Application of the reduced basis method to compressible single-phase flows in porous media

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Jana TARHINI

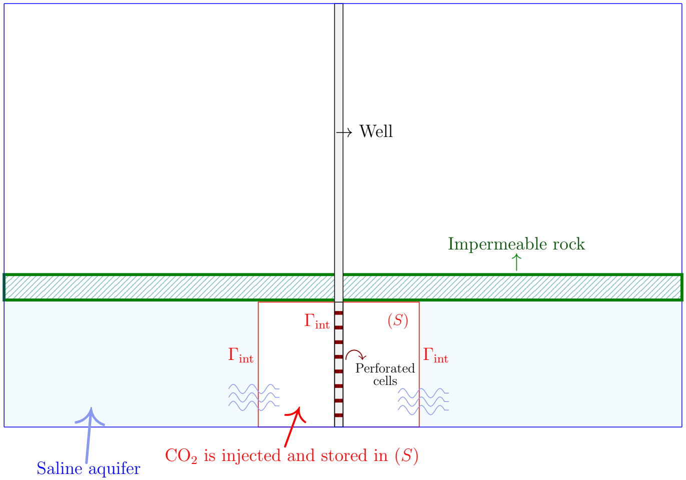
Doctoral dissertation supervised by

Sébastien BOYAVAL	ENPC & Inria	supervisor
Guillaume ENCHÉRY	IFPEN	advisor
Quang Huy TRAN	IFPEN	co-advisor

Outline

Outline

Modeling a CO_2 storage area

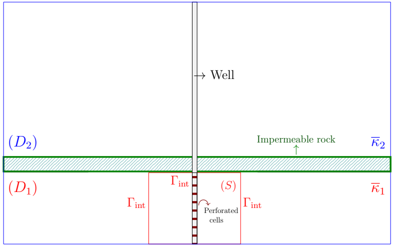


- CO_2 storage in saline aquifers
- Large-scale induced motions
- Large extended domain

Parametrized problem

- Uncertainty regarding certain parameters
  - Numerical resolution using a flow simulator
  - Several runs for *different* parameter values, e.g.,

$$\overline{\kappa}(\boldsymbol{x}) = \overline{\kappa}_1 \mathbb{1}_{D_1}(\boldsymbol{x}) + \overline{\kappa}_2 \mathbb{1}_{D_2}(\boldsymbol{x})$$



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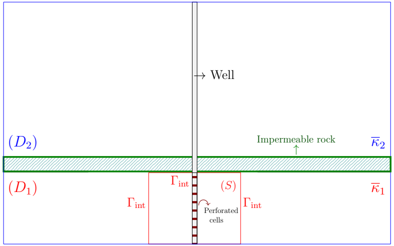
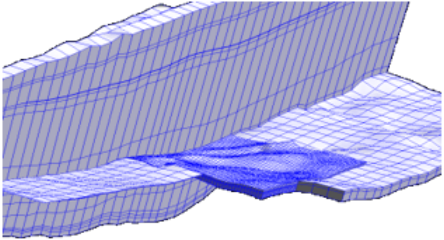


A possible way to handle large space domain



- Local grid refinement near the injection zone
- Still very costly computational effort

Objective

Methodology

- Reduced systems for *single-phase flow*
  - Lower **simulation costs** of many simulations for many parameters
  - Control errors in parameter independent norms
  - Control linear outputs of the solutions $\boldsymbol{p}$ and $\boldsymbol{u}$

Strategy

- Advantages of (RB) method
  - Replace **high-fidelity** calls by less expensive surrogate calls

Offline stage

- Use of high-fidelity solutions
  - Many degrees of freedom $\mathcal{N}$
- Construct **reduced bases**
  - **POD-Greedy** process

Online stage

- Construct reduced solutions
  - Construct a Galerkin system of small dimension $\mathbb{N}$
- Provide **cheap** *output* of interest

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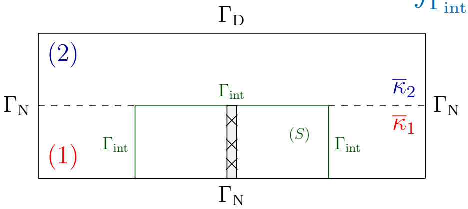
Single-phase flow

- Model

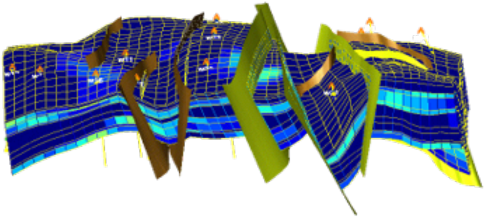
$$
\begin{aligned}
c_\Phi \partial_t p - \nabla \cdot (\mathbf{\Lambda}(\nabla p + \rho g \nabla z)) &= q(p) && \text{in }]0,T] \times \Omega \\
\mathbf{\Lambda}(\nabla p + \rho g \nabla z) \cdot \mathbf{n} &= 0 && \text{on }]0,T] \times \Gamma_{\mathrm{N}} \\
p &= p_{\mathrm{D}} && \text{on }]0,T] \times \Gamma_{\mathrm{D}} \\
p(x, t=0) &= p^0(x) && \text{in } \Omega
\end{aligned}
$$

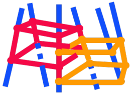
with uncertainty on $\mathbf{\Lambda} = \overline{\kappa}(\boldsymbol{x})/\mu$

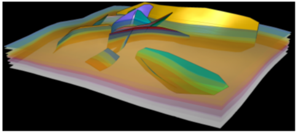
- Predict the impact of $\bar{\kappa}$ on the flux $s = -\int_{\Gamma_{\mathrm{int}}} \mathbf{\Lambda}(\overline{\kappa})(\boldsymbol{\nabla} p + \rho g \boldsymbol{\nabla} z) \cdot \mathbf{n}$

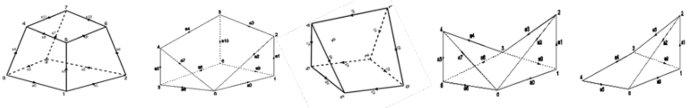


Complex meshes in porous medium simulations









- Geological layers using folders in a mesh of CPG-type

Discretization

- Time: Implicit Euler discretization
- Space: Average multi-point finite volume method
- Discrete **high-fidelity** system:

- $\Delta t = \frac{T}{N}$
- A parameter ξ

$$(\boldsymbol{M} + \Delta t \boldsymbol{A}(\xi))\boldsymbol{p}_{\mathcal{M}}^{n+1} = \boldsymbol{M}\boldsymbol{p}_{\mathcal{M}}^{n} + \Delta t \boldsymbol{b}(\xi) \qquad n = 0, \dots, N-1$$

$$s^{n+1} = \boldsymbol{l}^T(\xi)\boldsymbol{p}_{\mathcal{M}}^{n+1}$$

$$\begin{pmatrix} \boldsymbol{M}+\Delta t\boldsymbol{A} & 0 & 0 & \dots & \dots & 0 \\ -\boldsymbol{M} & \boldsymbol{M}+\Delta t\boldsymbol{A} & \ddots & \ddots & & \vdots \\ 0 & -\boldsymbol{M} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & -\boldsymbol{M} & \boldsymbol{M}+\Delta t\boldsymbol{A} \end{pmatrix} \begin{pmatrix} \boldsymbol{p}_{\mathcal{M}}^{1} \\ \boldsymbol{p}_{\mathcal{M}}^{2} \\ \vdots \\ \vdots \\ \boldsymbol{p}_{\mathcal{M}}^{N-1} \\ \boldsymbol{p}_{\mathcal{M}}^{N} \end{pmatrix} = \Delta t \begin{pmatrix} \boldsymbol{b}+\boldsymbol{M}\boldsymbol{p}_{\mathcal{M}}^{0} \\ \boldsymbol{b} \\ \vdots \\ \vdots \\ \boldsymbol{b} \\ \boldsymbol{b} \end{pmatrix}$$

State of the art and contribution

State of the art

- A goal oriented version of the RB method
  - Grepl & Patera. A posteriori error bounds for reduced-basis approximations of parametrized parabolic partial differential equations. M2AN, 2005
  - Haasdonk & Ohlberger. Reduced basis method for finite volume approximations of parametrized linear evolution equations. M2AN, 2008

Contribution

- New discrete a posteriori error estimations
- Reduction error estimated using the *same norm* of $L^2([0,T];H^1(\Omega))$-type *for all parameter* values
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Discrete Dual Problem

- $\forall\, n = N, \dots, 0$

$$
\begin{aligned}
\boldsymbol{M}\boldsymbol{\Psi}^N &= -\boldsymbol{l}(\xi) \\
(\boldsymbol{M} + \Delta t \boldsymbol{A}^T(\xi))\boldsymbol{\Psi}^n &= \boldsymbol{M}\boldsymbol{\Psi}^{n+1}
\end{aligned}
$$

$$
\begin{pmatrix}
\boldsymbol{M} + \Delta t \boldsymbol{A} & -\boldsymbol{M} & 0 & \dots & \dots & 0 \\
0 & \boldsymbol{M} + \Delta t \boldsymbol{A} & \ddots & \ddots & & \vdots \\
0 & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & & \ddots & \ddots & \ddots & -\boldsymbol{M} \\
0 & \dots & \dots & 0 & 0 & \boldsymbol{M} + \Delta t \boldsymbol{A}
\end{pmatrix}^T
\begin{pmatrix}
\boldsymbol{\Psi}^{N-1} \\ \boldsymbol{\Psi}^{N-2} \\ \vdots \\ \vdots \\ \boldsymbol{\Psi}^1 \\ \boldsymbol{\Psi}^0
\end{pmatrix}
=
\begin{pmatrix}
-\boldsymbol{l} \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 0
\end{pmatrix}
$$

Galerkin projection

Primal reduced problem

- *Galerkin* projection using $\boldsymbol{Z} \in \mathbb{R}^{\mathcal{N},\mathsf{N}_{\mathrm{pr}}}$

$$\boldsymbol{Z}^T(\boldsymbol{M} + \Delta t \boldsymbol{A}(\xi))\boldsymbol{Z}\widetilde{\boldsymbol{p}}^{n+1} = \boldsymbol{Z}^T\boldsymbol{M}\boldsymbol{Z}\widetilde{\boldsymbol{p}}^{n} + \Delta t \boldsymbol{Z}^T\boldsymbol{b}(\xi)$$

$\Rightarrow$ **Reduced** solution: $\boldsymbol{p}^{\mathsf{N},n+1} \approx \boldsymbol{Z}\widetilde{\boldsymbol{p}}^{n+1}$

Dual reduced problem

- *Galerkin* projection using $\boldsymbol{Z}_{\mathrm{du}} \in \mathbb{R}^{\mathcal{N},\mathsf{N}_{\mathrm{du}}}$

$$\left(\boldsymbol{Z}_{\mathrm{du}}^T\boldsymbol{M}\boldsymbol{Z}_{\mathrm{du}} + \Delta t\, \boldsymbol{Z}_{\mathrm{du}}^T\boldsymbol{A}^T(\xi)\boldsymbol{Z}_{\mathrm{du}}\right)\widetilde{\boldsymbol{\Psi}}^{n} = \boldsymbol{Z}_{\mathrm{du}}^T\boldsymbol{M}\boldsymbol{Z}_{\mathrm{du}}\widetilde{\boldsymbol{\Psi}}^{n+1}$$

$$\boldsymbol{Z}_{\mathrm{du}}^T\boldsymbol{M}\boldsymbol{Z}_{\mathrm{du}}\widetilde{\boldsymbol{\Psi}}^{N} = -\boldsymbol{Z}_{\mathrm{du}}^T\boldsymbol{l}(\xi)$$

$\Rightarrow$ **Reduced** solution: $\boldsymbol{\Psi}^{\mathsf{N}_{\mathrm{du}},n} = \boldsymbol{Z}_{\mathrm{du}}\widetilde{\boldsymbol{\Psi}}^{n}$

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$\Rightarrow$ **Reduced** solution: $\boldsymbol{\Psi}^{\mathsf{N}_{\mathrm{du}},n} = \boldsymbol{Z}_{\mathrm{du}}\, \widetilde{\boldsymbol{\Psi}}^{n}$

A posteriori error estimation for primal problem

- Reduction error $\boldsymbol{e}^n = \boldsymbol{p}^n_{\mathcal{M}} - \boldsymbol{p}^{\mathsf{N},n}$
- A new space-time energy norm $|||\cdot|||_{\text{pr}}$ independent of ξ

$$|||\boldsymbol{e}^N|||_{\text{pr}} := \left(\sum_{n=1}^{N} \langle \boldsymbol{e}^n, \boldsymbol{G}^* \boldsymbol{e}^n \rangle\right)^{1/2} \leq \Delta^N_{\text{pr}} := \left(\frac{\Delta t}{\alpha_{\boldsymbol{G},\text{LB}} \alpha_{\boldsymbol{A}_{\text{sym}},\text{LB}}} \sum_{n=1}^{N} (N+2-n) \|\boldsymbol{r}^n\|^2_{-1}\right)^{1/2}$$

- Primal residual

$$\boldsymbol{r}^{n+1} = \frac{1}{\Delta t}\left[(\boldsymbol{M} + \Delta t \boldsymbol{A})\boldsymbol{p}^{\mathsf{N},n+1} - \boldsymbol{M}\boldsymbol{p}^{\mathsf{N},n} - \Delta t \boldsymbol{b}\right]$$

$$\alpha_{\boldsymbol{A}_{\text{sym}},\text{LB}}(\xi) \leq \inf_{\boldsymbol{v} \in \mathbb{R}^{\mathcal{N}}} \frac{\boldsymbol{v}^T \boldsymbol{A}_{\text{sym}}(\xi) \boldsymbol{v}}{\|\boldsymbol{v}\|^2_{\boldsymbol{G}^*}} := \alpha_{\boldsymbol{A}_{\text{sym}}}(\xi)$$

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- $\boldsymbol{G}^* = \boldsymbol{M} + \Delta t \boldsymbol{A}_{\text{sym}}(\xi^*)$
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A posteriori error estimation for dual problem

- **Reduction** error $\boldsymbol{\varepsilon}^n_{\text{du}} = \boldsymbol{\Psi}^n_{\mathcal{M}} - \boldsymbol{\Psi}^{\mathsf{N}_{\text{du}},n}$
- **Energy** norm

$$|||\boldsymbol{\varepsilon}^N|||_{\text{du}} := \left(\sum_{n=1}^{N} \langle \boldsymbol{e}^n, \boldsymbol{G}^* \boldsymbol{e}^n \rangle \right)^{1/2} \leq \Delta^N_{\text{du}}$$

$$\Delta^N_{\text{du}} := \left(\frac{\Delta t}{\alpha_{\boldsymbol{A}_{\text{sym,LB}}} \alpha_{\boldsymbol{G},\text{LB}}} \sum_{m=0}^{N-1} (m+2) \|\boldsymbol{\varrho}^m\|^2_{-1} + \frac{(T + \Delta t)}{\Delta t} \frac{\|\boldsymbol{\varrho}^N\|^2_{-1}}{\alpha_{\boldsymbol{M}} \alpha_{\boldsymbol{G},\text{LB}}} \right)^{1/2}$$

- $\boldsymbol{\varrho}^n = \frac{1}{\Delta t} (\boldsymbol{M} + \Delta t \boldsymbol{A}^T) \boldsymbol{\Psi}^{\mathsf{N}_{\text{du}},n} - \boldsymbol{M} \boldsymbol{\Psi}^{\mathsf{N}_{\text{du}},n+1} \qquad n = N-1, \dots, 0$

▷ $\boldsymbol{\varrho}^N = -\boldsymbol{l} - \boldsymbol{M} \Psi^{\mathsf{N}_{\text{du}},N}$ ▷ $\|\boldsymbol{\varrho}^m\|_{-1} = \sup_{\boldsymbol{v} \in \mathbb{R}^{\mathcal{N}}} \frac{\langle \boldsymbol{\varrho}^m, \boldsymbol{v} \rangle}{\|\boldsymbol{v}\|_{\boldsymbol{G}^*}}$ ▷ $\alpha_{\boldsymbol{M}} := \inf_{\boldsymbol{v} \in \mathbb{R}^{\mathcal{N}}} \frac{\boldsymbol{v}^T \boldsymbol{M} \boldsymbol{v}}{\|\boldsymbol{v}\|^2_{\boldsymbol{G}^*}}$

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A posteriori error estimation for QOI

- Under the equality

$$\langle \boldsymbol{l}, \boldsymbol{p}^n_{\mathcal{M}} - \boldsymbol{p}^{\mathsf{N}_{\mathrm{pr}},n} \rangle = \Delta t \sum_{k=0}^{n-1} \langle \boldsymbol{r}^{k+1}, \boldsymbol{\varepsilon}^{N-n+k} \rangle + \Delta t \sum_{k=0}^{n-1} \langle \boldsymbol{r}^{k+1}, \boldsymbol{\Psi}^{\mathsf{N}_{\mathrm{du}},N-n+k} \rangle$$

- Reduced output

$$s^{\mathsf{N}_{\mathrm{s}},n} = \left\langle \boldsymbol{l}, \boldsymbol{p}^{\mathsf{N}_{\mathrm{pr}},n} \right\rangle + \Delta t \sum_{n'=0}^{n-1} \left\langle \boldsymbol{r}^{n'+1}, \boldsymbol{\Psi}^{\mathsf{N}_{\mathrm{du}},N-n+n'} \right\rangle$$

- Estimation

$$|s^N - s^{\mathsf{N}_{\mathrm{s}},N}| \leq \Delta t \left(\sum_{n=1}^{N} \|\boldsymbol{r}^n\|^2_{-1} \right)^{1/2} \Delta^N_{\mathrm{du}} =: \Delta^N_s$$

EIM and SCM

- Δ_{pr}^{N} has to be
  - reliable: $|||\boldsymbol{e}^N|||_{\mathrm{pr}} \leq \Delta_{\mathrm{pr}}^{N}$
  - $c := \dfrac{\Delta_{\mathrm{pr}}^{N}}{|||\boldsymbol{e}^N|||_{\mathrm{pr}}}$ depends weakly on ξ
  - computationally cheap
- EIM for $\boldsymbol{A}^{\mathsf{N}} = \boldsymbol{Z}^T \boldsymbol{A}(\xi)\boldsymbol{Z}$, $\boldsymbol{b}^{\mathsf{N}}(\xi) = \boldsymbol{Z}^T\boldsymbol{b}$

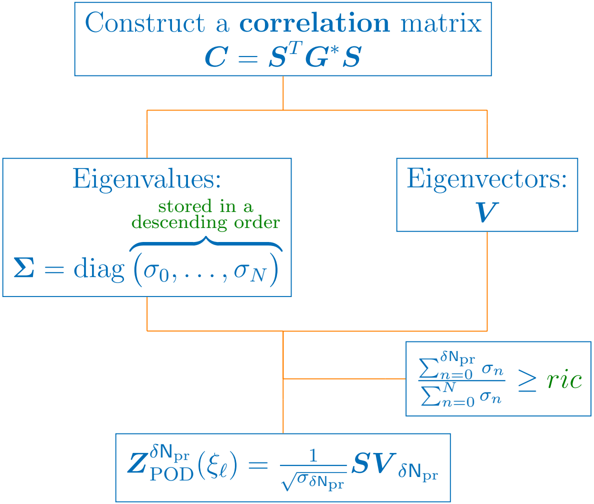
$$\boldsymbol{A}(\xi) \approx \sum_{d=1}^{\mathrm{D_a}} \theta_d^a(\xi)\boldsymbol{A}_d, \quad \boldsymbol{b}(\xi) \approx \sum_{d=1}^{\mathrm{D_b}} \theta_d^b(\xi)\boldsymbol{b}_d$$

- SCM on $\alpha_{\boldsymbol{A}_{\mathrm{sym}}}(\xi)$

$$\alpha_{\boldsymbol{A}_{\mathrm{sym,LB}}}(\xi) \leq \alpha_{\boldsymbol{A}_{\mathrm{sym}}}(\xi) \leq \alpha_{\boldsymbol{A}_{\mathrm{sym,UB}}}(\xi)$$

POD method

- $\boldsymbol{S}$: snapshots matrix
- $ric \in [0, 1]$



POD-Greedy algorithm

Algorithm 1: POD-greedy algorithm using Δ_{pr}^N

Input: $\mathsf{N}_{\max}, \epsilon_{\text{tol}}, \Xi, ric$

Data: $\mathsf{N}_{\text{pr}} = 1$, $\delta^{\mathsf{N}_{\text{pr}}} = \epsilon_{\text{tol}} + 1$

$\xi \in \Xi$, $\ell = 1$, $\Xi^\ell = \{\xi_1\}$

Initialize:

Compute $\boldsymbol{p}_{\mathcal{M}}^n(\xi_1)$ **for** $0 \leq n \leq N$

Set

$$\mathbf{S}_{\text{pr}} = \left(\boldsymbol{p}_{\mathcal{M}}^0(\xi_1) \;\middle|\; \dots \;\middle|\; \boldsymbol{p}_{\mathcal{M}}^N(\xi_1) \right)$$

Set $\boldsymbol{Z}^{\mathsf{N}_{\text{pr}}} = \mathbf{POD}(\mathbf{S}_{\text{pr}}, \boldsymbol{G}^*, ric)$

$\delta^{\mathsf{N}_{\text{pr}}} = \max_{\xi \in \Xi} \Delta_{\text{pr}}^N$ $\quad$ $\xi_{\ell+1} = \arg\max_{\xi \in \Xi} \Delta_{\text{pr}}^N$ $\quad$ $\Xi^{\ell+1} \leftarrow \Xi^\ell \cup \{\xi_{\ell+1}\}$

POD-Greedy algorithm

while $\delta^{\mathsf{N}_{\rm pr}} > \epsilon_{\rm tol}, \mathsf{N}_{\rm pr} < \mathsf{N}_{\max}$ **do**

Compute $\boldsymbol{p}^n_{\mathcal{M}}(\xi_\ell)$ for $0 \leq n \leq N$

Set

$$\mathbf{S}_{\rm pr} := \left[\boldsymbol{p}^0_{\mathcal{M}}(\xi_\ell) - \mathrm{Proj}_{\boldsymbol{Z}_{\rm pr}}(\boldsymbol{p}^0_{\mathcal{M}}(\xi_\ell))\Big| \dots \Big|\boldsymbol{p}^N_{\mathcal{M}}(\xi_\ell) - \mathrm{Proj}_{\boldsymbol{Z}_{\rm pr}}(\boldsymbol{p}^N_{\mathcal{M}}(\xi_\ell))\right]$$

Compute $[\boldsymbol{z}_1|\dots|\boldsymbol{z}_{\delta\mathsf{N}_{\rm pr}}] = \mathbf{POD}(\mathbf{S}_{\rm pr}, \boldsymbol{G}^*, ric)$

Define $\boldsymbol{Z}^{\mathsf{N}_{\rm pr}+\delta\mathsf{N}_{\rm pr}} := orthonormalize(\boldsymbol{Z}^{\mathsf{N}_{\rm pr}} \cup [\boldsymbol{z}_1|\dots|\boldsymbol{z}_{\delta\mathsf{N}_{\rm pr}}])$

$\delta^{\mathsf{N}_{\rm pr}} = \max_{\xi\in\Xi} \Delta^N_{\rm pr}$ $\quad \xi_{\ell+1} = \arg\max_{\xi\in\Xi} \Delta^N_{\rm pr}$ $\quad \Xi^{\ell+1} \leftarrow \Xi^\ell \cup \{\xi_{\ell+1}\}$

$\mathsf{N}_{\rm pr} \leftarrow \mathsf{N}_{\rm pr} + \delta\mathsf{N}_{\rm pr}$ $\quad \ell \leftarrow \ell + 1$

end

POD-Greedy algorithm

while $\delta^{\mathsf{N}_{\mathrm{pr}}} > \epsilon_{\mathrm{tol}}$, $\mathsf{N}_{\mathrm{pr}} < \mathsf{N}_{\max}$ **do**

Compute $\boldsymbol{p}^n_{\mathcal{M}}(\xi_\ell)$ for $0 \leq n \leq N$

Set

$$\mathbf{S}_{\mathrm{pr}} := \left[\boldsymbol{p}^0_{\mathcal{M}}(\xi_\ell) - \mathrm{Proj}_{\boldsymbol{Z}_{\mathrm{pr}}}(\boldsymbol{p}^0_{\mathcal{M}}(\xi_\ell))\middle| \ldots \middle|\boldsymbol{p}^N_{\mathcal{M}}(\xi_\ell) - \mathrm{Proj}_{\boldsymbol{Z}_{\mathrm{pr}}}(\boldsymbol{p}^N_{\mathcal{M}}(\xi_\ell))\right]$$

Compute $\left[\boldsymbol{z}_1 | \ldots | \boldsymbol{z}_{\delta\mathsf{N}_{\mathrm{pr}}}\right] = \mathbf{POD}(\mathbf{S}_{\mathrm{pr}}, \boldsymbol{G}^*, ric)$

Define $\boldsymbol{Z}^{\mathsf{N}_{\mathrm{pr}}+\delta\mathsf{N}_{\mathrm{pr}}} := orthonormalize(\boldsymbol{Z}^{\mathsf{N}_{\mathrm{pr}}} \cup \left[\boldsymbol{z}_1 | \ldots | \boldsymbol{z}_{\delta\mathsf{N}_{\mathrm{pr}}}\right])$

$\delta^{\mathsf{N}_{\mathrm{pr}}} = \max_{\xi \in \Xi} \Delta^N_{\mathrm{pr}}$ $\quad \xi_{\ell+1} = \arg\max_{\xi \in \Xi} \Delta^N_{\mathrm{pr}}$ $\quad \Xi^{\ell+1} \leftarrow \Xi^\ell \cup \{\xi_{\ell+1}\}$

$\mathsf{N}_{\mathrm{pr}} \leftarrow \mathsf{N}_{\mathrm{pr}} + \delta\mathsf{N}_{\mathrm{pr}}$ $\quad \ell \leftarrow \ell + 1$

end

POD-Greedy algorithm

while $\delta^{\mathsf{N}_{\rm pr}} > \epsilon_{\rm tol},\ \mathsf{N}_{\rm pr} < \mathsf{N}_{\max}$ **do**

Compute $\boldsymbol{p}^n_{\mathcal{M}}(\xi_\ell)$ for $0 \le n \le N$

Set

$$\mathbf{S}_{\rm pr} := \left[\boldsymbol{p}^0_{\mathcal{M}}(\xi_\ell) - \mathrm{Proj}_{\boldsymbol{Z}_{\rm pr}}(\boldsymbol{p}^0_{\mathcal{M}}(\xi_\ell))\middle| \ldots \middle|\boldsymbol{p}^N_{\mathcal{M}}(\xi_\ell) - \mathrm{Proj}_{\boldsymbol{Z}_{\rm pr}}(\boldsymbol{p}^N_{\mathcal{M}}(\xi_\ell))\right]$$

Compute $\left[\boldsymbol{z}_1|\ldots|\boldsymbol{z}_{\delta\mathsf{N}_{\rm pr}}\right] = \mathbf{POD}(\mathbf{S}_{\rm pr}, \boldsymbol{G}^*, ric)$

Define $\boldsymbol{Z}^{\mathsf{N}_{\rm pr}+\delta\mathsf{N}_{\rm pr}} := orthonormalize(\boldsymbol{Z}^{\mathsf{N}_{\rm pr}} \cup \left[\boldsymbol{z}_1|\ldots|\boldsymbol{z}_{\delta\mathsf{N}_{\rm pr}}\right])$

$\delta^{\mathsf{N}_{\rm pr}} = \max_{\xi\in\Xi} \Delta^N_{\rm pr}$ $\quad \xi_{\ell+1} = \arg\max_{\xi\in\Xi} \Delta^N_{\rm pr}$ $\quad \Xi^{\ell+1} \leftarrow \Xi^\ell \cup \{\xi_{\ell+1}\}$

$\mathsf{N}_{\rm pr} \leftarrow \mathsf{N}_{\rm pr} + \delta\mathsf{N}_{\rm pr}$ $\quad \ell \leftarrow \ell + 1$

end

POD-Greedy algorithm

while $\delta^{\mathsf{N}_{\mathrm{pr}}} > \epsilon_{\mathrm{tol}},\ \mathsf{N}_{\mathrm{pr}} < \mathsf{N}_{\max}$ **do**

Compute $\boldsymbol{p}^n_{\mathcal{M}}(\xi_\ell)$ for $0 \le n \le N$

Set

$$\mathbf{S}_{\mathrm{pr}} := \left[\boldsymbol{p}^0_{\mathcal{M}}(\xi_\ell) - \mathrm{Proj}_{\boldsymbol{Z}_{\mathrm{pr}}}(\boldsymbol{p}^0_{\mathcal{M}}(\xi_\ell)) \middle| \dots \middle| \boldsymbol{p}^N_{\mathcal{M}}(\xi_\ell) - \mathrm{Proj}_{\boldsymbol{Z}_{\mathrm{pr}}}(\boldsymbol{p}^N_{\mathcal{M}}(\xi_\ell))\right]$$

Compute $\left[\boldsymbol{z}_1 | \dots | \boldsymbol{z}_{\delta \mathsf{N}_{\mathrm{pr}}}\right] = \mathbf{POD}(\mathbf{S}_{\mathrm{pr}}, \boldsymbol{G}^*, ric)$

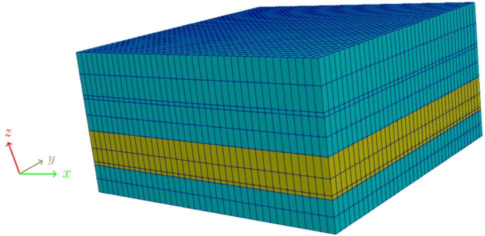
Define $\boldsymbol{Z}^{\mathsf{N}_{\mathrm{pr}}+\delta\mathsf{N}_{\mathrm{pr}}} := orthonormalize(\boldsymbol{Z}^{\mathsf{N}_{\mathrm{pr}}} \cup \left[\boldsymbol{z}_1 | \dots | \boldsymbol{z}_{\delta \mathsf{N}_{\mathrm{pr}}}\right])$

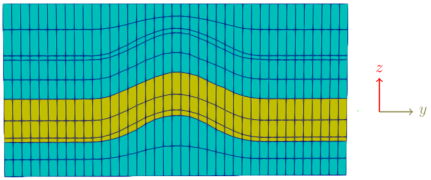
$\delta^{\mathsf{N}_{\mathrm{pr}}} = \max_{\xi \in \Xi} \Delta^N_{\mathrm{pr}}$ $\qquad \xi_{\ell+1} = \arg\max_{\xi \in \Xi} \Delta^N_{\mathrm{pr}}$ $\qquad \Xi^{\ell+1} \leftarrow \Xi^\ell \cup \{\xi_{\ell+1}\}$

$\mathsf{N}_{\mathrm{pr}} \leftarrow \mathsf{N}_{\mathrm{pr}} + \delta\mathsf{N}_{\mathrm{pr}}$ $\qquad \ell \leftarrow \ell + 1$

end

Numerical tests



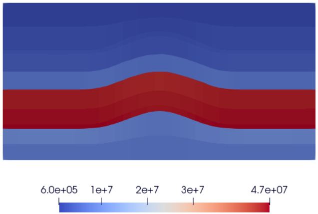


- Number of cells $\mathcal{N} = 15210$
- Yellow zone:
  $\kappa_1 \in [10^{-14}\ \mathrm{m}^2, 10^{-12}\ \mathrm{m}^2]$
- Blue zone: $\kappa_2 \in [10^{-17}\ \mathrm{m}^2, 10^{-15}\ \mathrm{m}^2]$

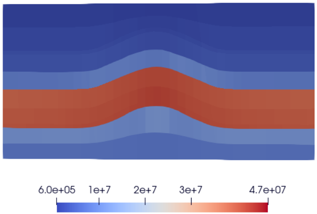
- $T = 200$ days, $\Delta t = 10$ days
- *Dirichlet* condition: $p_\mathrm{D} = 10^5$ Pa
- Well pressure: $p_{bh} = 4.3 \times 10^7$ Pa

Pressure profile

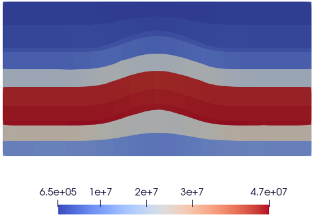
$\kappa_1 = 10^{-12}$, $\kappa_2 = 10^{-16}$



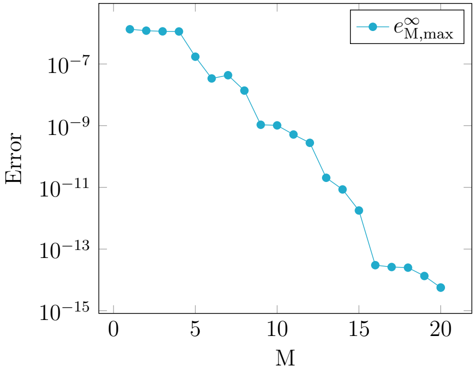
$\kappa_1 = 10^{-13}$, $\kappa_2 = 10^{-16}$



$\kappa_1 = 10^{-13}$, $\kappa_2 = 3 \times 10^{-17}$



Numerical tests: EIM

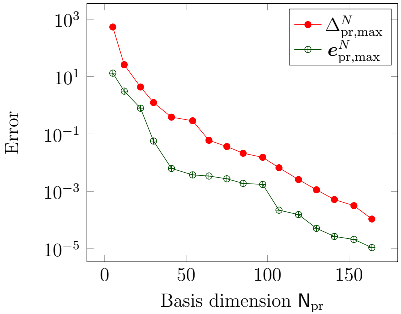


- $\Xi_{\text{training}} \in \mathbb{R}^{2\times 100}$
- $e^{\infty}_{\mathrm{M,max}} = \max_{\xi\in\Xi_{\text{training}}} \dfrac{\|\hat{\boldsymbol{v}}(\xi) - \mathcal{I}_{\mathrm{M}}[\hat{\boldsymbol{v}}(\xi)]\|_{L^\infty}}{\|\hat{\boldsymbol{v}}(\xi)\|_{L^\infty}}$

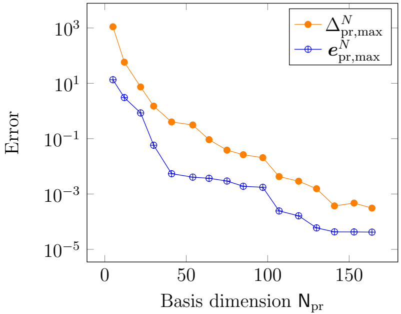
- $\hat{\boldsymbol{v}} = \Big((\alpha_{K,\sigma\sigma'})_{K\in\mathcal{T},\sigma\in\mathcal{E}_K,\sigma'\in\mathcal{S}_{K,\sigma}}, (\alpha_{K,\sigma\sigma'}\omega_{M,\sigma'})_{K\in\mathcal{T},M\in\mathcal{T}_{\sigma'},\sigma\in\mathcal{E}_K,\sigma'\in\mathcal{S}_{K,\sigma},\sigma'\in\mathcal{E}_{\text{int}}}, (\mathtt{WI})_{K\in\mathcal{T}} \Big)$

Numerical tests

Offline stage



Online stage

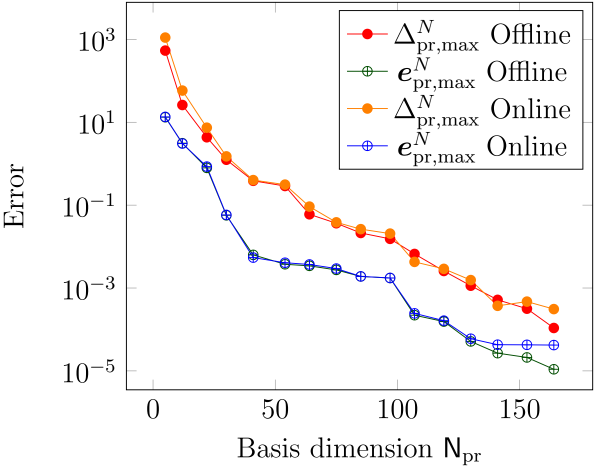


- $\xi = \{\kappa_1, \kappa_2\}$, $\Xi_{\mathrm{training}} \in \mathbb{R}^{2\times 100}$, $\Xi_{\mathrm{test}} \in \mathbb{R}^{2\times 100}$
- $\boldsymbol{e}^N_{\mathrm{pr,max}} = \max_{\xi} |||\boldsymbol{e}^N|||_{\mathrm{pr}}$
- $\Delta^N_{\mathrm{pr,max}} = \max_{\xi} \Delta^N_{\mathrm{pr}}$
- $\eta^N_{\mathrm{pr,max}} = \max_{\xi} \dfrac{\Delta^N_{\mathrm{pr}}}{|||\boldsymbol{e}^N|||_{\mathrm{pr}}}$

$\Rightarrow \mathsf{N}_{\mathrm{pr}} = 164$, $E^N_{\mathrm{pr,max}} = 7 \cdot 10^{-10}$

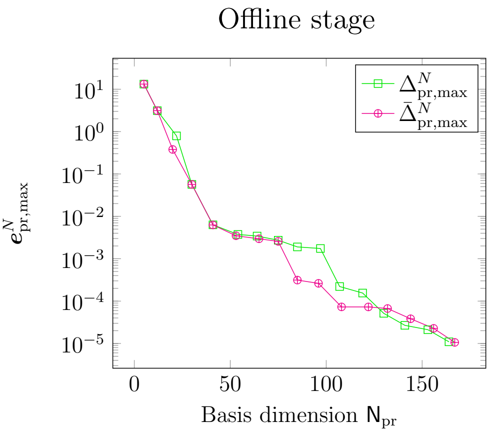
$\Rightarrow \eta^N_{\mathrm{pr,max}} \approx 10^3$

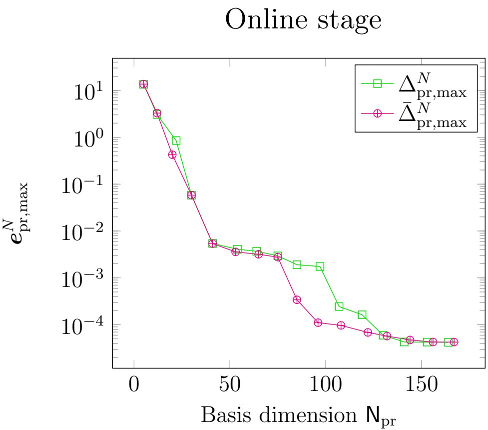
Numerical tests



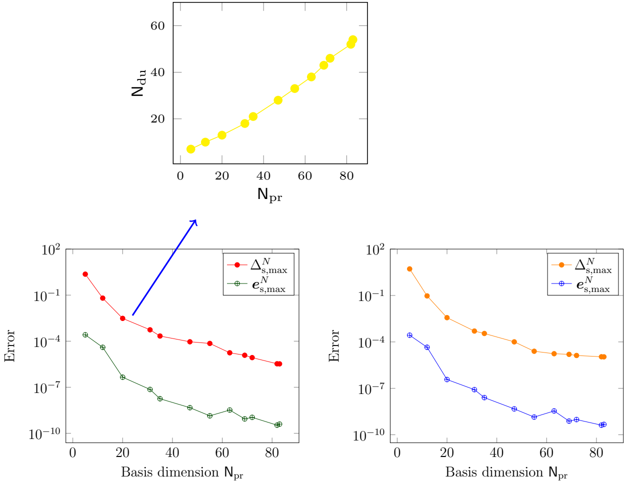
Numerical tests

- $\bar{\Delta}_{\mathrm{pr,max}} = \max_{\xi} \left(\frac{\Delta t}{\alpha_{\boldsymbol{A}_{\mathrm{sym,LB}}}} \sum_{n=1}^{N} \|\boldsymbol{r}^n\|_{-1}^2 \right)^{1/2}$ *(M. A. Grepl and A. T. Patera ,2005)*





Numerical tests

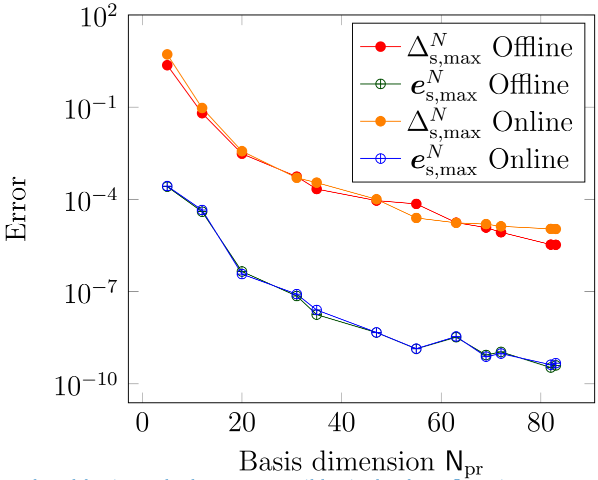


(a) Offline phase (b) Online phase

- $e^N_s = \left| s^N - s^{N_s,N} \right|$
- $e^N_{s,max} = \max_{\xi} \left| s^N - s^{N_s,N} \right|$
- $\Delta^N_{s,max} = \max_{\xi} \Delta^N_s$
- $\eta^N_{s,max} = \max_{\xi} \frac{\Delta^N_s}{e^N_s}$

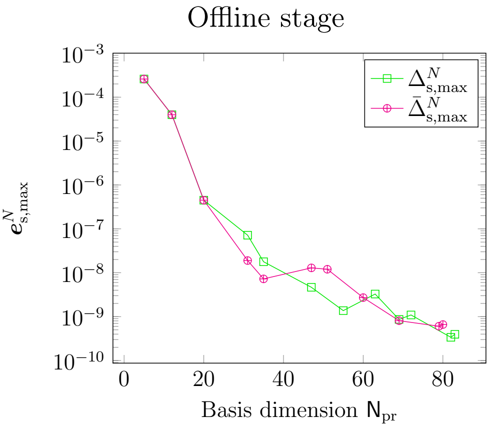
Effectivities $\eta^N_{s,max} \approx 10^6$

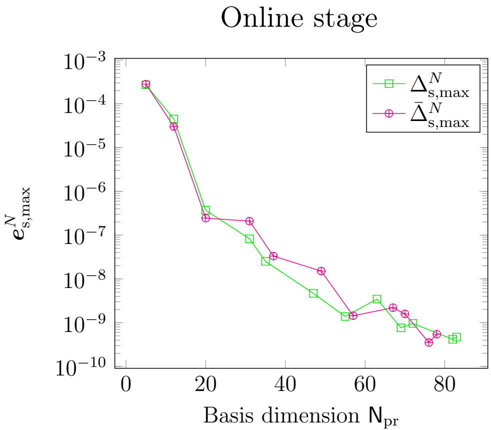
Numerical tests



Numerical tests

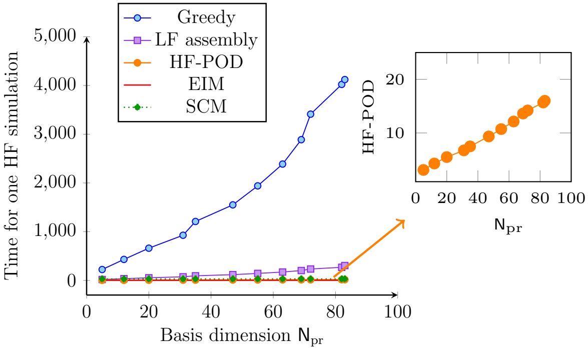
- $\bar{\Delta}_{\mathrm{s,max}} = \left(\sum_{n=1}^{N} \frac{\Delta t}{\alpha_{\boldsymbol{A}_{\mathrm{sym,LB}}}} \|\boldsymbol{r}^n\|_{-1}^2 \sum_{n=1}^{N} \frac{\Delta t}{\alpha_{\boldsymbol{A}_{\mathrm{sym,LB}}}} \|\boldsymbol{\varrho}^n\|_{-1}^2 \right)^{1/2}$ *(M. A. Grepl and A. T. Patera ,2005)*





Numerical tests

- **Offline** stage
  - Time computation of "Greedy" increases with respect to N_{pr}
  - Time computation of "POD" linear with respect to N_{pr}



- **Online** stage
  - Time computation reduced by a factor of $\frac{20 \text{ s}}{1.17 \text{ s}} = 17$

Outline

Conclusion

RB for SPF problem

- A space-time energy norm independent of the parameters
- A posteriori estimation for *linear* QOIs
- Submitted paper: Reduced Basis method for finite volume simulations of parabolic PDEs applied to porous media flows

Perspectives

SPF problem

- Modify the definition of the estimator for better efficiency ?
- Estimate different types of QOIs
- Consider a time-varying source term

Thank you for your attention!

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