Pricing and calibration with neural networks in finance

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Agenda

- Derivatives pricing, Feynman-Kac Theorem
- Fourier methods
  - Basics of COS method;
  - Parameter calibration
  - Initial attempts with neural networks.

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- Derivatives pricing, Feynman-Kac Theorem
- Fourier methods
  - Basics of COS method;
  - Parameter calibration
  - Initial attempts with neural networks.
- Joint work with Shuaiqiang Liu, Sander Bohte, Anastasia Borovykh

Feynman-Kac Theorem

- The linear partial differential equation:

$$\frac{\partial v(t,x)}{\partial t} + \mathcal{L}v(t,x) + g(t,x) = 0, \quad v(T,x) = h(x),$$

with operator

$$\mathcal{L}v(t,x) = \mu(x)Dv(t,x) + \frac{1}{2}\sigma^2(x)D^2v(t,x).$$

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Feynman-Kac theorem:

$$v(t,x) = \mathbb{E}\left[\int_t^T g(s,X_s)ds + h(X_T)\right],$$

where X_s is the solution to the FSDE

$$dX_s = \mu(X_s)ds + \sigma(X_s)d\omega_s,\ X_t = x.$$

Feynman-Kac Theorem (option pricing context)

Given the final condition problem

$$\begin{cases} \frac{\partial v}{\partial t} & + \quad \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + rS\frac{\partial v}{\partial S} - rv = 0, \\ v(T,S) & = \quad h(S_T) = \text{ given} \end{cases}$$

Then the value, $v(t,S)$, is the unique solution of

$$v(t,S) = e^{-r(T-t)}\mathbb{E}^Q\{v(T,S_T)|\mathcal{F}_t\}$$

with the sum of first derivatives square integrable, and $S = S_t$ satisfies the system of SDEs:

$$dS_t = rS_t dt + \sigma S_t d\omega_t^Q,$$

- Similar relations also hold for (multi-D) SDEs and PDEs!

A pricing approach; European options

$$v(t_0, S_0) = e^{-r(T-t_0)}\mathbb{E}^Q\{h(S_T)|\mathcal{F}_0\}$$

Quadrature:

$$v(t_0, S_0) = e^{-r(T-t_0)} \int_{\mathbb{R}} h(S_T) f(S_T, S_0) dS_T$$

- Trans. PDF, $f(S_T, S_0)$, typically not available, but the characteristic function, $\widehat{f}$, often is.

Motivation Fourier Methods

- Derive pricing methods that
    - are computationally fast
    - should work as long as we have a characteristic function,

$$\widehat{f}(u;x) = \int_{-\infty}^{\infty} e^{iux} f(x) dx;$$

      (available for Lévy processes and affine SDE systems).
    - The characteristic function of a Lévy process is known by means of the celebrated Lévy-Khinchine formula.

Mathematical models for option pricing

- The Black-Scholes asset model,

$$dS_t = rS_t dt + \sqrt{\nu_t} S_t d\omega_t^s, \quad S_{t_0} = S_0,$$

- The Heston model (considering stochastic volatility),

$$dS_t = rS_t dt + \sqrt{\nu_t} S_t d\omega_t^s, \quad S_{t_0} = S_0,$$
$$d\nu_t = \kappa(\bar{\nu} - \nu_t)dt + \gamma\sqrt{\nu_t} d\omega_t^\nu, \quad \nu_{t_0} = \nu_0,$$
$$d\omega_t^s d\omega_t^\nu = \rho_{x,\nu} dt,$$

- The Bates model (considering price jumps),

$$\frac{dS_t}{S_t} = \left(r - \lambda_J \mathbb{E}[e^J - 1]\right) dt + \sqrt{\nu_t} d\omega_t^x + \left(e^J - 1\right) dX_t^{\mathcal{P}},$$
$$d\nu_t = \kappa(\bar{\nu} - \nu_t)dt + \gamma\sqrt{\nu_t} d\omega_t^\nu, \quad \nu_{t_0} = \nu_0,$$
$$d\omega_t^s d\omega_t^\nu = \rho_{x,\nu} dt,$$

Heston option valuation PDE

- Calibrating is to fit 5 parameters, correlation coefficient ρ, long term variance $\bar{\nu}$, reversion speed κ, volatility of volatility γ, initial variance ν_0, given market option prices, v_c^{mkt}, v_p^{mkt}.

- The Heston option pricing PDE with these five parameters,

$$\begin{aligned}\frac{\partial v}{\partial t} &+ rS\frac{\partial v}{\partial S} + \kappa(\bar{\nu} - \nu)\frac{\partial v}{\partial \nu} + \frac{1}{2}\nu S^2 \frac{\partial^2 v}{\partial S^2} \\ &+ \rho\gamma S\nu \frac{\partial^2 v}{\partial S \partial \nu} + \frac{1}{2}\gamma^2 \nu \frac{\partial^2 v}{\partial \nu^2} - rv = 0.\end{aligned}$$

  where $v = v(t, S, \nu; K, T)$ is the option price at time t, with suitable terminal conditions.

- A European option payoff function: $v_c(T, S_T) = (S_T - K)^+$, $v_p(T, S_T) = (K - S_T)^+$, with strike price K.

Fourier-Cosine Expansions, COS Method (with Fang Fang)

- The COS method:
  - Exponential convergence;
  - Greeks (derivatives) are obtained at no additional cost.
- Based on the availability of a characteristic function.
- The basic idea:
  - Replace the density by its Fourier-cosine series expansion;
  - Coefficients have simple relation to characteristic function.

Series Coefficients of the Density and the ChF

- Fourier-Cosine expansion of density function on interval $[a, b]$:

$$f(x) = {\sum_{n=0}^{\infty}}' F_n \cos\left(n\pi \frac{x-a}{b-a}\right),$$

with $x \in [a, b] \subset \mathbb{R}$ and the coefficients defined as

$$F_n := \frac{2}{b-a} \int_a^b f(x) \cos\left(n\pi \frac{x-a}{b-a}\right) dx.$$

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- F_n has direct relation to the ChF, $\widehat{f}(u) := \int_{\mathbb{R}} f(x) e^{iux} dx$ ($\int_{\mathbb{R}\setminus[a,b]} f(x) \approx 0$),

$$\begin{aligned} F_n \approx P_n \quad &:= \quad \frac{2}{b-a} \int_{\mathbb{R}} f(x) \cos\left(n\pi \frac{x-a}{b-a}\right) dx \\ &= \quad \frac{2}{b-a} \Re\left\{ \widehat{f}\left(\frac{n\pi}{b-a}\right) \exp\left(-i \frac{na\pi}{b-a}\right) \right\}. \end{aligned}$$

Pricing European Options

- Start from the risk-neutral valuation formula:

$$v(t_0, x) = e^{-r\Delta t}\mathbb{E}^{\mathbb{Q}}\left[v(T, y)|\mathcal{F}_0\right] = e^{-r\Delta t}\int_{\mathbb{R}} v(T, y)f(y, x)dy.$$

- Truncate the integration range:

$$v(t_0, x) = e^{-r\Delta t}\int_{[a,b]} v(T, y)f(y, x)dy + \varepsilon.$$

- Replace the density by the COS approximation, and interchange summation and integration:

$$\hat{v}(t_0, x) = e^{-r\Delta t}{\sum_{n=0}^{N-1}}{}' \Re\left\{\widehat{f}\left(\frac{n\pi}{b-a}; x\right)e^{-in\pi\frac{a}{b-a}}\right\}\mathcal{H}_n,$$

where the series coefficients of the payoff, $\mathcal{H}_n$, are analytic.

Pricing European Options

- Log-asset prices: $x := \log(S_0/K) \quad \text{and} \quad y := \log(S_T/K)$.
- The payoff for European call options reads

$$v(T, y) \equiv \max\left(K(e^y - 1), 0\right).$$

- For a call option, we obtain

$$\begin{aligned} \mathcal{H}_k^{call} &= \frac{2}{b-a}\int_0^b K(e^y - 1)\cos\left(k\pi\frac{y-a}{b-a}\right) dy \\ &= \frac{2}{b-a} K\left(\chi_k(0,b) - \psi_k(0,b)\right). \end{aligned}$$

- For a vanilla put, we find

$$\mathcal{H}_k^{put} = \frac{2}{b-a} K\left(-\chi_k(a,0) + \psi_k(a,0)\right).$$

Results, Heston stochastic volatility PDE

- GPU computing: Multiple strikes for parallelism, 21 IC's.

Heston model				
	N	64	128	256
MATLAB	msec	3.850890	7.703350	15.556240
	max.abs.err	6.0991e-04	2.7601e-08	$< 10^{-14}$
GPU	msec	0.177860	0.209093	0.333786

Table 1: Maximum absolute error when pricing a vector of 21 strikes.

- Exponential convergence, Error analysis in our papers.
- Also work with wavelets instead of cosines.

Implied Volatility

Implied Volatility: "The wrong number in the wrong formula to get the right price". [Rebonato 1999]

Mathematically, we have:

$$v_c(t, S) = BS(\sigma, r, T, K, S_0)$$

where BS is monotonically increasing in σ (higher volatility corresponds to higher prices). Now, assume the existence of some inverse function

$$g_\sigma(\cdot) = BS^{-1}(\cdot)$$

where

$$\sigma_{impl} := g_\sigma(v_c, r, T, K, S_0).$$

Solving the inverse pricing model function

How to find implied volatility?

The inverse of the BS pricing function BS, $g_\sigma(\cdot)$, is not known in closed-form. A root-finding technique is used to solve the equation:

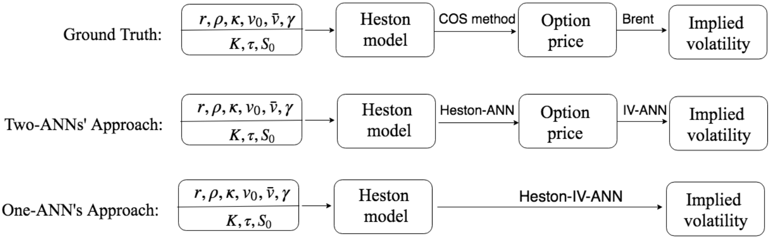
$$BS(\sigma_{impl}, r, T, K, S_0) - v_c^{mkt} = 0.$$

There are many ways to solve this equation, like "Newton-Raphson" or "Brent" iteration [1]. Since the options prices can move very quickly, it is often important to use the most efficient method when calculating implied volatilities.

[1] http://en.wikipedia.org/wiki/Brent's_method

CaNN for option pricing models

- CaNN consists of two stages, a forward pass and a backward pass. For example, Heston-CaNN:
- Forward pass:



“Neural Networks”

- Generate the sample data points for input parameters,
- Calculate the corresponding output with PDE or MC (option price or implied volatility), to form a complete set with in- and outputs,
- Split the above data set into a training and a test part,
- Train the ANN on the training data set,
- Evaluate the ANN on the test data set,
- Replace the original solver by the trained ANN in applications.

4.1 Implied volatility

- A gradient squashing technique is used to deal with an gradient in the volatilities wrt. option prices (see [Shuaiqiang et al, 2018]).
  - Obtain a **time value** by subtracting a **intrinsic value**,

$$\hat{V} = V_t^* - \max(S_t - Ke^{-r\tau}, 0)$$

  - Log-scale the intrinsic value, $\log\left(\hat{V}/K\right)$

	MSE	MAE	R^2
Input: S, K, τ, r, V/K Output: σ^*	$6.36 \cdot 10^{-4}$	$1.24 \cdot 10^{-2}$	0.97510
almond Input: S, K,τ, r, $\log(\tilde{V}/K)$ Output: σ^*	$1.55 \cdot 10^{-8}$	$9.73 \cdot 10^{-5}$	0.9999998

ANN-based model calibration

- Calculating IV is the most frequently executed numerical task in practice. The paper [S. Liu et al., 2019] developed a neural network solver to learn the 1D inversion of Black-Scholes.

Iterative algorithm	GPU (sec)	CPU (sec)	Robust
Newton-Raphson	19.68	23.06	No
Brent	52.08	60.67	Yes
Bi-section	337.94	390.91	Yes
IV-ANN	0.20	1.90	Yes

Table 2: The total time over 20,000 different cases. CPU (Intel i5) and GPU (Tesla P100). Robustness means no initial value is required.

Asset model calibration

- The difference between model value Q and market value Q^*,

$$J(\Theta) := \sum_{i=1}^{N} w_i ||Q(\tau_i, m_i; \Theta) - Q^*(\tau_i, m_i)|| + \bar{\lambda}||\Theta||,$$

  where Q could be either an option price or implied volatility (IV), with moneyness $m = S/K$ and time to maturity $\tau = T - t$, N the number of samples, $\bar{\lambda}$ a regularization factor.

- The objective function,

$$\underset{\Theta \in \mathbb{R}^n}{\operatorname{argmin}} J(\Theta),$$

with n the number of parameters to calibrate. For Heston, $\Theta := [\rho, \kappa, \gamma, \bar{\nu}, \nu_0]$; for Bates, $\Theta := [\rho, \kappa, \gamma, \bar{\nu}, \nu_0, \lambda_J, \mu_J, \sigma_J]$; for Black-Scholes, $\Theta := [\sigma]$;

Asset model calibration

- The inverse problem is computationally intensive, and the objective functions are often non-convex and non-linear, especially for high-dimensional model calibration.
- A fast and generic calibration framework should (at least) comprise three components, an efficient solver, a global optimizer and a parallel computing environment.

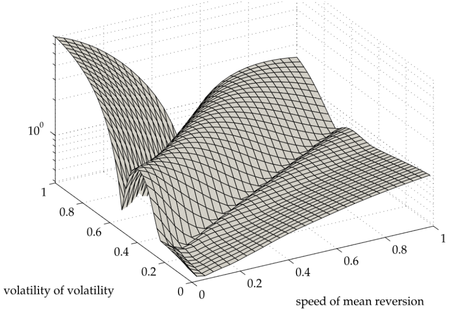
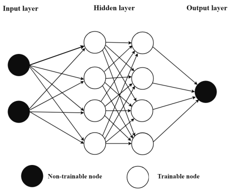


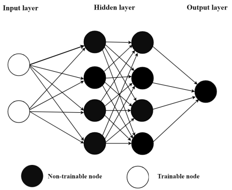
Figure 1: Multiple minima when calibrating Heston [Gilli and Schumann, 2011].

Calibration neural networks

- Training/prediction phases learn the numerical solvers, while the calibration phase inverts the trained ANN.
- The three phases are viewed as a whole, and the difference is just to change the learnable units.



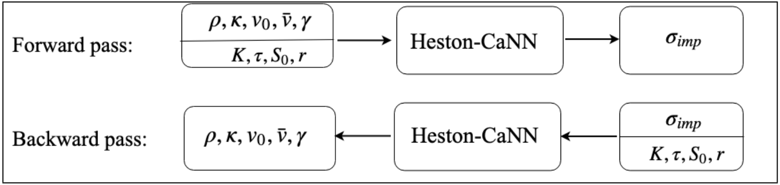
[Training phase (offline)]



[Calibration phase (online)]

CaNN for option pricing models

- Backward pass:



The forward pass

- The training data set with IV being the target quantity:

ANN	Parameters	Value Range	Method
ANN Input	Moneyness, $m = S_0/K$	[0.6, 1.4]	LHS
	Time to maturity, τ	[0.05, 3.0](year)	LHS
	Risk free rate, r	[0.0%, 5%]	LHS
	Correlation, ρ	[-0.90, 0.0]	LHS
	Reversion speed, κ	(0, 3.0]	LHS
	Volatility of volatility, γ	(0.01, 0.8]	LHS
	Long average variance, $\bar{\nu}$	(0.01, 0.5]	LHS
	Initial variance, ν_0	(0.05, 0.5]	LHS
-	European put price, v	$(0, 0.6)$	COS
ANN Output	implied volatility, σ	$(0, 0.76)$	Brent

Table 3: LHS=Latin Hypercube Sampling, COS [Fang and Oosterlee, 2008] to solve Heston, and Brent for implied vol.

- The evaluation result suggests no over-fitting.

Heston-CaNN	MSE	MAE	MAPE	R^2
Training	8.07×10^{-8}	2.15×10^{-4}	5.83×10^{-4}	0.9999936
Testing	1.23×10^{-7}	2.40×10^{-4}	7.20×10^{-4}	0.9999903

The backward pass of the CaNN

- Calibration on 35 samples (7 strike prices and 5 maturity time).
- Heston-CaNN averaged performance over 15,625 test cases.

Deviation from true Θ^*		Averaged Cost/Error	
$\|\nu_0^\dagger - \nu_0^*\|$	4.39×10^{-4}	CPU time (seconds)	0.85
$\|\bar{\nu}^\dagger - \bar{\nu}^*\|$	4.54×10^{-3}	GPU time (seconds)	0.48
$\|\gamma^\dagger - \gamma^*\|$	3.28×10^{-2}	Function evaluations	193, 249
$\|\rho^\dagger - \rho^*\|$	4.84×10^{-2}	Data points	35
$\|\kappa^\dagger - \kappa^*\|$	4.88×10^{-2}	Calibration error $J(\Theta)$	2.52×10^{-6}

parameter	lower	upper	points	CaNN search space
ρ	-0.75	-0.25	5	[-0.85,-0.05]
$\bar{\nu}$	0.15	0.35	5	[0.05, 0.45]
γ	0.3	0.5	5	[0.05, 0.75]
ν_0	0.15	0.35	5	[0.05, 0.45]
κ	0.5	1.0	5	[0.1, 2.0]

Summary

- The problem of financial model calibration is converted into a machine learning problem.

- We need robust components (many different parameter sets)!
- The robust and generic framework CaNN rapidly reaches a global solution with ANN's inherent parallelism.

- One neural network solves two problems, e.g., the forwards pass for a numerical solution of models, the backward pass for model calibration and sensitivity analysis. i
- Training is highly efficient with the COS method

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When we don't know the characteristic function I?

First discretize!

We can write the Euler, Milstein, and 2.0 weak Taylor discretization schemes in the following general form

$$X_{m+1}^{\Delta} = x + m(x)\Delta t + s(x)\Delta\omega_{m+1} + \kappa(x)(\Delta\omega_{m+1})^2, \quad X_m^{\Delta} = x.$$

For the Euler scheme:

$$m(x) = \mu(x), \quad s(x) = \sigma(x), \quad \kappa(x) = 0.$$

For the Milstein scheme:

$$m(x) = \mu(x) - \frac{1}{2}\sigma\sigma_x(x), \quad s(x) = \sigma(x), \quad \kappa(x) = \frac{1}{2}\sigma\sigma_x(x).$$

For the order 2.0 weak Taylor scheme:

$$\begin{aligned} m(x) &= \mu(x) - \tfrac{1}{2}\sigma\sigma_x(x) + \tfrac{1}{2}\left(\mu\mu_x(x) + \tfrac{1}{2}\mu_{xx}\sigma^2(x)\right)\Delta t, \\ s(x) &= \sigma(x) + \tfrac{1}{2}\left(\mu_x\sigma(x) + \mu\sigma_x(x) + \tfrac{1}{2}\sigma_{xx}\sigma^2(x)\right)\Delta t, \\ \kappa(x) &= \tfrac{1}{2}\sigma\sigma_x(x). \end{aligned}$$

Characteristic function

$$X^{\Delta}_{m+1} = x + m(x)\Delta t + \kappa(x)\left(\Delta\omega_{m+1} + \frac{1}{2}\frac{s(x)}{\kappa(x)}\right)^2 - \frac{1}{4}\frac{s^2(x)}{\kappa(x)}$$
$$\stackrel{d}{=} x + m(x)\Delta t - \frac{1}{4}\frac{s^2(x)}{\kappa(x)} + \kappa(x)\Delta t\left(U_{m+1} + \sqrt{\lambda(x)}\right)^2,$$

with $\lambda(x) := \frac{1}{4}\frac{s^2(x)}{\kappa^2(x)\Delta t}$, $U_{m+1} \sim \mathcal{N}(0,1)$. $(U_{m+1} + \sqrt{\lambda(x)})^2 \sim \chi_1'^2(\lambda(x))$ non-central chi-squared distributed.

The characteristic function of X^{Δ}_{m+1}, given $X^{\Delta}_m = x$

$$\widehat{f}_{X^{\Delta}_{m+1}}(u|X^{\Delta}_m = x) = \mathbb{E}\left[\exp\left(iuX^{\Delta}_{m+1}\right)\middle|X^{\Delta}_m = x\right]$$
$$= \exp\left(iux + ium(x)\Delta t - \frac{\frac{1}{2}u^2 s^2(x)\Delta t}{1-2iu\kappa(x)\Delta t}\right)(1-2iu\kappa(x)\Delta t)^{-1/2}.$$

When we do not know the ChF II? "Taylor expansion"

- In Pascucci's group (U. Bologna), the adjoint expansion method for the approximation of the ChF in local Lévy models was developed;
- Taylor expansion-based formulas for the ChF possess a structure that allows for the FFT in COS for early-exercise options.

⇒ An efficient second-order accurate Bermudan COS pricing formula results.

- J. Math. Anal. Applic. paper with A. Borovykh, A. Pascucci: "Pricing options under local Lévy models with default";