right)\right] \quad \forall \ i = 1, \ldots, r
\]

\( \text{VMS-ROM closure term} \)
2S-D2-VMS-ROM Algorithm:

- use the FOM data (snapshots)
- offline stage
- low-dimensional D2 operators
- coefficient-based ansatz:

\[
\text{VMS-ROM Closure term} \approx \tilde{A} a_r + a_r^\top \tilde{B} a_r
\]  

(14)

\[
\min_{\tilde{A}, \tilde{B}} \sum_{j=1}^{M} \left\| \text{Closure term}( (a_d)_{FOM}^j, (a_r)_{FOM}^j ) - \tilde{A}(a_r^{FOM})^j + ((a_r^{FOM})^j)^\top \tilde{B}(a_r^{FOM})^j \right\|^2
\]  

(15)

**2S-D2-VMS-ROM:**

\[
\dot{a} = (A + \tilde{A}) a_r + a_r^\top (B + \tilde{B}) a_r
\]  

(16)
3S-D2-VMS-ROM

- **large resolved** ROM scales \( X_1 := \text{span}\{\varphi_1, \ldots, \varphi_{r_1}\} \)
- **small resolved** ROM scales \( X_2 := \text{span}\{\varphi_{r_1+1}, \ldots, \varphi_r\} \)
- **unresolved** ROM scales \( X_3 := \text{span}\{\varphi_{r+1}, \ldots, \varphi_d\} \)

**three-scale decomposition:**
- \( u_L \in X_1 \) large resolved ROM component of \( u \)
- \( u_S \in X_2 \) small resolved ROM component of \( u \)
- \( u' \in X_3 \) unresolved ROM component of \( u \)

\[
\begin{align*}
\boldsymbol{u}_d &= \sum_{j=1}^{d} a_j \varphi_j = \sum_{j=1}^{r_1} (a_L)_j \varphi_j + \sum_{j=r_1+1}^{r} (a_S)_j \varphi_j + \sum_{j=r+1}^{d} (a')_j \varphi_j = u_L + u_S + u' = u_r + u'
\end{align*}
\]

**VMS-ROM closure terms**
- **large-small** VMS-ROM closure term:
  \[
  (\dot{u}_L, \varphi_i) = (f(u_r), \varphi_i) + [(f(u_d), \varphi_i) - (f(u_r), \varphi_i)] \quad \forall \ i = 1, \ldots, r_1,
  \]
- **small-unresolved** VMS-ROM closure term:
  \[
  (\dot{u}_S, \varphi_i) = (f(u_r), \varphi_i) + [(f(u_d), \varphi_i) - (f(u_r), \varphi_i)] \quad \forall \ i = r_1 + 1, \ldots, r,
  \]
• two different least squares problems:
  • 1st one: large and small resolved scales; produces $\tilde{A}_L$ and $\tilde{B}_L$
  • 2nd one: small resolved and unresolved scales; produces $\tilde{A}_S$ and $\tilde{B}_S$

$$\begin{bmatrix}
\dot{a}_L \\
\dot{a}_S
\end{bmatrix} = A a + a^T B a + \begin{bmatrix}
\tilde{A}_L a_L + a_L^T \tilde{B}_L a_L \\
\tilde{A}_S a_S + a_S^T \tilde{B}_S a_S
\end{bmatrix}, \quad (17)$$

• 2 truncated SVD
• more flexibility in choosing the VMS-ROM closure operators $\tilde{A}_L, \tilde{A}_S, \tilde{B}_L,$ and $\tilde{B}_S$ in the least squares problems.
Numerical Results

Numerical results are obtained by using the Burgers and NSE equations considering the reconstructive and predictive regimes.

\[
\begin{align*}
  u_t - \nu u_{xx} + uu_x &= 0 , \quad x \in [0,1], \ t \in [0,1], \\
u(0, t) = u(1, t) &= 0 , \quad t \in [0,1],
\end{align*}
\]

(18)

with the initial condition

\[
u_0(x) = \begin{cases} 
1, & x \in (0,1/2], \\
0, & x \in (1/2,1], 
\end{cases}
\]

(19)

**Snapshot Generation:** \( \nu = 10^{-3}, h = 1/2048, \Delta t = 10^{-3}, \) linear FE, CN.

<table>
<thead>
<tr>
<th>( r )</th>
<th>G-ROM ( E(L^2) )</th>
<th>2S-D2-VMS-ROM ( E(L^2) )</th>
<th>3S-D2-VMS-ROM ( E(L^2) )</th>
<th>( r_1 )</th>
<th>( tol_S )</th>
<th>( E(L^2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1.181e-01</td>
<td>7.278e-02</td>
<td>( r_1 )</td>
<td>1e+00</td>
<td>1.322e-02</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>1.828e-01</td>
<td>1.755e-01</td>
<td>2</td>
<td>1e-03</td>
<td>3.915e-03</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>1.258e-01</td>
<td>1.229e-01</td>
<td>1</td>
<td>1e-03</td>
<td>1.787e-03</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>6.551e-02</td>
<td>6.456e-02</td>
<td>1</td>
<td>1e-02</td>
<td>2.310e-03</td>
<td></td>
</tr>
</tbody>
</table>

**Table 1:** Reconstructive regime, \( tol = tol_L = 10^1 \), and optimal \( tol_S \).
Predictive regime: training set = [0,0.7] and testing set = [0.7,1].

<table>
<thead>
<tr>
<th>r</th>
<th>G-ROM ( E(L^2) )</th>
<th>2S-D2-VMS-ROM ( E(L^2) )</th>
<th>3S-D2-VMS-ROM ( E(L^2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2.185e-01</td>
<td>3.623e-02</td>
<td>3.029e-02</td>
</tr>
<tr>
<td>7</td>
<td>2.054e-01</td>
<td>2.004e-02</td>
<td>1.428e-02</td>
</tr>
<tr>
<td>11</td>
<td>1.620e-01</td>
<td>1.608e-02</td>
<td>1.418e-02</td>
</tr>
<tr>
<td>17</td>
<td>1.103e-01</td>
<td>1.524e-02</td>
<td>1.506e-02</td>
</tr>
</tbody>
</table>

Table 2: Predictive regime, optimal \( tol \), \( tol_S \), and \( tol_L \).

Conclusions:

- 3S-D2-VMS-ROM has more flexibility in choosing the VMS-ROM closure operators.
- 3S-D2-VMS-ROM is significantly more accurate than the 2S-D2-VMS-ROM for reconstructive regime.
- 3S-D2-VMS-ROM is more accurate than the 2S-D2-VMS-ROM for predictive regime.
Flow Past a Cylinder

We prescribe no-slip boundary conditions on the walls and cylinder, and the following inflow and outflow profiles:

\[
\begin{align*}
  u_1(0, y, t) &= u_1(2.2, y, t) = \frac{6}{0.41^2} y(0.41 - y), \\
  u_2(0, y, t) &= u_2(2.2, y, t) = 0,
\end{align*}
\]

where \( u = (u_1, u_2) \).

- DOF: 103\( K \) (102962) velocity and 76\( K \) (76725) pressure
- linearized BDF2 temporal discretization with \( \Delta t = 0.002 \)
- \( Re = 1000 \)

**ROM Construction:**

- use 10s of the FOM data, e.g., [13, 23] (one period=[13, 13.268])
- reconstructive regime: ROM basis=[13, 23], D2 operators=[13, 13.268]
- predictive regime: ROM basis=[13, 23], D2 operators=[13, 13.134] (half period)
Flow Past A Cylinder, $Re = 1000$

Figure 1: Flow past a cylinder, $Re = 1000$, reconstructive regime. Time evolution of the kinetic energy for G-ROM, 2S-DD-VMS-ROM, and 3S-DD-VMS-ROM for different $r$ values.
Figure 2: Flow past a cylinder, $Re = 1000$, predictive regime. Time evolution of the kinetic energy for G-ROM, 2S-DD-VMS-ROM, and 3S-DD-VMS-ROM for different $r$ values.

**Aim:** Create a ROM closure model depends on the ROM residual

For a given bilinear-linear form:

$$a(u^d, v^d) = \langle f, v^d \rangle$$  \hspace{1cm} (22)

Decompose (22) into two problems as:

$$a(u_L, v_L) + a(u_S, v_L) = \langle f, v_L \rangle$$ \hspace{1cm} (23a)

$$a(u_S, v_S) + a(u_L, v_S) = \langle f, v_S \rangle$$ \hspace{1cm} (23b)

The matrix-vector form of (23a)-(23b) is as follows:

$$A_{LL} a_L + A_{LS} a_S = b_L$$ \hspace{1cm} (24a)

$$A_{SL} a_L + A_{SS} a_S = b_S$$ \hspace{1cm} (24b)

$$\text{Closure}(a_S) \approx \text{Ansatz}(\text{Res}(a_L))$$.
where $\mathit{Res}_S(a_L) = \mathbf{b}_S - A_{SL} a_L$.

To construct the D2 operator in (25), we need to solve the following generalized minimization problem:

$$\min_{\tilde{A}} \sum_{j=1}^{M} \|a_S^j - \tilde{A} \mathit{Res}_S(a_L^j)\|_{L^2}^2.$$  \hspace{1cm} (26)

**R1-ROM:** \( \left( A_{LL} - A_{LS} \tilde{A} A_{SL} \right) a_L = \mathbf{b}_L - A_{LS} \tilde{A} \mathbf{b}_S. \)  \hspace{1cm} (27)
R-D2-VMS-ROM with two ansatizes

\[ a_S \approx \tilde{A}_1 \text{Res}_S(a_L), \]  
(28)

\[ \text{Res}_L(a_S) \approx \tilde{A}_2 \text{Res}_L(a_S^{\text{approx}}), \]  
(29)

where \( \text{Res}_L(a_S) = b_L - A_{LS} a_S \).

By using (26) and the following minimization problems to obtain the D2 operators \( \tilde{A}_1 \) and \( \tilde{A}_2 \), respectively.

\[
\min_{\tilde{A}_2} \sum_{j=1}^{M} \left\| \text{Res}_L(u_S^j) - \tilde{A}_2 \text{Res}_L(a_S^{\text{approx}}) \right\|_{L^2}^2.
\]  
(30)

\[
\text{R2-ROM : } (A_{LL} - \tilde{A}_2 A_{LS} \tilde{A}_1 A_{SL}) a_L = \tilde{A}_2 (b_L - A_{LS} \tilde{A}_1 b_S).
\]  
(31)
Numerical Results

We present numerical results for the parameter-dependent CD problem

\[
\begin{aligned}
-\mu \partial_{xx} u + c \partial_x u &= f \quad \text{for } x \in [0, 1], \\
u(0) &= 0 \quad \text{and} \quad u(1) = 0,
\end{aligned}
\]

with the following exact solution

\[
u(x, \mu) = \frac{\exp(cx/\mu) - 1}{\exp(c/\mu) - 1} - x,
\]

where \( \mu \) is a parameter. The force term is \( f = -c = -400 \).

Snapshot Generation \( \mu \in [1, 10], \ \Delta \mu = 1, \ \text{linear FE, } h = 1/4096 \).

ROM Construction Use \( \mu^{\text{training}} = 1, 2, ..., 9, 10 \) to generate the ROM basis functions and operators and train the D2-VMS-ROM operators. We test all the ROMs for \( \mu^{\text{testing}} = 0.5, 0.1, 0.05, 15 \), which fall outside the training range.
To compare the numerical accuracy of the methods, we use the following metric to compute the ROM errors:

$$
\mathcal{E}_{L^2}^2 = \left\| \mathbf{u}_L(\mu^{\text{testing}}) - \sum_{i=1}^{L} \left( \mathbf{u}^{\text{FOM}}(\mu^{\text{testing}}), \varphi_i \right)_{L^2} \varphi_i \right\|_{L^2}.
$$

(34)

For a fair comparison, we also consider the coefficient-based D2-VMS-ROM:

$$
\mathbf{A}_{LS} \mathbf{a}_S \approx \tilde{\mathbf{A}} \mathbf{a}_L,
$$

(35a)

C-ROM:

$$
(\mathbf{A}_{LL} + \tilde{\mathbf{A}}) \mathbf{a}_L = \mathbf{b}_L.
$$

(36)
Testing the ROMs for $\mu = 0.5$

<table>
<thead>
<tr>
<th>$L$</th>
<th>$G$</th>
<th>$C$</th>
<th>$R1$</th>
<th>$R2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.38e+00</td>
<td>5.72e+00</td>
<td>5.45e-04</td>
<td>3.04e-04</td>
</tr>
<tr>
<td>2</td>
<td>3.71e-01</td>
<td>3.66e-01</td>
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<td>8.88e-03</td>
</tr>
<tr>
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<td>1.68e-01</td>
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<tr>
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<td>1.61e-01</td>
<td>2.46e-04</td>
<td>1.22e-03</td>
</tr>
<tr>
<td>6</td>
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<td>9.92e-02</td>
<td>1.72e-04</td>
<td>8.84e-04</td>
</tr>
<tr>
<td>7</td>
<td>9.48e-02</td>
<td>9.48e-02</td>
<td>1.18e-04</td>
<td>6.80e-04</td>
</tr>
</tbody>
</table>

**Table 3:** $L^2$ error (34) for G-ROM, C-ROM, R1-ROM, and R2-ROM for various $L$ values.

<table>
<thead>
<tr>
<th>$L$</th>
<th>$A_{LS}a_S$</th>
<th>$C$</th>
<th>$L$</th>
<th>$A_{LS}a_S$</th>
<th>$C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.06e+02</td>
<td>7.10e+02</td>
<td>6</td>
<td>5.90e+02</td>
<td>2.40e-02</td>
</tr>
<tr>
<td>2</td>
<td>1.21e+03</td>
<td>1.77e+01</td>
<td>7</td>
<td>4.18e+02</td>
<td>9.31e-05</td>
</tr>
<tr>
<td>3</td>
<td>1.21e+03</td>
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<td>8</td>
<td>2.64e+02</td>
<td>7.84e-05</td>
</tr>
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<td>7.06e-01</td>
<td>9</td>
<td>1.26e+02</td>
<td>1.02e-02</td>
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<tr>
<td>5</td>
<td>7.85e+02</td>
<td>3.76e-01</td>
<td>10</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Table 4:** Consistency error comparison for C-D2-VMS-ROM with various $L$ values.
In Tables 4, 5, and 6, to investigate the ROM consistency, we list the norm of the closure term and its ansatz.

<table>
<thead>
<tr>
<th>$L$</th>
<th>$a_S$</th>
<th>R1</th>
<th>$L$</th>
<th>$a_S$</th>
<th>R1</th>
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<tr>
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<td>6.37e-02</td>
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<tr>
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<td>2.13e-03</td>
<td>5.12e-03</td>
</tr>
<tr>
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<td>1.14e-02</td>
<td>1.28e-02</td>
<td>8</td>
<td>1.50e-03</td>
<td>7.41e-04</td>
</tr>
<tr>
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<td>6.36e-03</td>
<td>8.77e-03</td>
<td>9</td>
<td>9.35e-04</td>
<td>4.23e-04</td>
</tr>
<tr>
<td>5</td>
<td>4.17e-03</td>
<td>7.11e-03</td>
<td>10</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Table 5**: Consistency error comparison for R1-D2-VMS-ROM with various $L$ values.

<table>
<thead>
<tr>
<th>$L$</th>
<th>$\text{Res}_L(a_S)$</th>
<th>R2</th>
<th>$L$</th>
<th>$\text{Res}_L(a_S)$</th>
<th>R2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.24e+01</td>
<td>4.07e+01</td>
<td>6</td>
<td>6.86e+02</td>
<td>9.31e+02</td>
</tr>
<tr>
<td>2</td>
<td>1.24e+03</td>
<td>1.32e+03</td>
<td>7</td>
<td>5.31e+02</td>
<td>8.24e+02</td>
</tr>
<tr>
<td>3</td>
<td>1.21e+03</td>
<td>1.23e+03</td>
<td>8</td>
<td>4.43e+02</td>
<td>6.97e+03</td>
</tr>
<tr>
<td>4</td>
<td>1.06e+03</td>
<td>1.16e+03</td>
<td>9</td>
<td>3.74e+02</td>
<td>3.57e+02</td>
</tr>
<tr>
<td>5</td>
<td>8.34e+02</td>
<td>1.00e+03</td>
<td>10</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Table 6**: Consistency error comparison for R2-D2-VMS-ROM with various $L$ values.
Testing for All $\mu$ Values

Instead of further investigating different $\mu^{\text{testing}}$ values that are out of the training set, we use the average $L^2$ error (37) to measure the ROM consistency for $\mu^{\text{testing}} = 0.5, 0.1, 0.05, 15.$

$$\mathcal{E}_{\text{avg}} = \frac{1}{M} \sum_{j=1}^{M} \left\| u_L(\mu^{\text{testing}}) - \sum_{i=1}^{r} \left( u^{\text{FOM}}(\mu^{\text{testing}}), \varphi_i \right) \right\|_{L^2}.$$  \hspace{1cm} (37)

<table>
<thead>
<tr>
<th>$L$</th>
<th>G-ROM</th>
<th>C-ROM</th>
<th>R1-ROM</th>
<th>R2-ROM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.95e+01</td>
<td>3.99e-03</td>
<td>8.98e-04</td>
<td>1.14e-03</td>
</tr>
<tr>
<td>2</td>
<td>4.17e-01</td>
<td>5.76e-03</td>
<td>2.11e-04</td>
<td>1.13e-02</td>
</tr>
<tr>
<td>3</td>
<td>3.31e+00</td>
<td>5.06e-03</td>
<td>2.92e-04</td>
<td>6.24e-03</td>
</tr>
<tr>
<td>4</td>
<td>3.07e-01</td>
<td>4.68e-03</td>
<td>3.36e-04</td>
<td>5.36e-03</td>
</tr>
<tr>
<td>5</td>
<td>1.93e+00</td>
<td>4.43e-03</td>
<td>1.03e-03</td>
<td>4.91e-03</td>
</tr>
<tr>
<td>6</td>
<td>2.64e-01</td>
<td>4.17e-03</td>
<td>8.30e-04</td>
<td>4.53e-03</td>
</tr>
<tr>
<td>7</td>
<td>1.49e+00</td>
<td>3.95e-03</td>
<td>6.54e-04</td>
<td>4.25e-03</td>
</tr>
</tbody>
</table>

**Table 7:** Average $L^2$ error (37) for G-ROM, C-ROM, R1-ROM, and R2-ROM for various $L$ values.
Conclusions

- The errors in Table 3, the G-ROM and C1-ROM yield the worst results among all the ROMs.
- Furthermore, R1-ROM give the lowest errors among all the ROMs.
- The order of magnitude of the C-ROM error quickly diminishes than the norm of the closure term.
- In Tables 5-6, we observe that the order of magnitude of the R1-ROM and R2-ROM are the same as their closure terms, and as $L$ goes to $d$. Thus, we conclude that R1-ROM and R2-ROM are equally consistent models.
- Based on the errors in Table 7, G-ROM and R1-ROM are the worst and most accurate, respectively.
- Although R2-ROM involves more information from the sub-scale equation, R1-ROM is more accurate than R2-ROM.