Bounds on Derivatives and Martingale Optimal Transportation

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Outline

Analytic formulation (Monge 1781)

- Initial distribution : probability measure μ

- Final distribution : probability measure ν

Problem : find an optimal transference plan T^*

$$P_2^M := \sup_{T \in \mathcal{T}(\mu,\nu)} \int c\big(x, T(x)\big)\mu(dx)$$

where $\mathcal{T}(\mu,\nu)$ of all maps $T : x \longmapsto y = T(x)$ such that

$$\nu \;=\; \mu \circ T^{-1}$$

Probabilistic formulation (Kantorovich 1942)

Randomization of transference plans :

$$\overline{P}_2^K := \sup_{\mathbb{P}\in\mathcal{P}_2(\mu,\nu)} \int c(x,y)\mathbb{P}(dx,dy)$$

where $\mathcal{P}_2(\mu,\nu)$ is the collection of all joint probability measures with marginals μ and ν

Example : $c(x,y) = -|x-y|^2 \Longrightarrow$ maximization of correlations :

$$\sup_{\mathbb{P}\in\mathcal{P}_2(\mu,\nu)} \mathbb{E}^{\mathbb{P}}[XY]$$

Kantorovich duality

Duality in linear programming, Legendre-Fenchel duality...

$$
\begin{aligned}
D_2^0 &:= \inf_{(\varphi,\psi)\in\mathcal{D}_2^0} \int \varphi(x)\mu(dx) + \int \psi(y)\nu(dy) \\
\mathcal{D}_2^0 &:= \left\{(\varphi,\psi) : \varphi^+ \in \mathbb{L}^1(\mu), \psi^+ \in \mathbb{L}^1(\nu), \varphi \oplus \psi \leq c\right\}
\end{aligned}
$$

where $\varphi \oplus \psi(x,y) := \varphi(x) + \psi(y)$

- Inequality $P_2^K \geq D_2^0$ obvious
- Reverse inequality needs Hahn-Banach theorem

Back to the original Monge formulation

- $P_2^K \geq P_2^M$: Kantorovitch formulation $\equiv$ relaxation of Monge one

Theorem (Y. Brenier)

Let $c \in C^1$ with $c_{xy} > 0$. Assume μ has no atoms. Then there is a unique optimal transference plan :

$$\mathbb{P}^*(dx, dy) = \mu(dx)\delta_{\{T^*(x)\}}(dy) \quad with \quad T^* = F_\nu^{-1} \circ F_\mu$$

Consequently $P_2^M = P_2^K$, and T^* solves both problems.

- T^* : monotone rearrangement, Frechet-Hoeffding coupling
- $c_{xy} > 0$: Spence-Mirrlees condition

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On the Spence Mirrlees condition

The solution of the Kantorovitch optimal transportation problem

$$\overline{P}_2^K := \sup_{\mathbb{P}\in\mathcal{P}_2(\mu,\nu)} \int c(x,y)\mathbb{P}(dx,dy)$$

is not modified by the change of performance criterion :

$$c(x,y) \longrightarrow \hat{c}(x,y) := c(x,y) + a(x) + b(y)$$

Notice that the Spence Mirrlees condition $c_{xy} > 0$ is stable by this transformation

Financial interpretation

- $X \sim \mu$ and $Y \sim \nu$ prices of two assets at time 1

- μ and ν identified from market prices of call options :

$$C_\mu(K) = \int (x-K)^+ \mu(dx), \qquad C_\nu(K) = \int (y-K)^+ \nu(dy)$$

(Breeden-Litzenberger 1978)

- $c(X, Y)$ payoff of derivative security

- Robust bounds on dervative's price :

$$\inf_{\mathbb{P} \in \mathcal{P}_2(\mu,\nu)} \mathbb{E}^{\mathbb{P}}[c(X,Y)] \quad \text{and} \quad \sup_{\mathbb{P} \in \mathcal{P}_2(\mu,\nu)} \mathbb{E}^{\mathbb{P}}[c(X,Y)]$$

Financial interpretation of the dual problem

- $\varphi(X), \psi(Y)$: optimal Vanilla position in Assets X and Y

- Can be expressed as a combination of calls/puts (Carr-Madan) :

$$g(s) = g(s^*) + (s - s^*)g'(s^*) + \int_0^{s^*} (K - s)^+ g''(K)dK + \int_{s^*}^{\infty} (s - K)^+ g''(K)dK$$

so their market market prices are $\int \varphi d\mu$ and $\int \psi d\nu$

- Then

$$D_2^0 = \inf_{(\varphi,\psi)\in\mathcal{D}_2^0} \int \varphi(x)\mu(dx) + \int \psi(y)\nu(dy)$$

is the cheapest static position in X and Y so as to superhedge $c(X, Y)$

Lower bound

Set $\bar{c}(\bar{x}, y) := -c(-\bar{x}, y)$. Then

$$\inf_{\mathbb{P}\in\mathcal{P}_2(\mu,\nu)} \mathbb{E}^{\mathbb{P}}[c(X,Y)] \;=\; -\sup_{\mathbb{P}\in\mathcal{P}_2(\mu,\nu)} \mathbb{E}^{\mathbb{P}}[\bar{c}(-\bar{X},Y)]$$

where

• $\bar{X} := -X \sim \bar{\mu}$ with c.d.f. $F_{\bar{\mu}}(\bar{x}) := 1 - F_\mu(-\bar{x})$

• $\bar{c}$ satisfies the Spence Mirrlees condition, whenever c does. So, the lower bound is attained by the anti-monotone transference plan :

$$\mathbb{P}_*(dx, dy) := \mu(dx)\delta_{\{T_*(x)\}}(dy), \qquad T_*(x) := F_\nu^{-1} \circ F_{\bar{\mu}}$$

Multimarginals Optimal transportation problem

- Gangbo and Święch 1998, Carlier 2003, Pass 2011

$$\sup_{\mathbb{P}\in\mathcal{P}_n(\mu)} \mathbb{E}[c(X)]$$

where $X = (X_1, \ldots, X_n)$, $\mu = (\mu_1, \ldots, \mu_n)$, and $\mathcal{P}_n(\mu) = \ldots$

- Pass 2012

$$\sup_{\mathbb{P}\in\mathcal{P}_\infty(\mu)} \mathbb{E}\Big[c\Big(\int_0^1 X_t dt\Big)\Big]$$

where $X = (X_t)_{t\in[0,1]}$, $\mu = (\mu_t)_{t\in[0,1]}$, and $\mathcal{P}_\infty(\mu) = \ldots$

Outline

One asset observed at two future dates

Our interest now is on the case where

$$X = X_0 \quad \text{and} \quad Y = X_1$$

are the prices of the same asset at two future dates 0 and 1

Interest rate is reduced to zero

This setting introduces a new feature :

- the possibility of dynamic trading the asset between times 0 and 1
- duality converts this possibility into the martingale condition $\mathbb{E}^{\mathbb{P}}[Y|X] = X$

The superhedging problem

- Start from initial capital V_0, hold $h(X)$ shares shares of $X \Longrightarrow$

$$V_0 + h(X)(Y - X)$$

- Trading in all European call options of any strike is possible $\Longrightarrow$

$$V_1^{H,\varphi,\psi} := V_0 + h(X)(Y - X) + \varphi(X) - \mu(\varphi) + \psi(Y) - \nu(\psi)$$

Superhedging problem :

$$v_0 := \inf\left\{V_0 : \ V_1^{h,\varphi,\psi} \geq c(X,Y) \text{ for some } h \in \mathbb{L}^0,\ \varphi \in \mathbb{L}^1(\mu),\ \psi \in \mathbb{L}^1(\nu)\right\}$$

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- Start from initial capital V_0, hold $h(X)$ shares shares of X $\Longrightarrow$

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Superhedging $\equiv$ Kantorovitch dual problem

Equivalently :

$$v_0 \quad = \quad D_2(\mu,\nu) \; = \inf_{(\varphi,\psi,h)\in\mathcal{D}_2} \left\{\mu(\varphi)+\nu(\psi)\right\}$$

where $\mu(\varphi)=\int\varphi d\mu$, $\mu(\psi)=\int\psi d\nu$, and

$$\mathcal{D}_2 := \Big\{(\varphi,\psi,h) \quad : \quad \varphi^+\in\mathbb{L}^1(\mu),\psi^+\in\mathbb{L}^1(\nu), h\in\mathbb{L}^0 \\ \varphi\oplus\psi+h^{\otimes}\geq c\Big\}$$

$$h^{\otimes}(x,y) := h(x)(y-x)$$

The Martingale Optimal Transportation Problem

The corresponding dual problem is :

$$P_2(\mu,\nu) \quad := \quad \sup_{\mathbb{P}\in\mathcal{M}_2(\mu,\nu)} \mathbb{E}^{\mathbb{P}}\big[c(X,Y)\big]$$

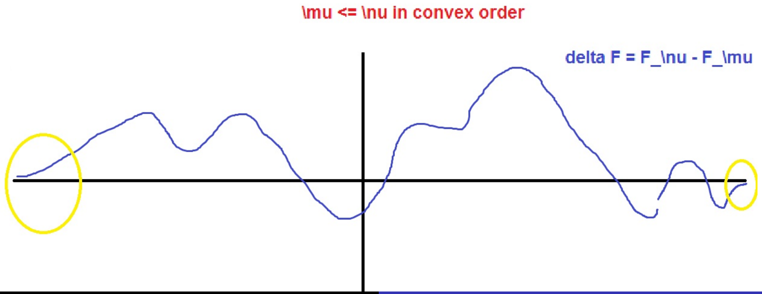
where $\mathcal{M}_2(\mu,\nu) := \big\{\mathbb{P}\in\mathcal{P}_2(\mu,\nu) : \mathbb{E}^{\mathbb{P}}[Y|X] = X\big\}$

and we recall $\mathcal{P}_2(\mu,\nu) := \big\{\mathbb{P}\in\mathcal{P}_{\mathbb{R}^2} : X\sim_{\mathbb{P}}\mu,\, Y\sim_{\mathbb{P}}\nu\big\}$

Implication of the convex ordering

Kellerer 1972 : $\mathcal{M}_2(\mu,\nu) \neq \emptyset$ iff μ and ν have same mean and $\mu \preceq \nu$ (convex), i.e. with $\delta F := F_\nu - F_\mu$

$$\int \delta F(\xi) d\xi = 0 \quad \text{and for all } k \quad \int_{(-\infty,k)} \delta F(\xi) d\xi \geq 0$$



Duality

Assume $c(x,y)$ USC with linear growth, and recall

$$\mathcal{M}_2(\mu,\nu) := \left\{\mathbb{P}:\ X \sim_{\mathbb{P}} \mu,\ \ Y \sim_{\mathbb{P}} \nu \text{ and } \mathbb{E}^{\mathbb{P}}[Y|X] = X\right\}$$
$$\mathcal{D}_2 := \left\{(\varphi,\psi,h) \in \mathbb{L}^1(\mu)\times\mathbb{L}^1(\nu)\times\mathbb{L}^0:\ \varphi\oplus\psi + h^{\otimes} \geq c\right\}$$

- The inequality $P_2(\mu,\nu) \geq D_2(\mu,\nu)$ is obvious

Theorem (Beiglbock, Henry-Labordère, Penkner 2011)

$P_2(\mu,\nu) = D_2(\mu,\nu)$. Moreover existence holds for the Martingale Transportation Problem $P_2(\mu,\nu)$

- existence for the dual problem $D_2(\mu,\nu)$ may fail
- duality result is not needed for our main result
- Continuous-time : Galichon, Henry-Labordère, T. 2011, Dolinsky & Soner 2012

Duality

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$$\begin{aligned} \mathcal{M}_2(\mu,\nu) &:= \left\{\mathbb{P}:\ X\sim_{\mathbb{P}}\mu,\ \ Y\sim_{\mathbb{P}}\nu \text{ and } \mathbb{E}^{\mathbb{P}}[Y|X]=X\right\} \\ \mathcal{D}_2 &:= \left\{(\varphi,\psi,h)\in\mathbb{L}^1(\mu)\times\mathbb{L}^1(\nu)\times\mathbb{L}^0:\ \varphi\oplus\psi+h^{\otimes}\geq c\right\} \end{aligned}$$

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Worst Case Financial Market – Brenier Theorem

- The solution $\mathbb{P}^* \in \mathcal{M}_2(\mu, \nu)$ always exists

- **Question 1** : Is there a transference plan, i.e. optimal transportation of μ to ν through a map T^* ? (Brenier Theorem)

Can not be a map, unless $\mu = \nu$!

- **Question 2** : Is there a transference plan along a minimal randomization

X →

$Y = T_u(X)$ with probability $q(X)$

$Y = T_d(X)$ with probability $1 - q(X)$

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- **Question 2** : Is there a transference plan along a minimal randomization

$$X \nearrow \quad Y = T_u(X) \text{ with probability } q(X)$$

$$X \searrow \quad Y = T_d(X) \text{ with probability } 1 - q(X)$$

Previous literature

Hobson and Neuberger (MF 2012) :

- Analyse the specific case $c(x, y) = |x - y|$
- They characterize an optimal $\mathbb{P}^*$ defined by a transference plan :

$$\mathbb{P}^*(dx, dy) = \mu(du)\big[q(x)\delta_{\{T_u(x)\}}(dy) + (1 - q)(x)\delta_{\{T_d(x)\}}(dy)\big]$$

where

$$q(x) := \frac{x - T_d(x)}{T_u(x) - T_d(x)}, \qquad T_d(x) \le x \le T_u(x)$$

and

T_u and T_d are non-decreasing

Previous literature : Beiglblock and Juillet (2012)

Definition

$\mathbb{P} \in \mathcal{M}_2(\mu,\nu)$ is **left-monotone** if $\mathbb{P}[(X,Y) \in \Gamma] = 1$, for some $\Gamma \subset \mathbb{R} \times \mathbb{R}$, and

$$\text{for all } (x, y_1), (x, y_2), (x', y') \in \Gamma : \quad x < x' \quad \Longrightarrow \quad y' \notin (y_1, y_2)$$

Definition

$\mathbb{P} \in \mathcal{M}_2(\mu,\nu)$ is called a **left curtain** if $\mathbb{P}$ is left-monotone and concentrated on two graphs

$$\mathbb{P} = \mu(dx)\big[q(x)\delta_{\{T_u(x)\}}(dy)(1-q)(x)\delta_{\{T_d(x)\}}(dy)\big]$$

Previous literature : Beiglblock and Juillet (2012)

Theorem

$\mu_2 \succeq \mu_1$, μ_1 without atoms. Then :
(i) there exists a unique left-monotone $\mathbb{P}^* \in \mathcal{M}_2(\mu, \nu)$, and $\mathbb{P}^*$ is a left-curtain
(ii) $\mathbb{P}^*$ is a solution $P_2(\mu, \nu)$ in the following cases :

- $c(x, y) = h(x - y)$ *with h' strictly convex,*
- $c(x, y) = \varphi(x)\psi(y)$, $\varphi, \psi \geq 0$, ψ *strict convex,* φ *decreasing*

Our objective :

- explicit derivation of $\mathbb{P}^*$
- extend the class of couplings c for which $\mathbb{P}^*$ is optimal
- extend to the multi-marginals case

Explicit left-monotone transference plan

Theorem

Let μ, ν have finite first moment, same mean, $\mu \preceq \nu$, and μ without atoms. Then, the unique left-monotone transference plan is

$$\mathbb{P}^*(dx, dy) = \left[q(x)\delta_{T_d(x)}(dx) + (1-q)(x)\delta_{T_u(x)}(dx)\right]\mu(dx)$$

where T_u, T_d are explicitly defined as follows...
In particular, outside jumps, T_u and T_d solve the following ODEs :

$$d(\delta F \circ T_d) = (1-q)dF_\mu, \quad d(F_\nu \circ T_u) = qdF_\mu$$

Duality and explicit Martingale Version of the Brenier Theorem

Theorem

Let μ, ν have finite first moment, same mean, $\mu \preceq \nu$, and μ without atoms. Assume that $c_{xyy} > 0$. Then

$$P_2 = D_2$$

and there is an explicit dual optimizer (φ^*, ψ^*, h^*) defined as follows...

The martingale version of the Spence-Mirrlees condition

... is $c_{xyy} > 0$:

• Notice that the solution of the Martingale Transport problem is not altered by the change of performance criterion :

$$c(x,y) \quad \longrightarrow \quad \hat{c}(x,y) := c(x,y) + a(x) + b(y) + h(x)(y-x)$$

• $\hat{c}_{xyy} = c_{xyy}$

• The conditions of Beiglblock and Juillet :

- $c(x,y) = h(x-y)$ with h' strictly convex,
- $c(x,y) = \varphi(x)\psi(y)$, $\varphi, \psi \geq 0$, ψ strict convex, φ decreasing

satisfy $c_{xyy} > 0$

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satisfy $c_{xyy} > 0$

Lower bound

Suppose $c_{xyy} > 0$. Then

$$\bar{c}(\bar{x}, \bar{y}) := -c(-\bar{x}, -\bar{y}) \quad \text{satisfies} \quad \bar{c}_{\bar{x}\bar{y}\bar{y}} > 0$$

We exploit this symmetry to derive the lower bound :

$$\begin{aligned} \inf_{\mathbb{P} \in \mathcal{M}_2(\mu,\nu)} \mathbb{E}^{\mathbb{P}}\left[c(X, Y)\right] &= - \sup_{\mathbb{P} \in \mathcal{M}_2(\mu,\nu)} \mathbb{E}^{\mathbb{P}}\left[\bar{c}(\bar{X}, \bar{Y})\right] \\ &= \mathbb{E}^{\mathbb{P}_*}\left[c(X, Y)\right] \end{aligned}$$

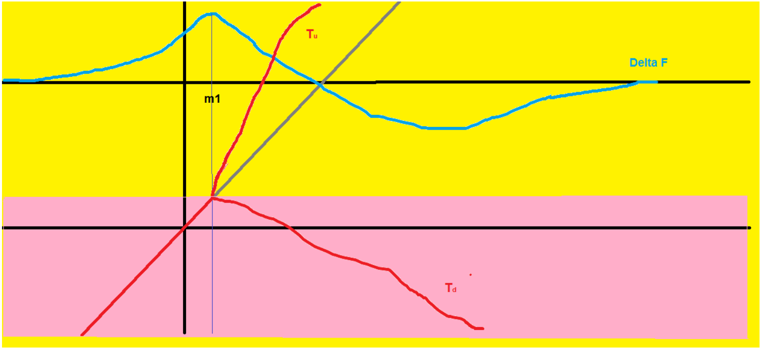
where $\mathbb{P}_*$ is the left-monotone transference plan constructed from

$$F_{\bar{\mu}}(\bar{x}) := 1 - F_{\mu}(-\bar{x}) \quad \text{and} \quad F_{\bar{\nu}}(\bar{y}) := 1 - F_{\nu}(-\bar{y}-)$$

Construction : One local maximizer of δF

Easy case : $T_u \nearrow$ and $T_d \searrow$ after m_1, and

$$\mathbb{P}^*(dx, dy) \;=\; \mu_0(dx)\Big[q(x)\delta_{\{T_u(x)\}}(dy) + (1 - q(x))\delta_{\{T_d(x)\}}(dy)\Big]$$



Martingale transportation constraints

- First marginal is μ_0, Martingale condition holds if $q \in [0,1]$
- Second marginal :
  - either $y \leq m_1$, then $\mathbb{P}_*[Y \in dy] = dF_\mu(y) + \mathbb{E}\big[(1-q)(X)\mathbb{1}_{\{T_d(X)\in dy\}}\big]$. So $Y \sim_{\mathbb{P}_*} \nu$ with decreasing T_d implies

$$d(\delta F \circ T_d) \quad = \quad -(1-q)dF_\mu,$$

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$$d(F_\nu \circ T_u) = q dF_\mu.$$

Characterizing T_u and T_d

By direct integration :

$$T_u(x) = g\big(x, T_d(x)\big)$$

and $T_d(x)$ is the unique solution in $(-\infty, m_1)$ of

$$\int_{-\infty}^{F_\nu^{-1}\circ F_\mu(x)} \xi dF_\nu(\xi) - \int_{-\infty}^{x} \xi dF_\mu(\xi) + \int_{-\infty}^{T_d(x)} \big(g(x,\xi) - \xi\big) d\delta F(\xi) = 0$$

where

$$g(x,y) \quad := \quad F_\nu^{-1}\big(F_\mu(x) + \delta F(y)\big)$$

(compare to Fréchet-Hoeffding coupling !)

T_u and T_d as solutions of ODEs

• Assume F_0 and F_1 are differentiable. Then

$$T_d' = -\frac{(1-q)F_0'}{\delta F' \circ T_d}, \quad T_u' = \frac{qF_0'}{F_1' \circ T_u} \quad \text{on} \quad (-\infty, m_1)$$

where $q = (x - T_d)/(T_u - T_d)$

• If in addition m_1 is a global maximizer of δF, and F_0, F_1 are twice differentiable near m_1,

$$T_d(m_1+) = m_1, \quad T_d'(m_1+) = -1/2 \quad \text{and} \quad T_d''(m_1) = +\infty$$

The Kantorovitch Dual Side

So far, we have :

$$\mathbb{E}^{\mathbb{P}_*}[c(X,Y)] \;\le\; \sup_{\mathcal{M}_2(\mu,\nu)} \mathbb{E}^{\mathbb{P}}[c(X,Y)] \le \inf_{\mathcal{D}_2} \{\mu(\varphi)+\nu(\psi)\}$$

Our next goal is to construct

$$(\varphi_*,\psi_*,h_*)\in\mathcal{D}_2 \quad \text{such that} \quad \mu(\varphi_*)+\nu(\psi_*)=\mathbb{E}^{\mathbb{P}_*}[c(X,Y)]$$

In particular, this would imply duality and existence hold

$$\Longrightarrow \varphi_*(X)+\psi_*(Y)+h_*(X)(Y-X)-c(X,Y)=0,\ \mathbb{P}_*\text{−a.s.}$$

$$\Longrightarrow \varphi_*(x)=\max_{y\in\mathbb{R}}\{c(x,y)-\psi_*(y)-h_*(x)(y-x)\},\ x\in\mathbb{R}$$

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$$(\varphi_*, \psi_*, h_*) \in \mathcal{D}_2 \quad \text{such that} \quad \mu(\varphi_*) + \nu(\psi_*) = \mathbb{E}^{\mathbb{P}_*}[c(X,Y)]$$

In particular, this would imply duality and existence hold

$$\Longrightarrow \varphi_*(X) + \psi_*(Y) + h_*(X)(Y-X) - c(X,Y) = 0, \; \mathbb{P}_* - \text{a.s.}$$

$$\Longrightarrow \varphi_*(x) = \max_{y\in\mathbb{R}}\{c(x,y) - \psi_*(y) - h_*(x)(y-x)\}, \; x \in \mathbb{R}$$

The Kantorovitch Dual Side

So far, we have :

$$\mathbb{E}^{\mathbb{P}_*}[c(X,Y)] \quad \leq \quad \sup_{\mathcal{M}_2(\mu,\nu)} \mathbb{E}^{\mathbb{P}}[c(X,Y)] \leq \inf_{\mathcal{D}_2} \{\mu(\varphi)+\nu(\psi)\}$$

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Explicit Dual Optimizers

- Dynamic hedging strategy :

$$\left| \begin{array}{l} h'_* = \frac{c_x(.,T_u)-c_x(.,T_d)}{T_u-T_d} \quad \text{on} \quad (m_1,\infty) \\ h_* = h_* \circ T_d^{-1} + c_y(.,.) - c_y(T_d^{-1},.) \quad \text{on} \quad (-\infty, m_1) \end{array} \right.$$

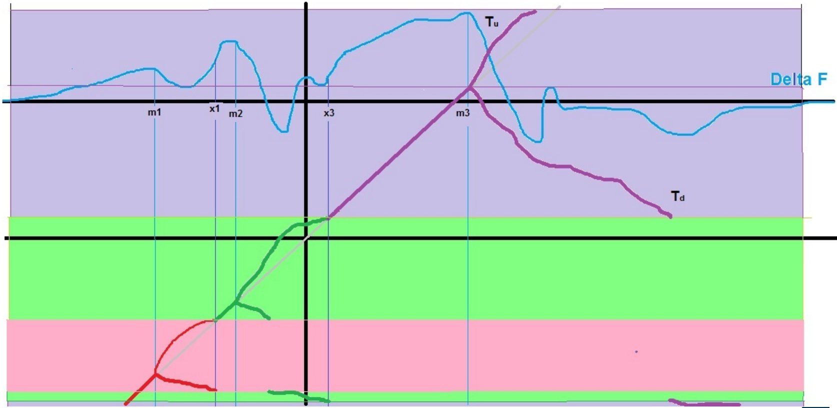
- Static hedging in Y :

$$\left| \begin{array}{l} \psi'_* = c_y(T_u^{-1},.) - h_* \circ T_u^{-1} \quad \text{on} \quad (m_1,\infty) \\ \psi'_* = c_y(T_d^{-1},.) - h_* \circ T_d^{-1} \quad \text{on} \quad (-\infty, m_1) \end{array} \right.$$

- Static hedging in X :

$$\varphi_*(x) = \mathbb{E}^{\mathbb{P}_*}\left[c(x,Y) - \psi_*(Y) | X = x\right] \quad x \in \mathbb{R}$$

Multiple local maxima of δF



Outline

Finitely many marginals martingale transportation

- Extension to finite discrete-time is immediate :
  - μ_i have same mean, and $\mu_n \succeq \ldots \succeq \mu_0$
  - Optimal transportation with n marginals constraint :

$$P_n(\mu) = \sup_{\mathbb{P}\in\mathcal{M}_n(\mu)} \mathbb{E}^{\mathbb{P}}[c(X)], \qquad c(x_1,\ldots,x_n) = \sum_{i=1}^{n-1} c^i(x_i, x_{i+1})$$

  - The dual problem :

$$D_n(\mu) \;:=\; \inf_{(u,h)\in\mathcal{D}_n} \sum_{i=1}^{n} \mu_i(u_i),$$

where

$$\mathcal{D}_n := \left\{(u,h) : (u_i)^+ \in \mathbb{L}^1(\mu_i) \text{ and } \oplus_{i=1}^n u_i + \sum_{i=1}^{n-1} h_i^{\otimes^i} \geq c\right\}.$$

Martingale Transportation under finitely many marginals constraints

Theorem

Suppose $\mu_1 \preceq \ldots \preceq \mu_n$ in convex order, with finite first moment, same mean, and $\mu_1, \ldots, \mu_{n-1}$ have no atoms. Assume further that $c^i_{xyy} > 0$. Then, the strong duality holds, the transference plan

$$\mathbb{P}^*_n(dx) = \mu_1(dx_1) \prod_{i=1}^{n-1} T^i_*(x_i, dx_{i+1})$$

is optimal for the martingale transportation problem $P_n(\mu)$, and (u^*, h^*) is optimal for the dual problem $D_n(\mu)$

Example : applies to the discrete monitoring variance swap : $c(x_1, \ldots, x_n) := \sum_{i=1}^{n} \left(\ln \frac{x_i}{x_{i-1}}\right)^2$

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One maximizer $m(t)$ of $\partial_t F(t,x)$: first guess

- Continuous-time limit : guess

$$T_u(t,x) = x + j^u(t,x)\Delta t \quad \text{and} \quad T_d(t,x) = x - j^d(t,x)$$

Plugg in the ODEs $d(\delta F \circ T_d) = (1-q)dF_\mu, \quad d(F_\nu \circ T_u) = qdF_\mu$

$$\partial_x j_d(t,x) = 1 + \frac{j_u(t,x)}{j_d(t,x)} \frac{f(t,x)}{\partial_t f(t, x - j_d(t,x))}, \quad x > m(t)$$

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- pointof view 2 : Fokker-Planck equation for the density f

What kind of continuous-time dynamics ?

The continuous-time dynamics

The equation satisfied by the density is :

$$\partial_t f = -\mathbb{1}_{\{x<m(t)\}} \frac{j_u f}{j_d(1-\partial_x j_d)}\left(t, T_d^{-1}(t,x)\right) - \mathbb{1}_{\{x>m(t)\}}\left(\frac{j_u f}{j_d} - \partial_x(j_u f)\right)$$

- Consider the pure (downward) jump Markov process in the spirit of local Lévy models (Carr-Madan-Geman-Yor)

$$X_t = X_0 - \int_0^t \mathbb{1}_{\{X_{t-}>m(t)\}} j_d(t, X_{t-})\left(dN_t - \nu_t dt\right)$$

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Theorem

The process X induces an a.c. measure with density f satisfying the above FP equation

- This is remarkable Peacock (PCOC) in the terminology of Yor

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