Primal-dual interior point methods for state and input constrained optimal control

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29/11/2022

Overview

Outline

Numerical optimization problem (1/2)

Original and penalized problem

Original problem

$$\min_{c(x)\leq 0} f(x)$$

Penalized problem

$$\min_{x} f(x) - \epsilon \log(-c(x))$$

Numerical optimization problem (2/2)

Primal penalized problem

The primal problem consists in solving for x the following first order conditions

$$f'(x) - c'(x).\frac{\epsilon}{c(x)} = 0$$

Primal dual penalized problem

The primal-dual problem consists in solving for x and λ the following first order conditions

$$f'(x) + c'(x).\lambda = 0$$

$$\lambda c(x) + \epsilon = 0 \Leftrightarrow \lambda - c(x) - \sqrt{\lambda^2 + c(x)^2 + 2\epsilon} = 0$$

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Objective of the paper

Objective of the paper

Adapt primal-dual interior point methods from numerical optimization to pure state and input constrained optimal control.

State of the art

- M. Weiser. *Interior Point Methods in Function Space*, SIAM Journal on Control and Optimization, 2005.
- J.F. Bonnans, Th. Guilbaud. *Using logarithmic penalties in the shooting algorithm for optimal control problems*, Optimal Control Applications and Methods, 2003.
- P. Malisani, F. Chaplais, N. Petit, *An interior penalty method for optimal control problems with state and input constraints of nonlinear systems*, Optimal Control Applications and Methods, 2014.

Outline

State and Input constrained optimal control problem

Problem Statement

$$\min_{u\in\mathcal{U}} J(u) = \int_0^T \ell(x_u(t), u(t))dt + \varphi(x_u(T))$$
$$\dot{x}_u(t) = f(x_u(t), u(t))$$
$$x_u(0) = x^0$$
$$\mathcal{U} := L^\infty([0,T]; U_{ad} \subset \mathbb{R}^m)$$
$$g(x_u(t)) \leq 0$$

Where $T > 0$, x^0 are fixed and x_u is the solution of the ODE with control u. And we define the classical pre-Hamiltonian function $H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \mapsto R$ as follows

$$H(x,u,p) := \ell(x,u) + p.f(x,u)$$

Main assumptions

Assumptions

(A1) The functions $\ell : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}$, $f : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ and $g_i : \mathbb{R}^n \mapsto \mathbb{R}$ are at least twice continuously differentiable.

(A2) The dynamics $f : \mathbb{R}^n \times \mathbb{R}^m$ from satisfies a sublinear growth property

$$\exists D < +\infty, \forall x \in \mathbb{R}^n, \forall u \in [-1,1]^m \text{ s.t. } \parallel f(x,u) \parallel \leq D(1+ \parallel x \parallel)$$

(A3) Interior accessibility

$$\{u \text{ s.t. } g(x_u) \leq 0\} \subseteq \mathrm{cl}_{L^1}(\{u \text{ s.t. } g(x_u) < 0\})$$

(A4) The OCP has a unique solution u^* and $\exists \beta \geq 0$ and $r > 0$ such that

$$J(u) - J(u^*) \geq \beta \left\| u - u^* \right\|_{L^2}^2, \quad \forall u \in B_{L^2}(u^*, r) \cap \mathcal{U}$$

Outline

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First order optimality conditions

State constrained Pontryagin Maximum Principle

Any optimal solution $(\bar{x}_u, \bar{u})$ of the presented COCP is a Pontryagin extremal, i.e. $(\bar{x}_u, \bar{u}, \bar{p}, \bar{\mu})$ is solution of

$$\begin{aligned}
\dot{\bar{x}}_u(t) &= f(\bar{x}_u(t), \bar{u}(t)) \\
-d\bar{p}(t) &= H'_x(\bar{x}_u(t), \bar{u}(t), \bar{p}(t))dt + g'(\bar{x}_u(t))d\bar{\mu}(t) \\
\bar{x}_u(0) &= x^0 \\
\bar{p}(T) &= \varphi'(\bar{x}_u(T)) \\
H(\bar{x}_u(t), \bar{u}(t), \bar{p}(t)) &= \inf_{v \in U_{ad}} H(\bar{x}_u(t), v, \bar{p}(t)) \\
d\bar{\mu} &\geq 0 \\
\int_0^T g(\bar{x}_u(t))d\bar{\mu}(t) &= 0
\end{aligned}$$

Outline

Penalized problem

Log-barrier OCP

Let $\epsilon > 0$, the penalized optimal control problem writes

$$\begin{aligned}
\min_{u\in\mathcal{U}} J_\epsilon(u) &= \int_0^T \ell(x_u(t), u(t)) - \epsilon \log(-g(x_u(t)))dt + \varphi(x_u(T)) \\
\dot{x}_u(t) &= f(x_u(t), u(t)) \\
x_u(0) &= x^0 \\
\mathcal{U} &:= L^\infty([0,T]; U_{ad})
\end{aligned}$$

and the corresponding penalized pre-Hamiltonian is

$$H[\epsilon](x,u,p) = H(x,u,p) - \epsilon \log(-g(x))$$

Statement of main result

Main result (1/2)

Any Pontryagin extremal $(x_{u,\epsilon}, u_\epsilon, p_\epsilon)$ of the penalized problem, i.e. solution of

$$\begin{aligned} \dot{x}_{u,\epsilon}(t) &= f(x_{u,\epsilon}(t), u_\epsilon(t)) \\ \dot{p}_\epsilon(t) &= -H[\epsilon]'_x(x_{u,\epsilon}(t), u_\epsilon(t), p_\epsilon(t)) \\ x_{u,\epsilon}(0) &= x^0 \\ p_\epsilon(T) &= \varphi'(x_{u,\epsilon}(T)) \\ H[\epsilon](x_{u,\epsilon}(t), u_\epsilon(t), p_\epsilon(t)) &= \inf_{v \in U_{ad}} H[\epsilon](x_{u,\epsilon}(t), v, p_\epsilon(t)) \end{aligned}$$

Statement of main result

Main result (2/2)

converges to $(\bar{x}_u, \bar{u}, \bar{p}, \bar{\mu})$ to a Pontryagin extremal of the original problem as follows

$$\lim_{\epsilon \downarrow 0} \|u_\epsilon - \bar{u}\|_{L^2} = 0$$

$$\lim_{\epsilon \downarrow 0} \|x_{u,\epsilon} - \bar{x}_u\|_{L^\infty} = 0$$

$$\lim_{\epsilon \downarrow 0} \|p_\epsilon - \bar{p}\|_{L^1} = 0$$

$$\lim_{\epsilon \downarrow 0} \frac{-\epsilon}{g(x_{u,\epsilon})} dt \overset{*}{\rightharpoonup} d\bar{\mu}$$

Outline

Preliminary Results (1/3)

State Lipschitz continuity

From the sublinear growth property, $\forall u_1, u_2 \in \mathcal{U}$, $\exists \text{const}(f) < +\infty$ such that

$$\| x_{u_1} - x_{u_2} \|_{L^\infty} \leq \text{const}(f) \| u_1 - u_2 \|_{L^1}$$

State-constraint measure

For all $u \in \mathcal{U}$ and For all $E \subseteq g \circ x[u]([0,T])$ we note $m[u,g]$ the push-forward g-measure of E defined as follows

$$m[u,g](E) := \text{meas}\left((g \circ x_u)^{-1} (E) \right)$$

Proposition

For all $u \in \mathcal{U}$, let $E \subseteq g \circ x_u([0,T]) \subset \mathbb{R}$ be a Lebesgue-measurable set, the g-measure is lower bounded as follows

$$m[u,g](E) \geq \text{const}(f,g)\text{meas}(E)$$

Preliminary results (2/3)

Set of state saturated control

Let us define the set of saturated-state control $\mathcal{U}_g^0$ as follows

$$\mathcal{U}_g^0 := \{u \in \mathcal{U} \text{ s.t. } \sup_t g(x_u(t)) = 0\}$$

Set of near state-saturated times

For all u s.t. $g(x_u) \leq 0$ and $\forall \delta \geq 0$ we define the set of near state-saturated times, noted $S_u(\delta)$ as follows

$$S_u(\delta) := (g \circ x_u)^{-1} ([-\delta, +\infty))$$

Preliminary results (3/3)

Proposition

There exists $\Gamma_g > 0$ and $v \in \mathcal{U}$ satisfying

$$\sup_t g(x_v(t)) \leq -2\Gamma_g \tag{1}$$

Thus, for all $u \in \mathcal{U}_g^0$ this yields

$$g(x_v(t)) \leq g(x_u(t)) - \Gamma_g, \ \forall t \in S_u(\Gamma_g) \tag{2}$$

Proof

Let $\delta > 0$

$$\Gamma_g = -\frac{1}{2} \sup_{u \in \mathcal{U}_g^0} \left\{ \inf_{v \in B_{L^1}(u,\delta) \cap \mathcal{U}} \left\{ \sup_t g(x[v](t)) \right\} \right\}$$

Outline

State penalized optimal control

State-penalized problem

$$\min_u G_\epsilon(u) = \int_0^T \ell(x_u(t), u(t)) - \epsilon \log(-g(x_u(t)))dt$$

$$\dot{x}_u(t) = f(x_u(t), u(t))$$

$$x_u(0) = x^0$$

$$u \in \mathcal{U}$$

State Interiority

Lemma

Any optimal solution of state penalized problem u_ϵ satisfies

$$g(x_{u_\epsilon}(t)) < 0, \forall t \in [0, T]$$

and $\exists K_\psi < +\infty$ such that $\forall \epsilon \in (0, \epsilon_0)$ one has

$$\left\| \frac{\epsilon}{g(x_{u_\epsilon})} \right\|_{L^1} \leq K_\psi$$

State Interiority proof (1/6)

$$G_\epsilon(v) - G_\epsilon(u_\epsilon) = \Delta_\ell(u_\epsilon, v) + \epsilon\Delta_{\log}(u_\epsilon, v)$$

with

$$\Delta_\ell(u_\epsilon, v) = \int \ell(x_v, v) - \ell(x_{u_\epsilon}, u_\epsilon)dt$$

$$\Delta_{\log}(u_\epsilon, v) = \int -\log(-g(x_v)) + \log(-g(x_{u_\epsilon}))dt$$

State Interiority proof (2/6)

$$\begin{aligned}\Delta_\ell(u_\epsilon, v) &= \int \ell(x_v, v) - \ell(x_{u_\epsilon}, u_\epsilon)dt \\ &\le \int \mathrm{const}(\ell)(\|x_v(t) - x_{u_\epsilon}(t)\| + \|v(t) - u_\epsilon(t)\|)dt \\ &\le \mathrm{const}(\ell, f, T, \Gamma_g)\end{aligned}$$

$$\begin{aligned}\epsilon\Delta_{\log}(u_\epsilon, v) =& \epsilon\int_{S_{u_\epsilon}(\Gamma_g)} -\log(-g(x_v)) + \log(-g(x_{u_\epsilon}))dt \\ &+ \epsilon\int_{[0,T]\setminus S_{u_\epsilon}(\Gamma_g)} -\log(-g(x_v)) + \log(-g(x_{u_\epsilon}))dt \\ :=& \epsilon\Delta_S + \epsilon\Delta_{S^c}\end{aligned}$$

State Interiority proof (3/6)

$$
\begin{aligned}
\epsilon\Delta_{S^c} \leq & \text{const}(g, f, T, \Gamma_g, \epsilon_0, \psi) \\
\epsilon\Delta_S = & \epsilon \int_{S_{u_\epsilon}(\Gamma_g)} -\log(-g(x_v)) + \log(-g(x_{u_\epsilon}))dt \\
= & \epsilon \int_{S_{u_\epsilon}(\Gamma_g)} -\frac{1}{g(x[u_\epsilon]))} \left(g(x[v]) - g(x[u_\epsilon])\right) dt \\
\leq & -\epsilon\Gamma_g \int_{S_{u_\epsilon}(\Gamma_g)} -\frac{1}{g(x[u_\epsilon]))} dt
\end{aligned}
$$

State Interiority proof (4/6)

Now, let us prove interiority of u_ϵ by contradiction. Assume u_ϵ an optimal solution such that $\forall \rho \in (0, \Gamma_g)$, $(g \circ x_{u_\epsilon})^{-1}((-\Gamma_g, -\rho]) \subset [0, T]$

$$\epsilon \int_{S_{u_\epsilon}(\Gamma_g)} -\frac{1}{g(x[u_\epsilon]))} dt \geq \epsilon \int_{(g \circ x[u_\epsilon])^{-1}((-\Gamma_g, -\rho))} -\frac{1}{g(x[u_\epsilon]))} dt$$
$$= \epsilon \int_{-\Gamma_g}^{-\rho} -\frac{1}{s} m[u_\epsilon, g](ds)$$

Using the lower bound on the state-constraint measure

$$\epsilon \int_{S_{u_\epsilon}(\Gamma_g)} -\frac{1}{g(x[u_\epsilon]))} dt \geq \epsilon \mathrm{const}(f, g) \int_{-\Gamma_g}^{-\rho} -\frac{1}{s} ds$$
$$\geq \epsilon \mathrm{const}(f, g) \left(-\log(\rho) + \log(\Gamma_g)\right)$$

State Interiority proof (5/6)

For ρ small enough

$$G_\epsilon(v) - G_\epsilon(u_\epsilon) \leq \text{const}(\ell, f, g, T, \Gamma_g, \epsilon_0, \psi) + \epsilon \text{const}(f, g, \Gamma_g) \log(\rho) < 0$$

which contradicts the optimality of u_ϵ and proves interiority.

State Interiority proof (6/6)

Let us prove the L^1-boundedness by contradiction. Assume u_ϵ optimal such that $\forall K_\psi > 0, \exists \epsilon > 0, \|\epsilon \psi' \circ g(x[u_\epsilon])\|_{L^1} \geq K_\psi$. First, we have

$$(g \circ x[u_\epsilon]))^{-1}(\{0\}) = \emptyset$$

thus

$$\left\| \frac{\epsilon}{g(x[u_\epsilon]))} \right\|_{L^1} := \lim_{\rho \to 0} \int_{-\infty}^{-\rho} \frac{\epsilon}{g(x[u_\epsilon]))} m[u_\epsilon, g](ds) > K_\psi + \frac{\epsilon_0 T}{\Gamma_g}$$

Finally, we have:

$$G_\epsilon(v) - G_\epsilon(u_\epsilon) \leq \text{const}(\ell, f, g, T, \Gamma_g, \psi, \epsilon_0) - \Gamma_g K_\psi$$

Then $\exists K_\psi > 0$, such that $G_\epsilon(v) - G_\epsilon(u_\epsilon) < 0$, which contradicts the optimality of u_ϵ.

Outline

Primal variables convergence

Primal variables

Primal variables convergence is a well established result

$$\lim_{\epsilon \downarrow 0} \|u^* - u_\epsilon\|_{L^2} = 0$$

$$\lim_{\epsilon \downarrow 0} \|x_{u^*} - x_{u_\epsilon}\|_{L^\infty} = 0$$

Convergence of state penalties (1/2)

Convergence of state penalties

The sequence $(\frac{\epsilon_n}{g(x_{u_{\epsilon_n}})})_n$ is uniformly L^1-bounded by K_ψ. Thus

$$\forall \phi \in L^\infty, |T_{u_{\epsilon_n}}(\phi)| \leq K_\psi \|\phi\|_{L^\infty}$$

hence $\|T_{u_{\epsilon_n}}\|_{\mathcal{M}} \leq K_\psi$. By weak $*$ compacity of the unit ball of $\mathcal{M}([0,T])$, $\exists \mu \in \mathcal{M}([0,T])$ such that

$$\lim_{k\to+\infty} -\frac{\epsilon_{n_k}}{g(x_{u_{\epsilon_{n_k}}})} dt \overset{*}{\rightharpoonup} d\mu$$

Convergence of state penalties (2/2)

Complementarity conditions

Positivity of μ

$$-\frac{\epsilon_n}{g(x_{u_{\epsilon_n}})} \geq 0 \Rightarrow d\mu \geq 0$$

Complementarity conditions

$$\begin{aligned}
\langle \mu, g(x[u^*]) \rangle &= \lim_{n \to +\infty} \int -\frac{\epsilon_n}{g(x_{u_{\epsilon_n}})} g(x_{u^*}) dt \\
&= \lim_{n \to +\infty} \int -\frac{\epsilon_n}{g(x_{u_{\epsilon_n}})} g(x_{u_{\epsilon_n}}) dt \\
&= \lim_{n \to +\infty} -\epsilon_n T \\
&= 0
\end{aligned}$$

Convergence of adjoint state

The optimal adjoint state satisfies the following ODE

$$-dp^* = \left(\ell'_x(x[u^*],u^*) - f'_x(x[u^*],u^*).p^*\right)dt + g'(x[u^*])d\mu$$

with $p^*(T)=0$ and the penalized adjoint state is solution of

$$\begin{aligned}\dot{p}[u_{\epsilon_n}] = & -\ell'_x(x[u_{\epsilon_n}],u_{\epsilon_n}) - f'_x(x[u_{\epsilon_n}],u_{\epsilon_n}).p[u_{\epsilon_n}] \\ & + \epsilon g'(x[u_{\epsilon_n}](t))\frac{1}{g(x[u_{\epsilon_n}])} \\ p[u_{\epsilon_n}](T) = 0 &\end{aligned}$$

Then $p[u_{\epsilon_n}]$ converges pointwise to p^*, and the boundedness of both $p[u_{\epsilon_n}]$ and p^* proves $\lim_{n\to+\infty}\|p[u_{\epsilon_n}]-p^*\|_{L^1}$

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Primal Dual TPBVP

Primal Dual TPBVP

Any solution $(x_{u_\epsilon^*}, u_\epsilon^*, p_\epsilon^*, \lambda_\epsilon^*)$ of the following Primal Dual TPBVP

$$\begin{aligned}
\dot{x}_{u_\epsilon^*} &= f(x_{u_\epsilon^*}, u_\epsilon^*) \\
\dot{p}_\epsilon^* &= -H'_x(x_{u_\epsilon^*}, u_\epsilon^*, p_\epsilon^*) - \lambda_\epsilon^* g'(x_{u_\epsilon^*}) \\
H(x_{u,\epsilon}, u_\epsilon^*, p_\epsilon^*) &= \inf_{v \in U_{ad}} H(x_{u,\epsilon}, v, p_\epsilon^*) \\
0 &= \lambda_\epsilon^* - g(x_{u_\epsilon^*}) - \sqrt{\lambda_\epsilon^{*2} + g(x_{u_\epsilon^*})^2 + 2\epsilon} \\
x[u_\epsilon^*](0) &= x^0 \quad ; \quad p_\epsilon^*(T) = \varphi'(x_{u_\epsilon^*}(T))
\end{aligned}$$

converges to (x_{u^*}, u^*, p^*) as follows

$$\lim_{\epsilon \downarrow 0} \|u_\epsilon^* - u^*\|_{L^2} = 0 \; ; \; \lim_{\epsilon \downarrow 0} \left\|x_{u_\epsilon^*} - x[u^*]\right\|_{L^\infty} = 0 \quad ; \quad \lim_{\epsilon \downarrow 0} \|p_\epsilon^* - p^*\|_{L^1} = 0$$

$$\lim_{\epsilon \downarrow 0} \lambda_\epsilon^* dt \overset{*}{\rightharpoonup} d\mu$$

Outline

Solving Algorithm

We note

$$S(\epsilon) = (x_{u_\epsilon^*}, u_\epsilon^*, p_\epsilon^*, \lambda_\epsilon^*)$$

a solution of the primal-dual TPBVP.

Primal dual Algorithm

1: Define $\epsilon_0 > 0,\ \alpha \in (0,1),\ \mathrm{tol} = o(1),\ k = 0$
2: **while** $\epsilon_k > \mathrm{tol}$ **do**
3: $\quad S(\epsilon_{k+1}) \leftarrow$solution of primal-dual TPBVP initialized with $S(\epsilon_k)$
4: $\quad \epsilon_{k+1} \leftarrow \alpha\epsilon_k$
5: $\quad k \leftarrow k+1$
6: **end while**
7: **return** $S(\epsilon_k)$

Outline

Battery management problem

Optimal control problem

$$\min_u \int_0^T \mathrm{spot}(t)\left(\max\{p_{\mathrm{compteur}}(t);0\} - \min\{p_{\mathrm{compteur}}(t);0\}\right)dt$$

under the following constraints

$$\begin{aligned}
p_{\mathrm{compteur}} &:= \mathrm{cons} - PV + \frac{1}{\rho_c}\max\{u;0\} + \rho_d \min\{u;0\} \\
\dot{x}(t) &= u(t) \\
x(0) &= \frac{x^+}{2} \\
x(t) &\in [0;x^+] \\
u(t) &\in [u^-;\rho_c u^+] \\
T &:= 8760h
\end{aligned}$$

Comparison with discretize then optimize approach

Discretize and optimize

- Numerical scheme : Lobatto IIIa
- Solver : IPOPT+ MUMPS
- Gradients and jacobians provided and vectorized
- Computation time $\approx$ 9500s

Primal dual OCP

- Numerical scheme: Lobatto IIIa
- Residual error on ODEs: 10^{-3}
- Linear solver: SPLU + umfpack
- Gradients and jacobians provided and vectorized
- Computation time $\approx$ 25s

Outline

Conclusion and perspectives

- Extend theoritical results to mixed state and input constraints and to more general input constraints
- Provide a predictor-corrector in function space type algorithm to automatically adapt the weighting parameter of IPMs.