Espaces grossiers adaptatifs pour les méthodes de décomposition de domaine

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Outline

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(Recall) An introduction to DDM I

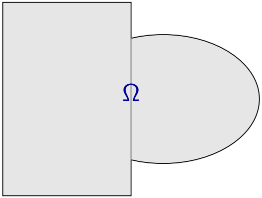
Consider the discretized Poisson problem: $Au = f \in \mathbb{R}^n$.
Given a decomposition of $[\![1;n]\!]$, $(\mathcal{N}_1, \mathcal{N}_2)$, define:

- the restriction operator R_i from $\mathbb{R}^{[\![1;n]\!]}$ into $\mathbb{R}^{\mathcal{N}_i}$,
- R_i^T as the extension by 0 from $\mathbb{R}^{\mathcal{N}_i}$ into $\mathbb{R}^{[\![1;n]\!]}$.

$u^m \longrightarrow u^{m+1}$ by solving concurrently:

$$u_1^{m+1} = u_1^m + A_1^{-1} R_1 (f - Au^m) \qquad u_2^{m+1} = u_2^m + A_2^{-1} R_2 (f - Au^m)$$

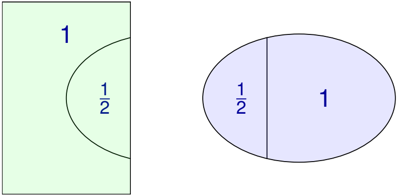
where $u_i^m = R_i u^m$ and $A_i := R_i A R_i^T$.



We have effectively *divided*, but we have yet to *conquer*.

Duplicated unknowns coupled via a *partition of unity*:

$$I = \sum_{i=1}^{N} R_i^T D_i R_i .$$



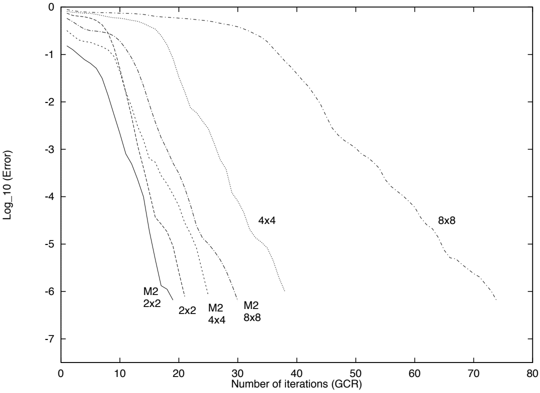
Then, $u^{m+1} = \sum_{i=1}^{N} R_i^T D_i u_i^{m+1}$.

$$M_{RAS}^{-1} = \sum_{i=1}^{N} R_i^T D_i A_i^{-1} R_i$$

+ Krylov acceleration $\Rightarrow$ RAS algorithm (Cai & Sarkis, 1999).
Symmetrized version $M_{ASM}^{-1} := \sum_{i=1}^{N} R_i^T A_i^{-1} R_i$ as a preconditioner.

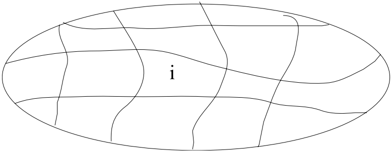
Convergence curves- more subdomains

Plateaus appear in the convergence of the Krylov methods.



Stagnation corresponds to a few very low eigenvalues in the spectrum of the preconditioned problem. They are due to the lack of a global exchange of information in the preconditioner.

$$-\Delta u = f \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$



The mean value of the solution in domain i depends on the value of f on all subdomains.

How to achieve scalability

Introduction of a coarse problem that couples all subdomains. This is closely related to deflation technique classical in linear algebra (see Y. Saad, J. Erhel, Nabben and Vuik) and multigrid techniques.

Strongly related to Multiscale FEM and Reduced Order Model.

Adding a coarse space

One level methods are not scalable: $M_{ASM}^{-1} := \sum_{i=1}^{N} R_i^T A_i^{-1} R_i$.

We add a coarse space correction (*aka* second level). Let V_H be the coarse space and Z be a basis, $V_H = \mathrm{span}\, Z$, we define the two level preconditioner as:

$$M_{ASM,2}^{-1} := Z\,(Z^T A Z)^{-1} Z^T + \sum_{i=1}^{N} R_i^T A_i^{-1} R_i .$$

For constant per subdomain coefficients, Nicolaides approach (1987) is to use the near-kernel of the local operators to build the coarse space:

$$Z := (R_i^T\, D_i R_i \mathbf{1})_{1 \le i \le N} .$$

Key notion: Stable splitting (J. Xu, 1989)
Great for (locally) smooth problems

Theoretical convergence result

Theorem (Widlund, Dryija)

Let $M^{-1}_{ASM,2}$ be the two-level additive Schwarz method:

$$\kappa(M^{-1}_{ASM,2}\,A) \le C\left(1 + \frac{H}{\delta}\right)$$

where δ is the size of the overlap between the subdomains and H the subdomain size.

This does indeed work very well

Number of subdomains	8	16	32	64
ASM	18	35	66	128
ASM + Nicolaides	20	27	28	27

Fails for highly heterogeneous problems
You need a larger and adaptive coarse space

Introduction to GenEO

Adaptive Coarse space for highly heterogeneous Darcy and (compressible) elasticity problems:
Geneo .EVP per subdomain:

Find $V_{j,k} \in \mathbb{R}^{\mathcal{N}_j}$ and $\lambda_{j,k} \geq 0$:

$$A_j^{Neu} V_{j,k} = \lambda_{j,k} D_j R_j A R_j^T D_j V_{j,k}$$

In the two-level ASM, let τ be a user chosen parameter in $[0, 1)$:
Choose eigenvectors $\lambda_{j,k} \leq \tau$ per subdomain:

$$Z := (R_j^T D_j V_{j,k})_{\lambda_{j,k} \leq \tau}^{j=1,\ldots,N}$$

This automatically includes Nicolaides CS made of Zero Energy Modes.

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Energy Modes.

Theory of GenEO

Under two technical assumptions:

Theorem (Spillane, Dolean, Hauret, N., Pechstein, Scheichl (Num. Math. 2013))

If for all j: $0 < \lambda_{j,m_{j+1}} < \infty$*:*

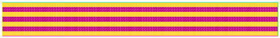
$$\kappa(M_{ASM,2}^{-1} A) \; \leq \; (1 + k_0)\left[2 + k_0\,(2k_0 + 1)\left(1 + \frac{1}{\tau}\right)\right]$$

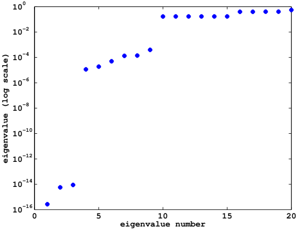
Possible criterion for picking τ: (used in our Numerics)

$$\tau := \max_{j=1,\dots,N} \frac{\delta_j}{H_j}$$

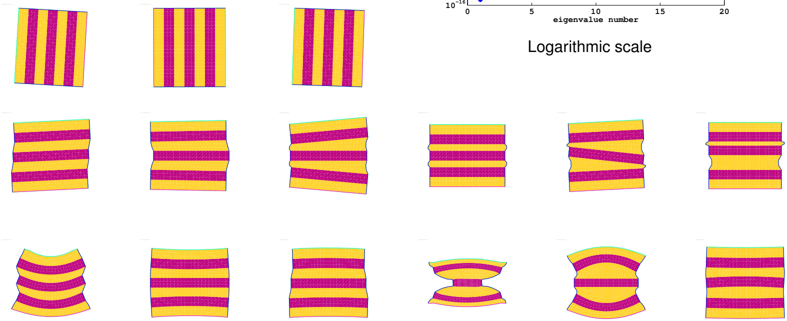
$H_j \dots$ subdomain diameter, $\delta_j \dots$ overlap

Eigenvalues and eigenvectors (Elasticity)





Logarithmic scale

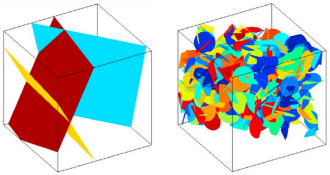


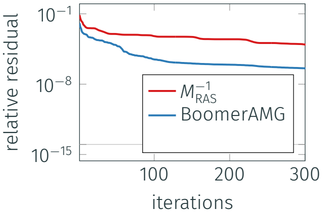
Multiscale FEM based locally *harmonic* spectral coarse space by Ma, Scheichl, Dodwell (2022) and Bénézech, Seelinger, Bastian, Butler, Dodwell, Ma, Scheichl, (2024)

Highly Heterogeneous diffusion problem

Use case for SPD systems I

- HHO for Darcy's law [Ern et al. 2022]
- discrete fracture networks from Inria SERENA
- 140M d.o.f., 4k MPI processes





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Joint work with P. Jolivet, M. Kern, G. Pichot, et al.

Tailoring multigrid method via a PhD: 3 years and is "fragile"

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- HHO for Darcy's law [Ern et al. 2022]
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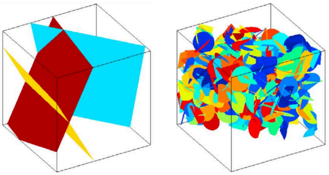
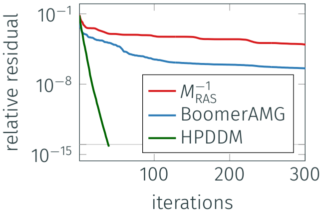
- 140M d.o.f., 4k MPI processes
- 1.002 GC and 1.2 OC (80 sec setup, 20 sec solve)





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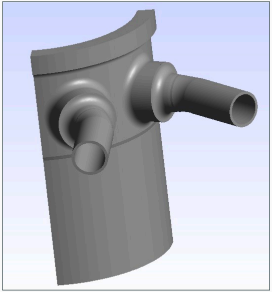
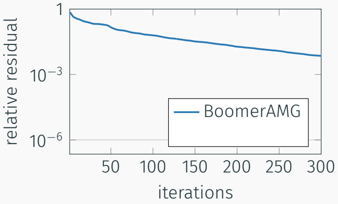
Use case for SPD systems II

- elasticity for stress corrosion cracking at ÉDF
- Code_Aster for the discretization
- 68M d.o.f., 1.2k MPI processes





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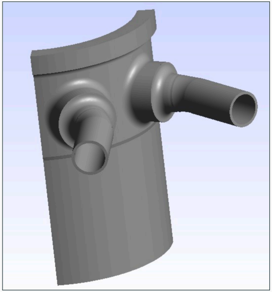
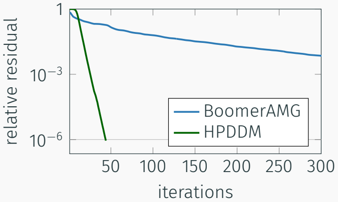
By courtesy of EDF (N. Tardieu)

Use case for SPD systems II

- elasticity for stress corrosion cracking at ÉDF
- Code_Aster for the discretization
- 68M d.o.f., 1.2k MPI processes
- 1.0007 GC and 1.002 OC (1 min setup, 20 sec solve)





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Outline

Extension of GenEO to Saddle Point problem I

Inverting $\mathcal{A}$ (e.g. Stokes, Nearly inc. elasticity):

$$\mathcal{A} := \begin{pmatrix} A & B^T \\ B & -C \end{pmatrix} = \begin{pmatrix} I_d & 0 \\ BA^{-1} & I_d \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & -(C + BA^{-1}B^T) \end{pmatrix} \begin{pmatrix} I_d & A^{-1}B^T \\ 0 & I_d \end{pmatrix}$$

is equivalent to inverting A and $S := C + BA^{-1}B^T$. Starting with $A^{-1} \approx^1 M_{ASM2}^{-1}$ as above, we have

$$S \approx C + BM_{ASM2}^{-1}B^T = S_0 + \underbrace{\sum_{i=1}^{N} \tilde{R}_i^T (\tilde{C}_i + \tilde{B}_i (R_i A R_i^T)^{-1} \tilde{B}_i^T) \tilde{R}_i}_{S_1},$$

where $S_0 := B\,Z_{GenEO}\,(Z_{GenEO}^T\,A\,Z_{GenEO})^{-1}\,Z_{GenEO}^T\,B^T$.

The operator S_1 is itself preconditioned by a Neumann-Neumann/FETI preconditioner with a GenEO type correction.

[1] $\approx$ means controlled provable spectral equivalence

Outline

Nearly incompressible elasticity

The mechanical properties of a solid are characterized by its elastic energy:

$$\int_\Omega 2\,\mu\,\underline{\underline{\varepsilon}}(\boldsymbol{u}) : \underline{\underline{\varepsilon}}(\boldsymbol{u}) + \lambda\,|\mathrm{div}\,(\boldsymbol{u})|^2$$

where the Lamé coefficients λ and μ are defined in terms of the Young modulus E and Poisson ratio ν:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \text{ and } \mu = \frac{E}{2(1+\nu)}\,.$$

As ν is close to $1/2^-$, $\lambda \to \infty$ so that $\mathrm{div}(u) \to 0$, but the pressure p:

$$p := \lambda\,\mathrm{div}\,(\boldsymbol{u}) \to p_{incompressibility}$$

and has thus to be introduced for stability, e.g. $\nu_{rubber} = 0.4999$.

Saddle point system

The resulting discretized variational formulation reads:

$$\begin{cases} \int_\Omega 2\,\mu\,\underline{\underline{\varepsilon}}(\boldsymbol{u}_h) : \underline{\underline{\varepsilon}}(\boldsymbol{v}_h)dx & - \int_\Omega p_h \,\mathrm{div}\,(\boldsymbol{v}_h)dx = \int_\Omega \boldsymbol{f}\boldsymbol{v}_h dx \\ & \\ -\int_\Omega \mathrm{div}\,(\boldsymbol{u}_h) q_h dx & - \int_\Omega \frac{1}{\lambda} p_h q_h = 0. \end{cases} \quad (1)$$

where we take the lowest order Taylor-Hood finite element $C0P2 - C0P1$ so that the pressure p_h is continuous. In matrix form we have:

$$\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_h \\ p_h \end{pmatrix} = \begin{pmatrix} \boldsymbol{F}_h \\ 0 \end{pmatrix}.$$

with an arbitrary domain decomposition .

Mechanical test case

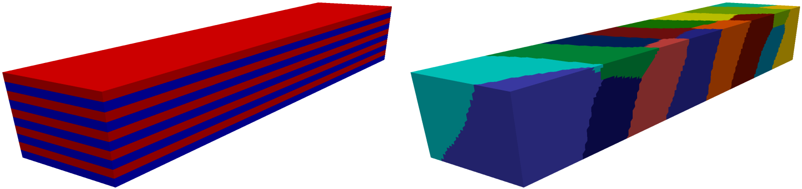

Figure: Heterogeneous beam of rubber and steel. Coefficient distribution (left) and mesh partitioning by the automatic graph partitioner *Metis* (right).

Rubber is nearly incompressible $\nu_{rubber} = 0.4999$ and soft $E_{rubber} = 0.01\,GPa$ whereas steel is compressible $\nu_{steel} = 0.35$ and hard $E_{steel} = 200.GPa$.

Weak scalability

#cores	n	$dim(V_0)$	$dim(\tilde{W}_0)$	setup(s)	#It	gmres(s)	total(s)	#It N_S^{-1}
262	15 987 380	5 383	3 319	710.7	24	631.6	1342.3	11
525	27 545 495	9 959	2 669	526.6	21	519.5	1046.1	12
1 050	64 982 431	17 837	4 587	675.2	22	665.9	1341.1	11
2 100	126 569 042	32 361	7 995	689.2	25	733.8	1423.0	10
4 200	218 337 384	59 704	13 912	593.0	27	705.4	1298.4	10
8 400	515 921 881	141 421	25 949	735.8	32	1152.5	1888.3	10
16 800	1 006 250 208	260 348	41 341	819.2	29	1717.9	2537.1	12

Table: Weak scaling experiment.

Reproducible script with the open source DSL FreeFem[2]
https://github.com/FreeFem/FreeFem-sources/blob/develop/examples/ffddm/elasticity_saddlepoint.edp

[2]Hecht since 1987

Comparison with AMG *GAMG (PETSc)*

Comparisons on the velocity (only) formulation since we were unable to run GAMG on the saddle point formulation.

525 cores	GAMG		DD solver				
ν	#It	total(s)	$dim(V_0)$	setup(s)	#It	gmres(s)	total(s)
0.48	56	25.5	41 766	60.4	18	5.0	65.4
0.485	60	26.1	41 984	60.9	20	5.3	66.2
0.49	116	33.3	42 000	60.4	23	5.9	66.3
0.495	$>$2000	/	42 000	60.4	32	7.6	68.1
0.499	$>$2000	/	42 000	60.6	95	20.3	81.0

Table: GAMG (PETSc) versus standard GenEO for a homogeneous beam discretized with 7.9 million unknowns.

As ν gets close to 0.5, GAMG fails to compute a solution.

Outline

Availability of the GenEO method in HPDDM

- focused on domain decomposition [Jolivet, Hecht, Nataf and Prud'homme 2013]
- block Krylov methods [Jolivet and Tournier 2016]
- POD (Plain Old Data) library, Open-sourced in 2014
- Written in C++, interfaces in C, Python, and Fortran
- integrated in PETSc: KSPHPDDM and PCHPDDM [Jolivet, Roman, Zampini 2021]
- Running on GENCI machines (Irene Rome, A64FX, Jean Zay GPU), porting to Frontier in process
- Used in academia and the industry (FreeFem++, EDF PRISME and ERMES, Denso, ABB, MOOSE)
- Thermomechanics with 300 millions de d.o.f. on 2000 cores for which multigrid fails.

Outline

Spectral Coarse Space for General Sparse problems

In a 2024 Arxiv preprint with E. Parolin, we develop a general theory for:

$$M^{-1} := \sum_{i=1}^{N} R_i^T \, S_i \, R_i$$

(S_i is a local operator) applied to a well posed, real or complex valued, *sparse* problem $A\,x = b$.

Examples:

- RAS algorithm: $S_i = D_i \, A_i^{-1}$
- RAS algorithm with inexact local solves: e.g. $S_i = D_i \; \mathrm{ILU0}_i^{-1}$
- ASM algorithm: $S_i = A_i^{-1}$

Coarse Space for General Sparse problems

Let C be a Hermitian matrix such that

$$\rho := \|I - C^{-1}A\|_C < 1\,,$$

and $\tilde{L}_i := \tilde{D}_i - \tilde{S}_i\,\tilde{R}_i\,A\tilde{R}_i^*$, (tilde matrices are built on a subdomain enclosing subdomain i, oversampling subdomain), the coarse space is built from local generalized eigenvalues:

$$\tilde{L}_i^*\,\tilde{R}_i\,C\,\tilde{R}_i^*\,\tilde{L}_i\,\tilde{u}_{ik} = \lambda_{ik}\,\tilde{C}_i\,\tilde{u}_{ik}$$

Then, let $\tau > 0$ be some user-chosen threshold, we can define a two-level preconditioner M_2^{-1} with a coarse space built from the eigenvalues larger than τ such that:

$$\|I - M_2^{-1}\,A\|_C \leq \frac{\sqrt{k_0\,k_1\,\tau}}{1-\rho}\,.$$

Connections with other works

Key remark: if $S_i := D_i\, A_i^{-1}$ as in RAS, then the local generalized eigenvalues of:

$$\tilde{L}_i^* \, \tilde{R}_i \, C \, \tilde{R}_i^* \, \tilde{L}_i \, \tilde{u}_{ik} = \lambda_{ik} \, \tilde{C}_i \, \tilde{u}_{ik}$$

are harmonic in the subdomains as in

- *Ma, R. Scheichl, and T. Dodwell, 2022* for Multiscale FEM
- *Q. Hu and Z. Li, 2024* and *Strehlow, Ma, Scheichl, 2024* where the two-level RAS preconditioner with exact local solves is studied

Note that our theory applies to the RAS algorithm with inexact local solves as well as to other DD preconditioners:

- additive Schwarz Method (ASM): $S^i := A_i^{-1}$
- Symmetrized Optimized Restricted Additive Schwarz (SORAS): $S^i := D_i\, A_i^{-1}\, D_i$

First numerical results

- Smaller coarse space than the GenEO coarse space.
- Efficiency with Incomplete Cholevski (ICC)
- More on the way

Conclusion and Prospects

"Past" works

- Spectral Coarse space for SPD problems
- Iterative solver for saddle point problem with highly heterogeneous linear elasticity, Stokes systems, ...
- Available via HPDDM with or without its interface to PETSc
- Available to FreeFem user

Recent works (2024) on coarse space constructions

- Nataf, Parolin, "Coarse spaces for non-symmetric two-level preconditioners based on local generalized eigenproblems", 2024
- Al Daas, Jolivet, Nataf, Tournier (2024). "A robust two-level Schwarz preconditioner for sparse matrices", submitted for publication. (not shown here)

See also *Heinlein, Smetana, 2022*

Acknowledgements

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- F Nataf and P.-H. Tournier, ”A GenEO Domain Decomposition method for Saddle Point problems”, Comptes Rendus. Académie des Sciences, 2023
- F Nataf and E. Parolin, ”Coarse spaces for non-symmetric two-level preconditioners based on local generalized eigenproblems”, Arxiv, 2024