



High-order flux reconstruction schemes for combustion and AMR

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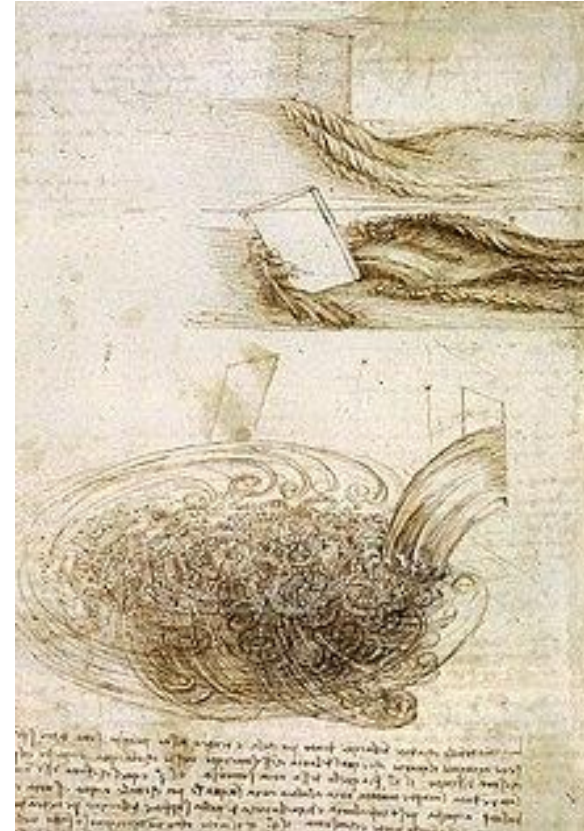
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CONTEXT

- The modeling of turbulent combustion requires an accurate numerical **method** and **modeling** of the effects of turbulence
- For Large Eddy Simulation, it is necessary to use **high-order** methods in order to reduce **numerical dissipation**

Objectives of the thesis:

Developing high-order numerical methods for LES modeling of low-dispersion/dissipation combustion



Leonard de Vinci

HIGH-ORDER METHODS: STATE OF THE ART

- Discontinuous Galerkin by Reed and Hill 1973
 - weak formulation, mass matrix bloc-diagonal
 - expensive if standard Gaussian quadrature rules are employed
- Spectral Difference by Kopriva and Kolas 1996
 - strong formulation, exponential convergence for smooth solutions
 - difficult to implement on complex geometries
- **Flux Reconstruction** by Huynh 2007
 - strong formulation
 - set of schemes including collocation-based nodal DG and SD schemes in linear cases with many properties (stability, dissipation, CFL)

$$\|U_{approx} - U_{exact}\|_{L^2} = \mathcal{O}(\Delta x^{p+1})$$

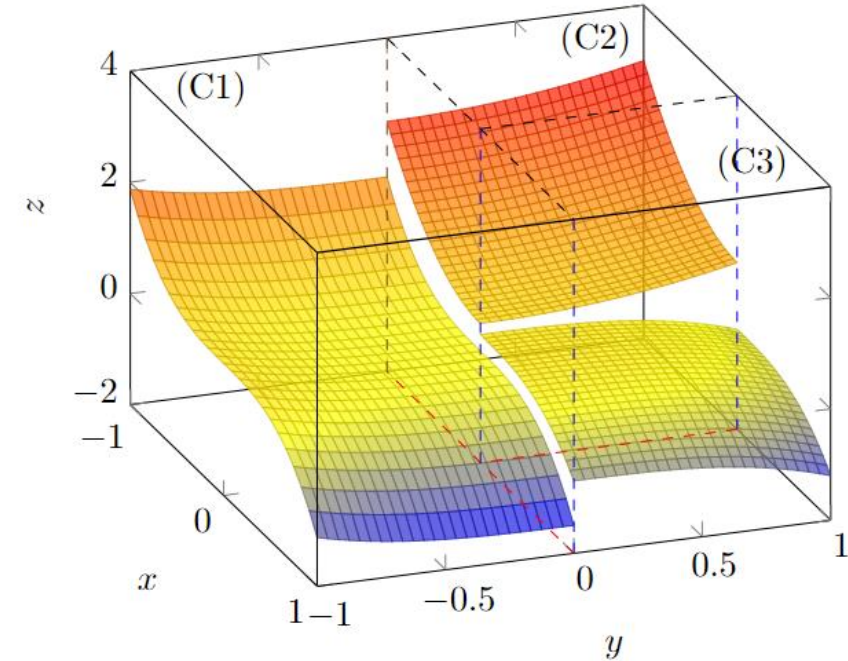
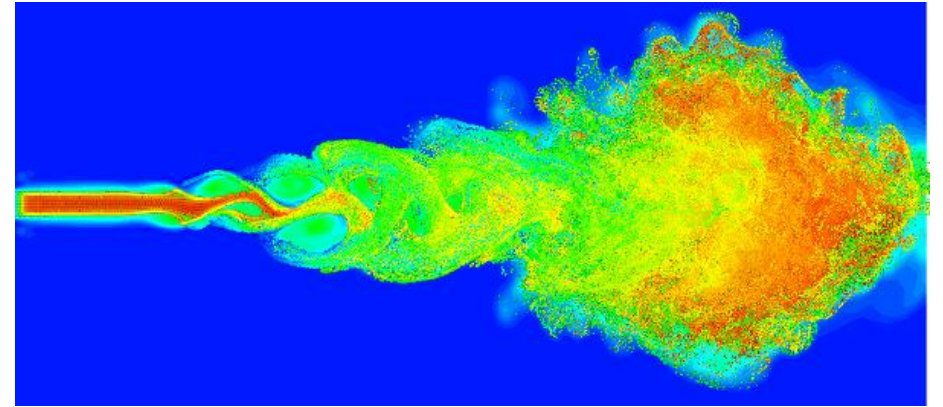


Figure 2 : Representation of a **discontinuous solution** over three cells

HIGH-ORDER METHODS: APPLICATIONS

- Discontinuous Galerkin by Reed and Hill 1973
 - turbulent jets, Anghan et al.[1]
 - laminar-turbulent transition, Beck et al. [2]
- Spectral Difference by Kopriva and Kolas 1996
 - turbulent flows limited to the walls, Chapelier et al.[3]
 - shocks and detonations, Gupta et al. [4]
 - SD method for laminar and turbulent combustion, T. Marchal et al [5]
- Flux Reconstruction by Huynh 2007
 - flows in turbomachines, flow around a wing and a cylinder Vincent, Jameson et al [6]
 - double flux method to the FR discretization of the multi-species Euler equations, Peyvan et al [7]
 - **no work on combustion**



Development of turbulent flow in a jet

OUTLINE

- **Principles of Flux Reconstruction method**
- Development of FR schemes for non-reactive 3D flows in AEROSOL
- Development of FR schemes for non-reactive and reactive 3D flows in CONVERGE
- Development of a coupling methodology for FR schemes and AMR in CONVERGE
- Conclusion and future works

PRINCIPLES OF FLUX RECONSTRUCTION METHOD FOR $\partial_t u + \partial_x f(u) = 0$

1. Write conservation laws on each Ω_n , $\Omega = \cup_{n=1}^N \Omega_n$

- $\partial_t u_n^{\delta D} + \partial_x f_n^\delta(u_n^\delta) = 0$

2. Isoparametric map $\Theta_n: \Omega_n \longrightarrow \Omega_s$

- $\partial_t \hat{u}_n^{\delta D} + \partial_\zeta \hat{f}_n^\delta(\hat{u}_n^\delta) = 0$

- $\hat{u}_n^{\delta D} = J_n u_n^{\delta D}, \quad \hat{f}_n^\delta = f_n^\delta$

3. Polynomial approximation

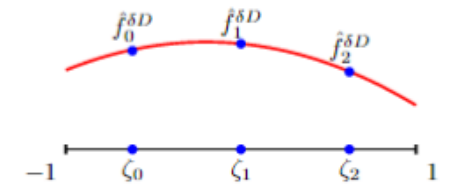
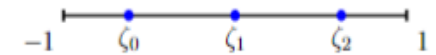
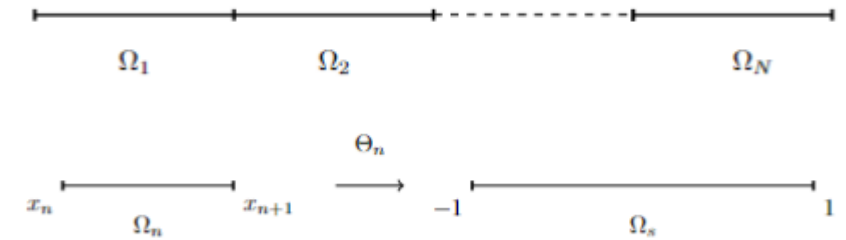
- $p + 1$ solution points $\zeta_k / k = 0, \dots, p$

- Interpolation $(\zeta_k, \hat{u}^{\delta D}(t, \zeta_k)) = (\zeta_k, \hat{u}_k^{\delta D})$

$$\hat{u}^{\delta D}(\zeta) = \sum_{0 \leq k \leq p} \hat{u}_k^{\delta D} l_k(\zeta), \quad \hat{f}^{\delta D}(\zeta) = \sum_{0 \leq k \leq p} \hat{f}_k^{\delta D} l_k(\zeta)$$

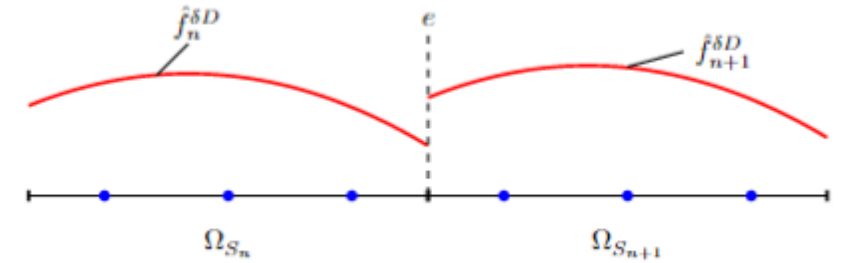
degree p

degree p



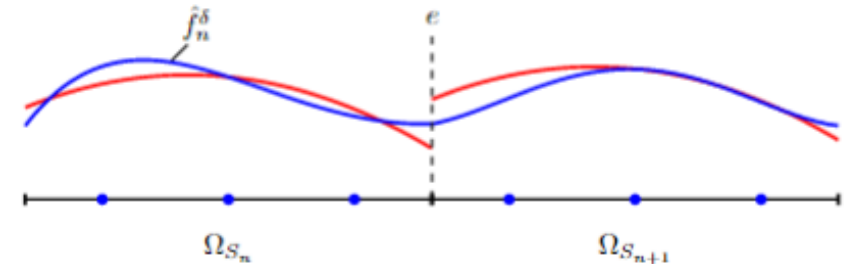
PRINCIPLES OF FLUX RECONSTRUCTION METHOD

- $\hat{u}^{\delta D}$ and $\hat{f}^{\delta D}$ may be discontinuous at the interfaces



- The transfer of information is ensured by **flux reconstruction**

$$\hat{f}_n^\delta(\zeta) = \underbrace{\hat{f}_n^{\delta D}(\zeta) + \hat{f}_n^{\delta C}(\zeta, g_L, g_R)}_{\text{degree } p + 1}$$



PRINCIPLES OF FLUX RECONSTRUCTION METHOD

- Vincent-Castonguay-Jameson-Huynh VCJH correction functions

- $$g_L = \frac{(-1)^p}{2} \left(L_p - \frac{\eta_p L_{p-1} + L_{p+1}}{1 + \eta_p} \right), \quad g_R = \frac{1}{2} \left(L_p + \frac{\eta_p L_{p-1} + L_{p+1}}{1 + \eta_p} \right)$$

- L_p : Legendre polynomial of degree p

- $$\eta_p = \frac{c(2p+1)(a_p p!)^2}{2}, \quad a_p = \frac{2p!}{2^p (p!)^2}$$

- Set of schemes parameterized by c : $\frac{-2}{(2p+1)(a_p p!)^2} < c < +\infty$

- $c = 0$: **Nodal Discontinuous Galerkin** scheme

- $c = \frac{2p}{(2p+1)(p+1)(a_p p!)^2}$: **Spectral Difference** scheme

- $c = \frac{2(p+1)}{(2p+1)p(a_p p!)^2}$: **Huynh** scheme

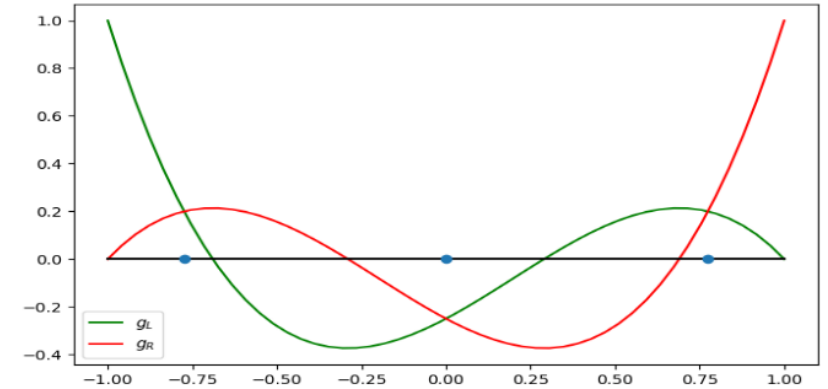


Figure 3 : VCJH Correction functions for $c = 0$ and $p = 2$

OUTLINE

- Principles of Flux Reconstruction method
- **Development of FR schemes for non-reactive 3D flows in AEROSOL**
- Development of FR schemes for non-reactive and reactive 3D flows in CONVERGE
- Development of a coupling methodology for FR schemes and AMR in CONVERGE
- Conclusion and future works

DEVELOPMENT OF FR SCHEMES IN AEROSOL

General form of the Navier-Stokes equations

$$\partial_t \mathbf{U} + \nabla \cdot \mathbf{F}_{\text{conv}}(\mathbf{U}) + \nabla \cdot \mathbf{F}_{\text{vis}}(\mathbf{U}, \nabla \mathbf{U}) = \mathbf{S}, \quad (t, \mathbf{x}) \in [0; T_f] \times \Omega$$
$$\mathbf{U} = \mathbf{g}, \quad (t, \mathbf{x}) \in [0, T_f] \times \Gamma$$

$$\mathbf{U} = (\rho, \rho \mathbf{V}, \rho E)$$

$$\mathbf{F}_{\text{conv}} = (\rho \mathbf{V}, \rho \mathbf{V} \otimes \mathbf{V} + p \mathbf{I}, \rho E \mathbf{V} + p \mathbf{V})$$

$$\mathbf{F}_{\text{vis}} = (0, -\nabla \cdot \boldsymbol{\tau}, -(\nabla \cdot (\boldsymbol{\tau} \cdot \mathbf{V}) + \nabla \cdot (\kappa \nabla T)))$$

$$\tau_{ij} = \nu (\partial_{x_i} V_j + \partial_{x_j} V_i + \lambda \delta_{ij} \nabla \cdot \mathbf{V})$$

$$E = \frac{P}{\rho(\gamma - 1)} + \frac{1}{2} (V_1^2 + V_2^2 + V_3^2)$$

NUMERICAL EXPERIMENTS: NAVIER-STOKES EQUATIONS 2D

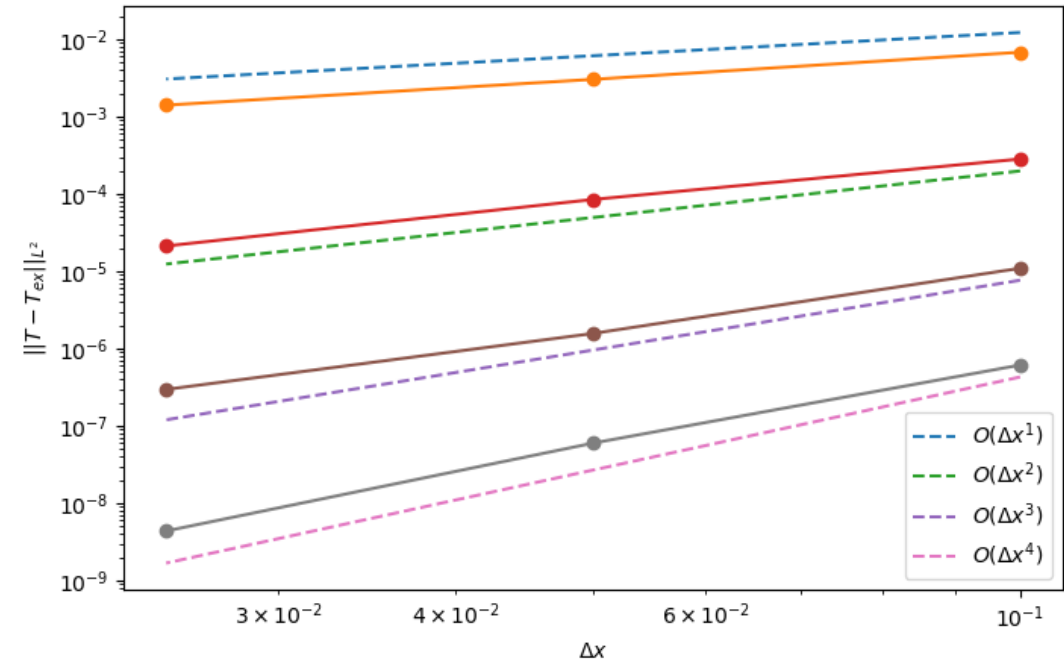
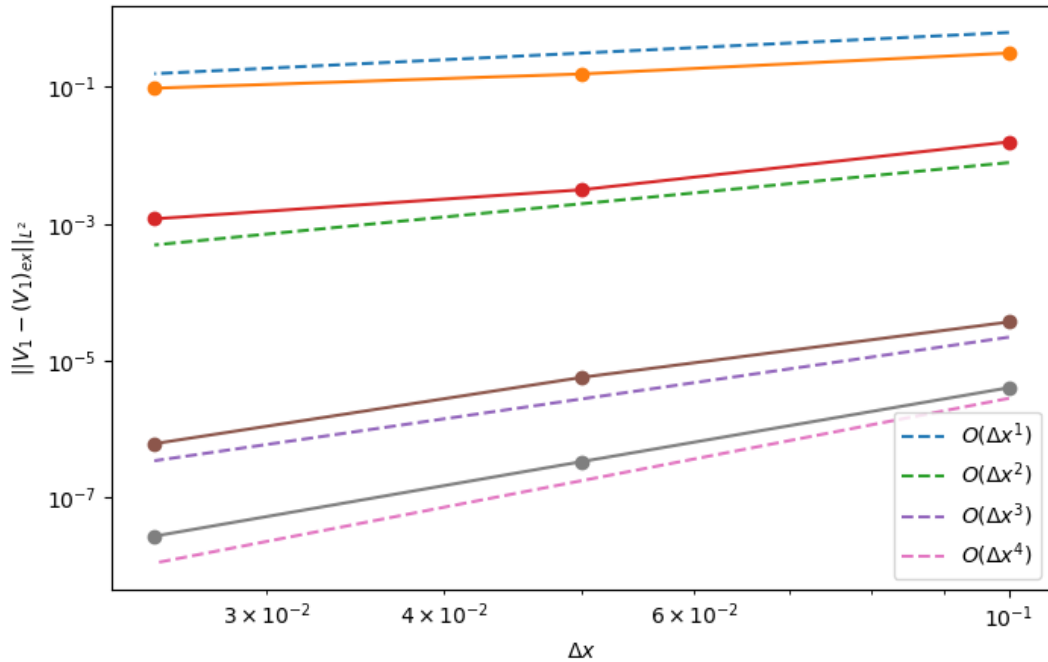
Poiseuille Flow

- Source term: $\mathbf{S} = (0, F_V, F_V \cdot V)$
 - $F_V = (8 \nu u_c, 0)$
 - $F_V \cdot V = 32 \nu u_c^2 y(1 - y)$
- Isothermal boundary conditions : $(V_1, V_2, T) = (0, 0, T_W)$
 - $g = (\rho_{ref}, 0, 0, \rho_{ref} c_v T_W)$
- Exact solution
 - $V = (4u_c y(1 - y), 0)$
 - $T(x, y) = T_W \left[1 + \frac{16 Pr Ma^2 (\gamma - 1)}{3} y(1 - y) \left(y^2 - y + \frac{1}{2} \right) \right]$
- Discretisation
 - $\Omega = [0, 1] \times [0, 1], T_W = 1, u_c = 1, \rho_{ref} = 0.001, \nu = 0.001, Pr = 0.7, Ma = 0.2, Re = 1$
 - Numerical flux Lax-Friedrichs (advective), LdG (viscous); RK45, DG via FR
 - $CFL = \frac{\delta_t \nu}{\delta_x^2}, p \in \{0, 1, 2, 3\}, N^2 \in \{10^2, 20^2, 40^2\}$

NUMERICAL EXPERIMENTS: NAVIER-STOKES EQUATIONS 2D

Convergence curves with DG via FR

- $p \in \{0,1,2,3\}$
- $N^2 \in \{10^2, 20^2, 40^2\}$



$$\|U - U^\delta\|_{L^2} = O(\Delta x^{p+1})$$

NUMERICAL EXPERIMENTS: NAVIER-STOKES EQUATIONS 3D

Taylor-Green Vortex (TGV)

- Initial condition

- $V_1 = u_0 \sin\left(\frac{x}{L_0}\right) \cos\left(\frac{y}{L_0}\right) \cos\left(\frac{z}{L_0}\right)$
- $V_2 = -u_0 \cos\left(\frac{x}{L_0}\right) \sin\left(\frac{y}{L_0}\right) \cos\left(\frac{z}{L_0}\right)$
- $V_3 = 0$
- $P = P_0 + \frac{\rho_0 V_0^2}{16} \left(\cos\left(\frac{2x}{L_0}\right) + \cos\left(\frac{2y}{L_0}\right) \right) \left(\cos\left(\frac{2z}{L_0}\right) + 2 \right)$

- Boundary condition

- Periodic

- Discretisation

- Numerical flux Lax-Friedrichs (advective), LdG (viscous); RK45, DG via FR
- $\Omega = [-\pi L_0, \pi L_0] \times [-\pi L_0, \pi L_0] \times [-\pi L_0, \pi L_0], T_f = 12s.$
- $L_0 = \frac{1}{\pi}, u_0 = \frac{1}{\pi}, \rho_0 = 1, T_0 = 1, Pr = 0.71, Ma = 0.1, Re = 1600.$

NUMERICAL EXPERIMENTS: TGV at Re = 1600

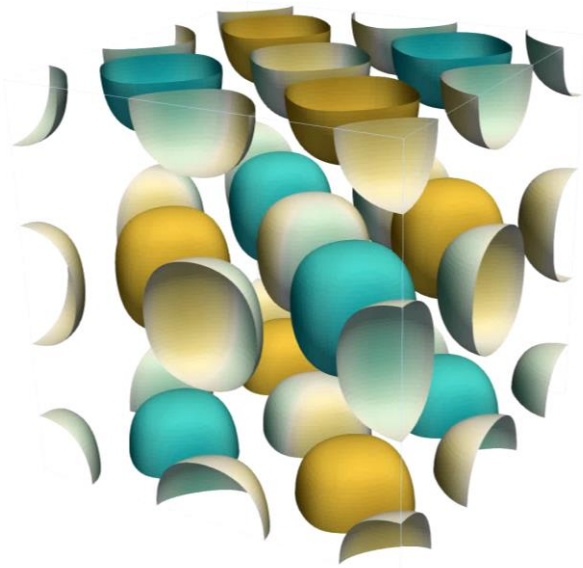
Isosurface of Q criterion colored by the z-vorticity

- Order = 4
- $N^3 = 96^3$

h,p -refinement

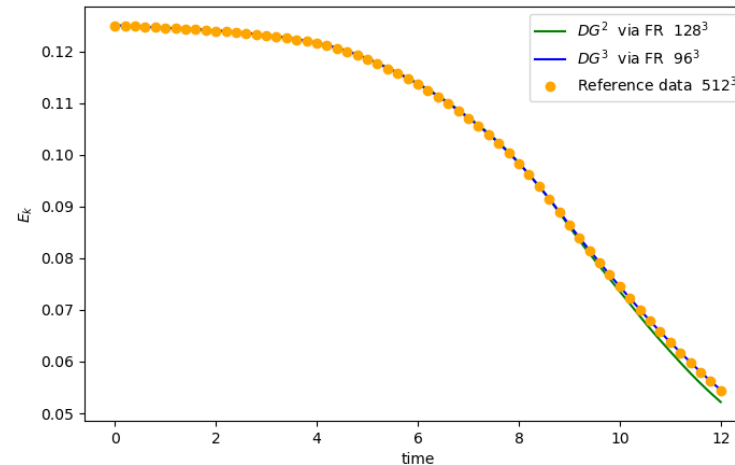
- Order $\in \{3,4\}$
- $N^3 \in \{96^3, 128^3\}$

Time: 0.000000



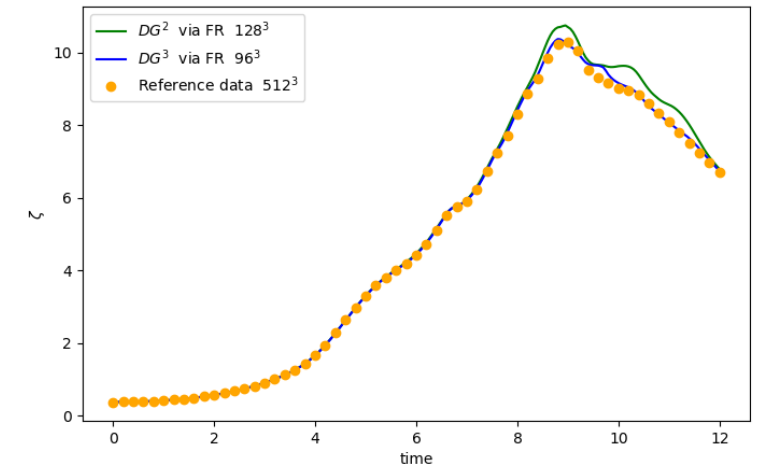
Kinetic Energy

$$E_k = \frac{1}{K} \int_K \frac{V \cdot V}{2}$$



Enstrophy

$$\zeta = \frac{1}{K} \int_K \frac{(\nabla \times V) \cdot (\nabla \times V)}{2}$$



- Good agreement with reference data and with less DoF
- The results are better with high-order for constant number of DoF

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NAVIER-STOKES EQUATIONS: GLOBAL CONSERVATION OF MASS

General form of Navier-Stokes equations

- $\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (t, x) \in [0; T_f] \times \Omega$
- $\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + P\mathbf{I}) + \nabla \cdot (-\boldsymbol{\tau}) = 0, \quad (t, x) \in [0; T_f] \times \Omega$
- $\partial_t (\rho E) + \nabla \cdot (\rho E \mathbf{u} + P\mathbf{u}) + \nabla \cdot \left(-((\boldsymbol{\tau} \cdot \mathbf{u}) + (\kappa \nabla T) - \rho \sum_k h_{sk} \mathbf{Y}_k \mathbf{V}_k) \right) = \dot{\omega}_T, \quad (t, x) \in [0; T_f] \times \Omega$
- $\partial_t \rho_k + \nabla \cdot (\rho_k \mathbf{u}) = \nabla \cdot \left(\rho D_k \frac{W_k}{W} \nabla X_k - \rho_k \mathbf{V}^c \right) + \dot{\omega}_k, \quad (t, x) \in [0; T_f] \times \Omega, \quad k \in \{1, 2, \dots, N_S\}$

$$Y_k \mathbf{V}_k = -D_k \frac{W_k}{W} \nabla X_k$$

$$\mathbf{V}^c = \sum_j \frac{D_j W_j}{W} \nabla X_j$$

$$\frac{W_k}{W} \nabla X_k = \nabla Y_k - Y_k W \sum_S \frac{\nabla Y_S}{W_S}$$

$$E = e_s + \frac{1}{2} \mathbf{u}^2,$$

$$h_s = \sum Y_k h_{sk}$$

$$h_{sk} = \Delta h_{f,k}^0 + \int_{T_0}^T C_{pk}(T') dT'$$

CHALLENGES OF FR SCHEMES IN MULTI-SPECIES SIMULATION

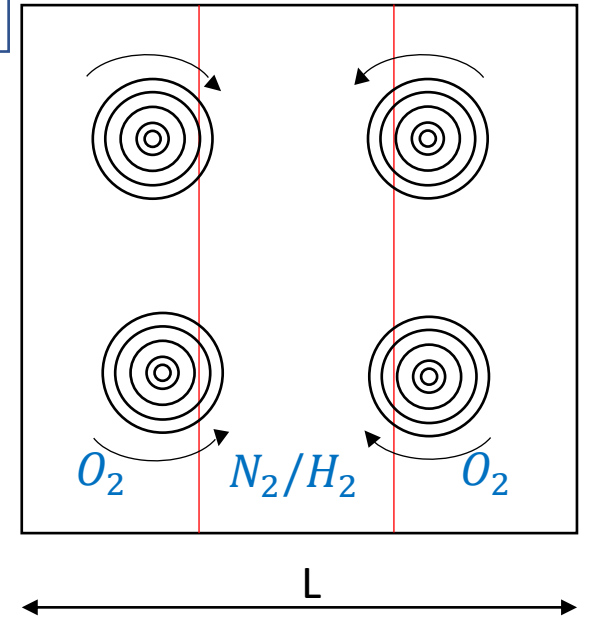
HOW TO COMPUTE NUMERICAL FLUXES $F^{\delta I}$?

- Conservative variables: $U = (\rho, \rho V_1, \rho V_2, \rho V_3, \rho E, \rho Y_1, \rho Y_2, \dots, \rho Y_{N_s})$
- Primitives variables: $Q = (P, V_1, V_2, V_3, T, Y_1, Y_2, \dots, Y_{N_s})$
- CONS approach $U_{SP} \rightarrow U_{FP} \rightarrow F^{\delta I}$
 - Extrapolate the conservative variables to the flux points: $U_{SP} \rightarrow U_{FP}$
 - Compute $F^{\delta I}$ from U_{FP} .
- TUPY approach $U_{SP} \rightarrow Q_{SP} \rightarrow Q_{FP} \rightarrow U_{FP} \rightarrow F^{\delta I}$
 - Compute the primitive variables from the conservative variables at the solution points: $U_{SP} \rightarrow Q_{SP}$
 - Extrapolate the primitive variables to the flux points: $Q_{SP} \rightarrow Q_{FP}$
 - Compute the conservative variables from the primitive variables at the flux points: $Q_{FP} \rightarrow U_{FP}$
 - Compute $F^{\delta I}$ from U_{FP} .

NUMERICAL EXPERIMENTS: REACTIVE NAVIER-STOKES EQUATIONS 2D

H2/O2 combustion kinetic scheme Boivin et al. (9 species | 12 reactions)
SAGE is used to solve kinetic chemistry.

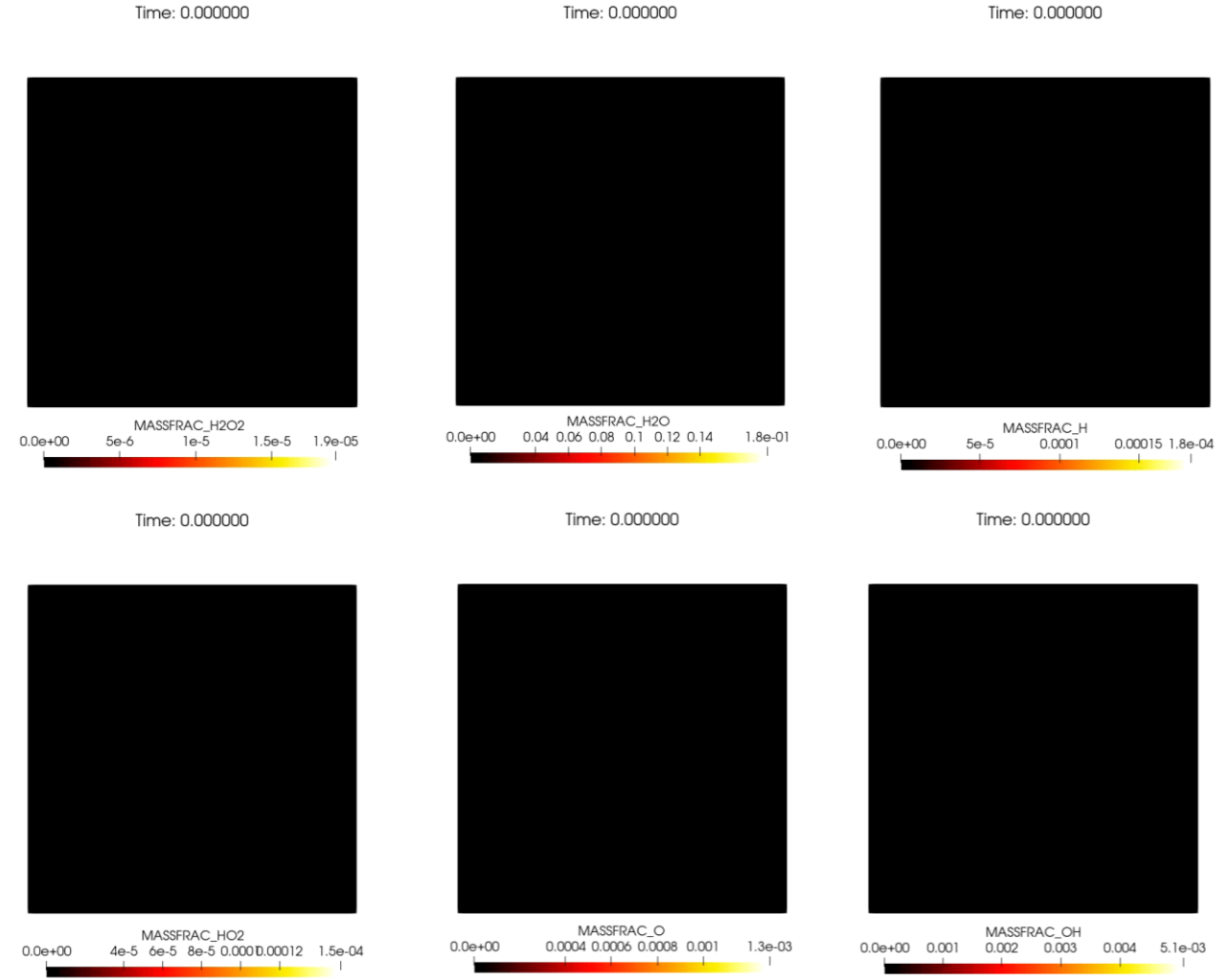
- $Y_{O_2}^0, Y_{N_2}^0, Y_{H_2}^0, P = 101325 \text{ Pa}, T = 300 \text{ K}, T_f = 1910 \text{ K}$ at the interface between the O_2 and $H_2 + N_2$
- $V_1 = u_0 \sin\left(\frac{x}{L_0}\right) \cos\left(\frac{y}{L_0}\right), V_2 = -u_0 \cos\left(\frac{x}{L_0}\right) \sin\left(\frac{y}{L_0}\right)$
- Boundary condition
 - Periodic
- Discretisation
 - Time: Explicit RK45, four order DG via FR vs PISO
 - Numerical flux Lax-Friedrichs, LdG (viscous)
- Setup
 - $\Omega = [-\pi L_0, \pi L_0] \times [-\pi L_0, \pi L_0], L_0 = 1 \text{ mm}, L = 2\pi L_0, u_0 = 4,$
 - $\tau_{ref} = \frac{L_0}{u_0} = 0.25 \text{ ms}, T_{end} = 10 \times \tau_{ref}, Ma = 0.1, Re = 267, N^2 \in \{64^2, 128^2\}, CFL=0.2$



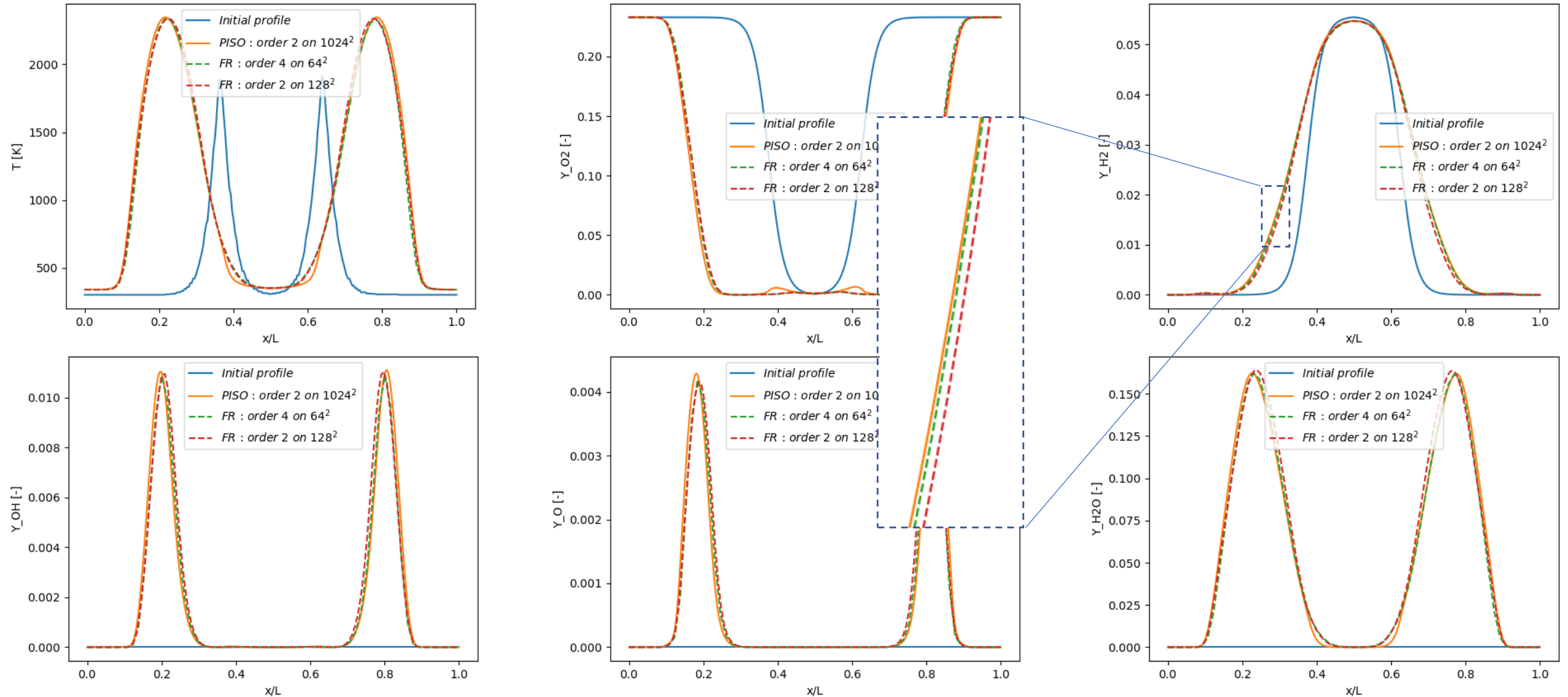
CHEMICAL REACTION: TRANSPORT - COMBUSTION

Reactive test case

- The simulation is stable $\forall t \geq 0$
- We observe the consumption and production of chemical species
- $\sum_k Y_k(t, \cdot) = 1, \forall t \geq 0$
- $Y_{k,min}^0 \leq Y_k(t, \cdot) \leq Y_{k,max}^0, \forall k \in \llbracket 1, N_s \rrbracket, t \geq 0$

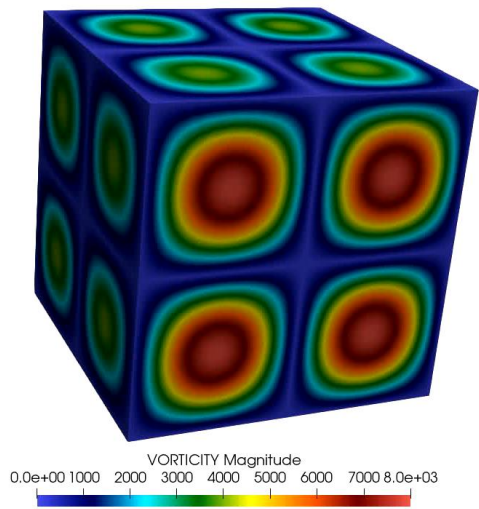


NUMERICAL EXPERIMENTS: 1D profiles at $t = \tau_{ref}$, $y = 0.5L$

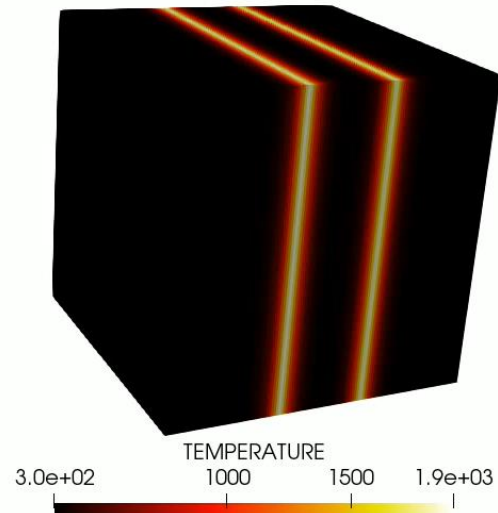


NUMERICAL EXPERIMENTS: TUPY APPROACH IN 3D

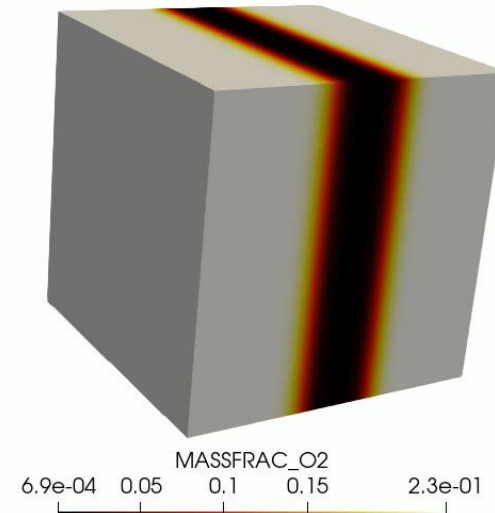
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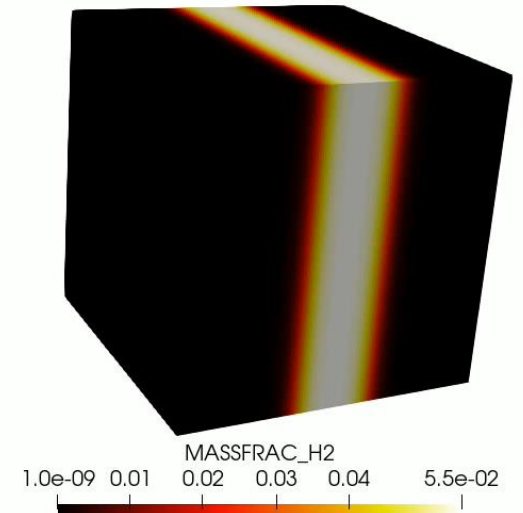
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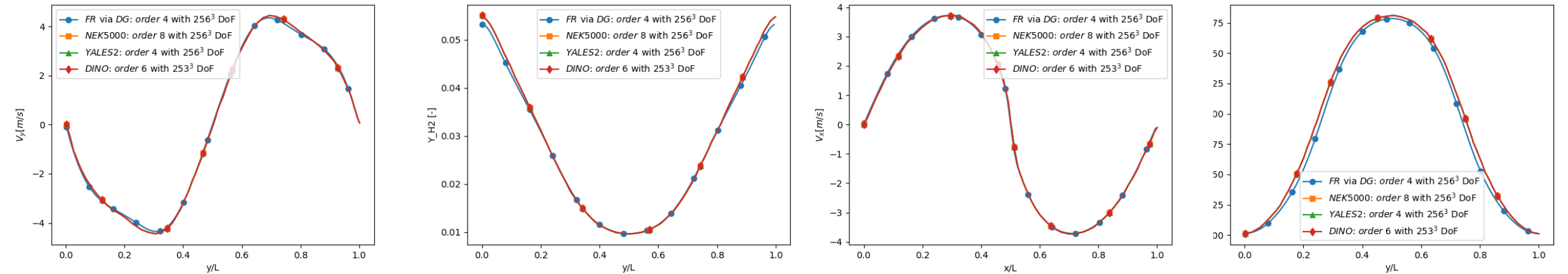
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NUMERICAL EXPERIMENTS: TUPY APPROACH IN 3D



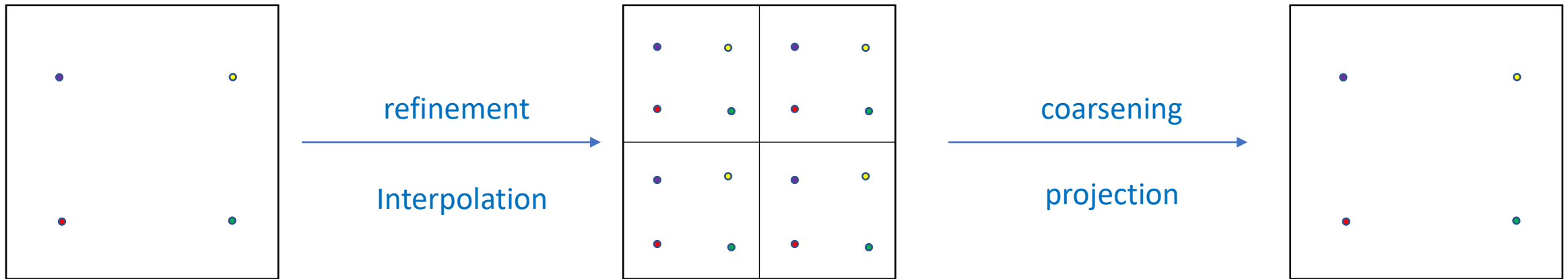
$x = 0.5 L$, and $z = 0.5 L$ for 3-D non-reacting multi-species flow at $t = 2 \tau_{ref}$

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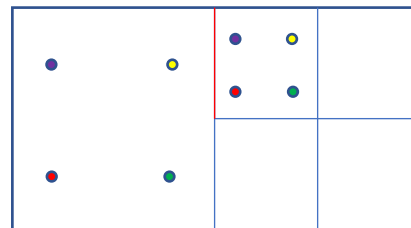
HIGH ORDER INTERPOLATION AND PROJECTION OPERATORS FOR AMR

- To ensure the conservativeness of the scheme, we need to define the high-order operators
 - To transfer data after refinement or coarsening

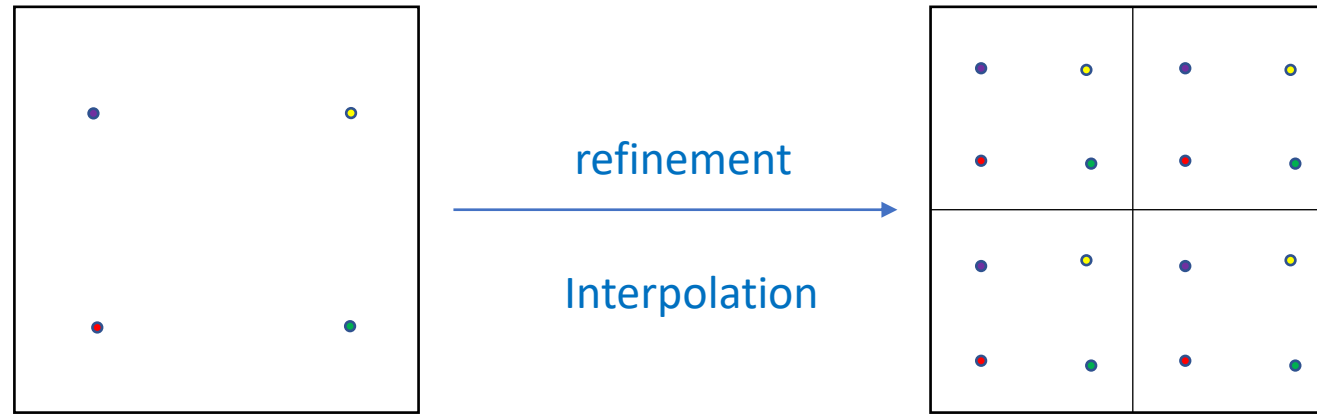


Example $p = 1$

- To calculate the numerical flux on the faces separating two cells of different sizes



HIGH ORDER INTERPOLATION AND PROJECTION OPERATORS FOR AMR



$$\mathbf{u}^P(\zeta) = \sum_{i=1}^{N_p} l_i(\zeta) \mathbf{u}_i^P$$

$$\mathbf{u}_m^K(\zeta) = ?$$

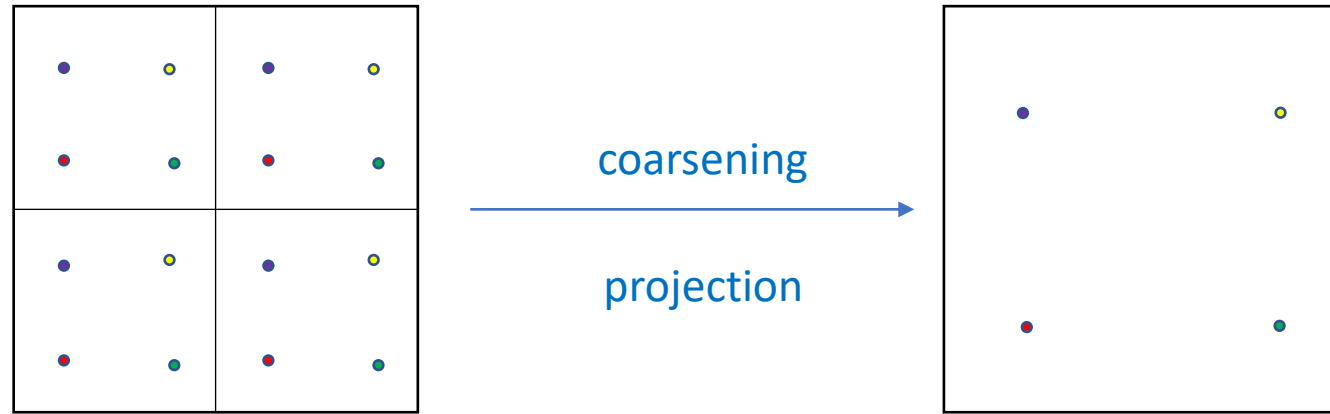
Interpolation operator:

$$\mathbf{u}_m^K(\zeta) = I^c(\mathbf{u}^P)(\zeta) = \sum_{i=1}^{N_p} l_i(\Phi^m(\zeta)) \mathbf{u}_i^P, \quad m \in \{1, 2, \dots, 2^D\}$$

With isoparametric map:

$$\Phi^m(\zeta) = \left(\frac{\zeta_1 - o_1^{(m)}}{s}, \frac{\zeta_2 - o_2^{(m)}}{s} \right), \quad m \in \{1, 2, \dots, 2^D\}$$

HIGH ORDER INTERPOLATION AND PROJECTION OPERATORS FOR AMR



$$\mathbf{u}_m^K(\zeta) = \sum_{i=1}^{N_p} \hat{l}_i^m(\zeta) \mathbf{u}_{m,i}^K$$

$$\mathbf{u}^P(\zeta) = ?$$

Projection operator:

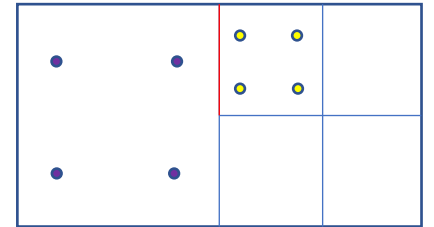
$$\mathbf{u}^P = \operatorname{argmin}_{\{V \in \hat{L}_p\}} \left\| V - \sum_{m=1}^{2^D} \mathbf{u}_m^K \right\|_{L^2} \Rightarrow \mathbf{u}^P = \Pi^c(\mathbf{u}_m^K) = \sum_{m=1}^{2^D} \mathbf{M}^{-1} \mathbf{S}_m \mathbf{u}_m^K$$

With mass matrix:

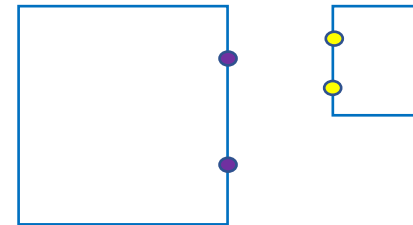
$$M_{ji} = \int_{\hat{\Omega}} l_j l_i d\zeta \quad S_{m,ji} = \int_{\hat{\Omega}} l_j \hat{l}_i^m d\zeta$$

HIGH ORDER INTERPOLATION AND PROJECTION OPERATORS FOR AMR

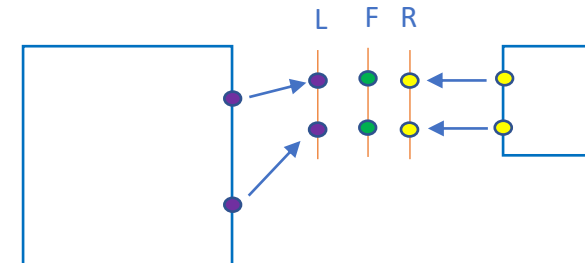
- The numerical flux on the faces separating two cells of different sizes



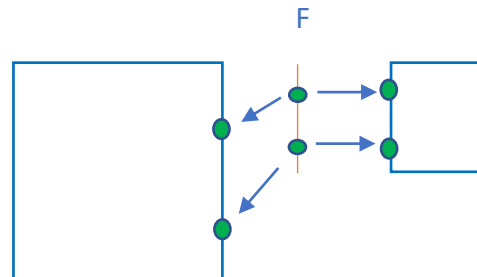
- Interpolation on the real faces of each element: I^f



- Projection on mortar face: $(\Pi^f)^{-1}$



- Projection from mortar face: Π^f



NUMERICAL EXPERIMENTS: EULER EQUATIONS 2D

Vortex convection

- Initial condition

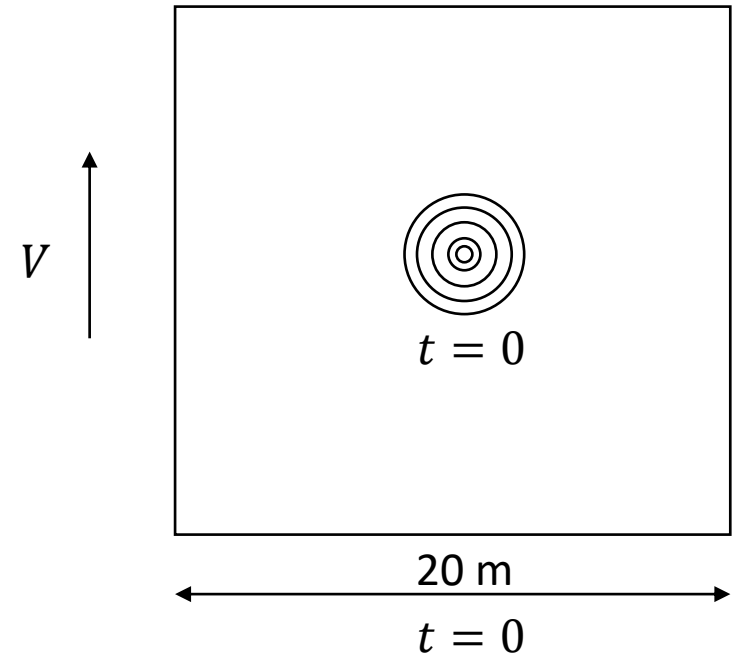
- $\rho = 1, V = (0,1),$
- $\delta V = \frac{\beta}{2\pi} (-(y - y_c), (x - x_c)) \exp(0.5(1 - r^2))$
- $\delta T = -\frac{(\gamma-1)\beta^2}{8\gamma\pi^2} \exp((1 - r^2))$
- $E = \frac{P}{\rho(\gamma-1)} + \frac{1}{2} (V_1^2 + V_2^2), \gamma = 1.4$

- Boundary condition

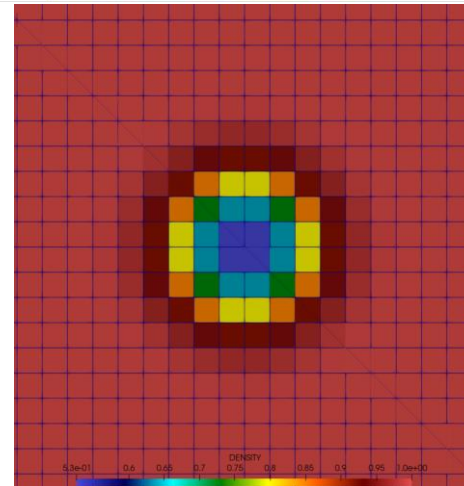
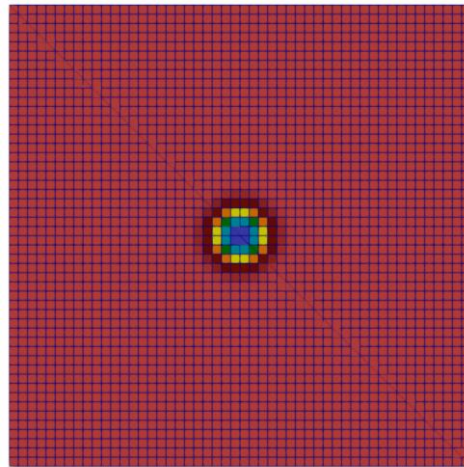
- Periodic

- Discretisation

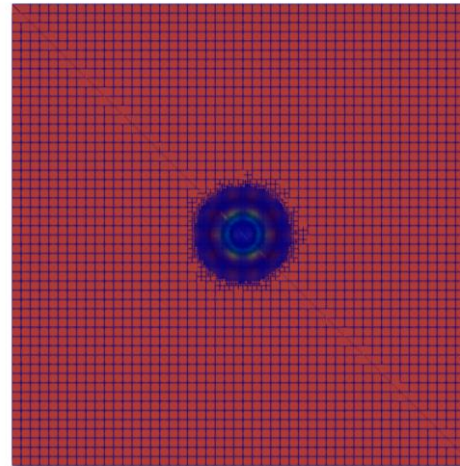
- Numerical flux Lax-Friedrichs (advective) , RK33, **DG** via FR second order ($p=3$)
- $\Omega = [-10,10] \times [-10,10], N^2 = 50^2, T_f = 20s, CFL(p) = \frac{1}{p+1}$



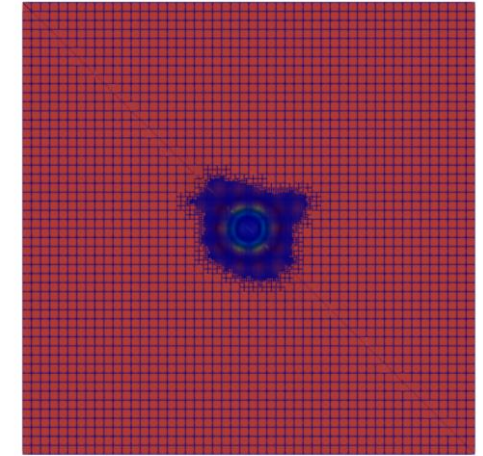
NUMERICAL EXPERIMENTS: EULER EQUATIONS 2D



Density profile at $t = 0$ sec
without AMR



Density profile at $t = 0$ sec
with AMR



Density profile at $t = 20$ sec
with AMR

OUTLINE

- Principles of Flux Reconstruction method
- Development of FR schemes for non-reactive 3D flows in AEROSOL
- Development of FR schemes for non-reactive and reactive 3D flows in CONVERGE
- Development of a coupling methodology for FR schemes and AMR in CONVERGE
- **Conclusion and future works**

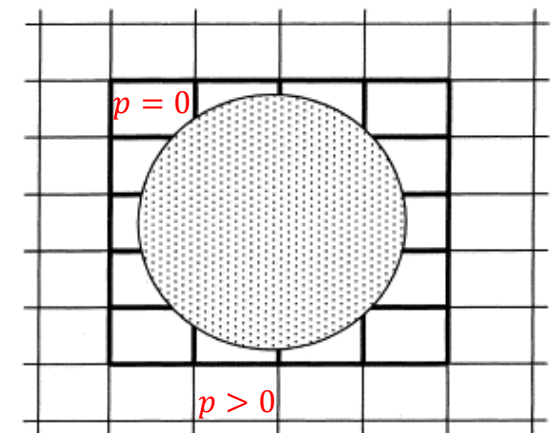
CONCLUSION AND FUTURE WORKS

During this work we have:

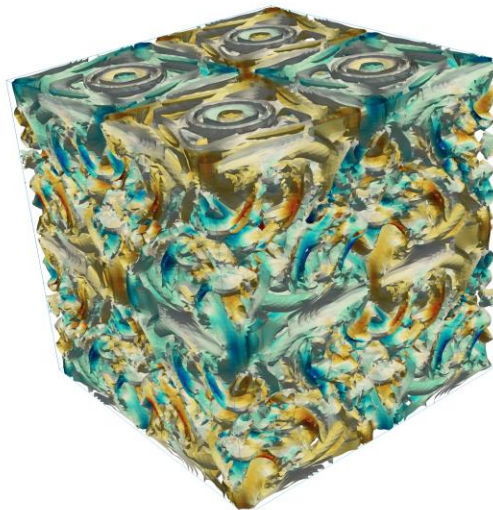
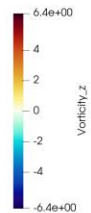
- Developed up to the sixth order of accuracy of Flux Reconstruction scheme for 3D non-reactive Navier-Stokes equations (AEROSOL CFD)
- Developed up to the sixth order of accuracy of Flux Reconstruction scheme for 3D multispecies reactive Navier-Stokes equations (CONVERGE CFD Version 3.1 and 3.2)
 - TUPY approach is stable, preserves global mass conservation and the positivity of mass fractions
- Developed high order Galerkin projection for AMR (h-refinement) coupled with FR Schemes (CONVERGE CFD Version 3.2)

Future work :

- Adapt FR schemes for order refinement: h, p -refinement
- Adapt FR Schemes for CutCells method: Add wall / inlet / outlet BC ($p=0$)



Cut Cells on the surface of a cylinder



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