Mean Field Games: Numerical Methods

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$$
\begin{cases}
\dfrac{\partial u}{\partial t} - \nu \Delta u + H(x, \nabla u) = \Phi[m] & \text{in } (0, T] \times \mathbb{T} \\
\dfrac{\partial m}{\partial t} + \nu \Delta m + \operatorname{div}\left(m \dfrac{\partial H}{\partial p}(x, \nabla u) \right) = 0 & \text{in } [0, T) \times \mathbb{T} \\
u(t=0) = \Phi_0[m(t=0)] \\
m(t=T) = m_\circ
\end{cases}
\qquad (*)
$$

where

$$
H(x,p) = \sup_{\gamma \in \mathbb{R}^d} \left(p \cdot \gamma - L(x,\gamma) \right).
$$

Except when mentioned,

$$
\mathbb{T} = (\mathbb{R}/\mathbb{Z})^d \quad \text{(periodic problem)}.
$$

Most of what follows holds with Neumann or Dirichlet conditions.

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Realistic models may include **congestion**, i.e. L depends on m, for example

$$L(x, m, \gamma) = \ell(x) + (c_1 + c_2 m)^q |\gamma|^\beta.$$

This induces a stronger coupling between u and m in $(*)$.

A simple case

Framework

- $d = 1$
- L is strictly convex

$$H(x, p) = \sup_{\gamma \in \mathbb{R}} (p \cdot \gamma - L(x, \gamma))$$

- $\Phi[m](x) = F(m(x))$ and $F = W'$ where $W : \mathbb{R} \to \mathbb{R}$ is a strictly convex function
- $\Phi_0[m](x) = F_0(m(x))$ and $F_0 = W_0'$ where $W_0 : \mathbb{R} \to \mathbb{R}$ is a strictly convex function

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(*) can be found as the optimality conditions of an optimal control problem on a transport equation.

Optimal control problem

Minimize

$$J(m,\gamma) = \int_0^T \int_{\mathbb{T}} \Big(m(t,x) L(x,\gamma(t,x)) + W(m(t,x)) \Big) dx dt + \int_{\mathbb{T}} W_0(m(x,0)) dx$$

subject to the constraints

$$\left\{ \begin{array}{rcl} \frac{\partial m}{\partial t} + \nu \Delta m + \mathrm{div}(m\,\gamma) & = & 0, \quad \text{in } (0,T)\times\mathbb{T}, \\ m(T,x) & = & m_T(x) \quad \text{in } \mathbb{T}. \end{array} \right.$$

Optimality conditions

$\delta\gamma \mapsto \delta m \mapsto \delta J$

$$\left\{ \begin{array}{rcl} \partial_t \delta m + \nu \Delta \delta m + \mathrm{div}(\delta m\, \gamma) &=& -\mathrm{div}(m\, \delta\gamma), \quad \text{in } (0,T)\times \mathbb{T}, \\ \delta m(T,x) &=& 0 \quad \text{in } \mathbb{T}. \end{array} \right.$$

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$$\delta J(m,\gamma) = \int_0^T \int_{\mathbb{T}} \delta m(t,x) \Big(L(x,\gamma(t,x)) + F(m(t,x)) \Big)$$
$$+ \int_0^T \int_{\mathbb{T}} \delta\gamma(t,x) m(t,x) \frac{\partial L}{\partial \gamma}(x,\gamma(t,x)) + \int_{\mathbb{T}} \delta m(0,x) F_0(m(0,x)).$$

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$$\delta J(m,\gamma) = \int_0^T \int_{\mathbb{T}} \delta m(t,x) \Big(L(x,\gamma(t,x)) + F(m(t,x)) \Big) + \int_0^T \int_{\mathbb{T}} \delta\gamma(t,x) m(t,x) \frac{\partial L}{\partial \gamma}(x,\gamma(t,x)) + \int_{\mathbb{T}} \delta m(0,x) F_0(m(0,x)).$$

Adjoint problem

$$\left\{ \begin{array}{l} \dfrac{\partial u}{\partial t} - \nu \Delta u + \gamma \cdot \nabla u = L(x,\gamma) + F(m) \qquad \text{in } (0,T]\times \mathbb{T} \\ u(t=0) = F_0(m|_{t=0}) \end{array} \right.$$

Variation of J

$$
\begin{aligned}
\delta J(m,\gamma) &= \int_0^T \int_{\mathbb{T}} -u(t,x)\left(\partial_t \delta m + \nu\Delta \delta m + \operatorname{div}(\delta m\,\gamma)\right) \\
&\quad + \int_0^T \int_{\mathbb{T}} m(t,x)\delta\gamma(t,x)\frac{\partial L}{\partial \gamma}(x,\gamma(t,x)) \\
&= \int_0^T \int_{\mathbb{T}} m(t,x)\ \left(\frac{\partial L}{\partial \gamma}(x,\gamma(t,x)) - \nabla u(t,x)\right)\delta\gamma(t,x).
\end{aligned}
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$$+ \int_0^T \int_{\mathbb{T}} m(t,x)\delta\gamma(t,x)\frac{\partial L}{\partial \gamma}(x,\gamma(t,x))$$
$$= \int_0^T \int_{\mathbb{T}} m(t,x)\,\left(\frac{\partial L}{\partial \gamma}(x,\gamma(t,x)) - \nabla u(t,x)\right)\delta\gamma(t,x).$$

Optimality conditions

- $\nabla u(t,x) = \frac{\partial L}{\partial \gamma}(x,\gamma^*(t,x))$
- $\gamma^*(t,x)$ achieves the max. in $H(x,p) = \sup_\gamma \left(p\cdot\gamma - L(x,\gamma)\right)$ and
$$\gamma^*(t,x) = H_p(x,\nabla u(t,x))$$
- $\Rightarrow$ MFG system of PDEs

A discrete scheme when $L(x,\gamma) = f(x) + \ell(\gamma)$

- Assume that ℓ is strictly convex and $\ell(0) = \ell'(0) = 0$
- Uniform grid: $x_i = ih$, $t_n = n\Delta t$

The transport equation for m

- γ is discretized on a staggered grid: $\gamma^n_{i+1/2} \approx \gamma(t_n, x_i + h/2)$
- upwind scheme (explicit w.r.t γ)

$$0 = \frac{m_i^{n+1} - m_i^n}{\Delta t} + \nu(\Delta_h m^n)_i + \gamma^{n+1,+}_{i+1/2} m^n_{i+1} - \gamma^{n+1,-}_{i+1/2} m^n_i - \gamma^{n+1,+}_{i-1/2} m^n_i + \gamma^{n+1,-}_{i-1/2} m^n_{i-1}.$$

The scheme is conservative and preserves positivity: it is L^1 stable.

Discrete version of J: many possible choices

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- To preserve the structure of the PDE system, we rather choose:

$$J_h = h\Delta t \sum_n \sum_i m_i^n \left(f(x_i) + \ell(\gamma_{i-1/2}^{n+1,+}) + \ell(-\gamma_{i+1/2}^{n+1,-}) \right) + h\Delta t \sum_n \sum_i W(m_i^n) + h \sum_i W_0(m_i^n)$$

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Adjoint equation

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \nu(\Delta_h u^{n+1})_i + \gamma_{i-1/2}^{n+1,+} \frac{u_i^{n+1} - u_{i-1}^{n+1}}{h} - \gamma_{i+1/2}^{n+1,-} \frac{u_{i+1}^{n+1} - u_i^{n+1}}{h} = f(x_i) + \ell(\gamma_{i-1/2}^{n+1,+}) + \ell(-\gamma_{i+1/2}^{n+1,-}) + F(m_i^n)$$

Optimality conditions for the discrete problem

$$\frac{\partial \ell}{\partial \gamma}(\gamma_{i+1/2}^{n+1,*}) = (u_{i+1}^{n+1} - u_i^{n+1})/h.$$

Kushner-Dupuis numerical Hamiltonian:

$$g(x, p_1, p_2) = -f(x) + \max_{\gamma \in \mathbb{R}} \left(-p_1^- \gamma + p_2^+ \gamma - \ell(\gamma) \right)$$

Then

$$\gamma_{i+1/2}^{n+1,*,-} = -\frac{\partial g}{\partial p_1} \left(x_i, (u_{i+1}^{n+1} - u_i^{n+1})/h, (u_i^{n+1} - u_{i-1}^{n+1})/h \right),$$

$$\gamma_{i-1/2}^{n+1,*,+} = \frac{\partial g}{\partial p_2} \left(x_i, (u_{i+1}^{n+1} - u_i^{n+1})/h, (u_i^{n+1} - u_{i-1}^{n+1})/h \right).$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \nu (\Delta_h u^{n+1})_i + g\left(x_i, \frac{u_{i+1}^{n+1} - u_i^{n+1}}{h}, \frac{u_i^{n+1} - u_{i-1}^{n+1}}{h} \right) = F(m_i^n)$$

Direct discretization of (*)

Take $d = 2$.

- Let $\mathbb{T}_h$ be a uniform grid on the torus with mesh step h, and x_{ij} be a generic point in $\mathbb{T}_h$
- Uniform time grid: $\Delta t = T/N_T$, $t_n = n\Delta t$
- The values of u and m at $(x_{i,j}, t_n)$ are approximated by $u^n_{i,j}$ and $m^n_{i,j}$

Notation

- The discrete Laplace operator:

$$(\Delta_h w)_{i,j} = \frac{1}{h^2}(w_{i+1,j} + w_{i-1,j} + w_{i,j+1} + w_{i,j-1} - 4w_{i,j})$$

- Right-sided finite difference formulas for $\frac{\partial w}{\partial x_1}(x_{i,j})$ and $\frac{\partial w}{\partial x_2}(x_{i,j})$

$$(D_1 w)_{i,j} = \frac{w_{i+1,j} - w_{i,j}}{h}, \quad \text{and} \quad (D_2 w)_{i,j} = \frac{w_{i,j+1} - w_{i,j}}{h}$$

- The collection of the 4 first order finite difference formulas at $x_{i,j}$

$$[D_h w]_{i,j} = \left\{ (D_1 w)_{i,j}, (D_1 w)_{i-1,j}, (D_2 w)_{i,j}, (D_2 w)_{i,j-1} \right\}$$

For the Bellman equation, a semi-implicit monotone scheme

$$\frac{\partial u}{\partial t} - \nu \Delta u + H(x, \nabla u) = \Phi[m]$$

$$\downarrow$$

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} - \nu (\Delta_h u^{n+1})_{i,j} + g(x_{i,j}, [D_h u^{n+1}]_{i,j}) = (\Phi_h[m^n])_{i,j}$$

where $[D_h u]_{i,j} \in \mathbb{R}^4$ is the collection of the two first order finite difference formulas at $x_{i,j}$ for $\partial_x u$ and for $\partial_y u$.

$$\begin{aligned} & g(x_{i,j}, [D_h u^{n+1}]_{i,j}) \\ =& g\Big(x_{i,j}, (D_1 u^{n+1})_{i,j}, (D_1 u^{n+1})_{i-1,j}, (D_2 u^{n+1})_{i,j}, (D_2 u^{n+1})_{i,j-1}\Big) \end{aligned}$$

Assumptions on the discrete Hamiltonian g

$$(q_1, q_2, q_3, q_4) \rightarrow g\left(x, q_1, q_2, q_3, q_4\right).$$

- **Monotonicity:**
  - g is nonincreasing with respect to q_1 and q_3
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$$g(x, q_1, q_1, q_3, q_3) = H(x, q), \quad \forall x \in \mathbb{T}, \forall q = (q_1, q_3) \in \mathbb{R}^2$$

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- **Differentiability:** g is of class $\mathcal{C}^1$
- **Convexity (for uniqueness and stability):**
  $$(q_1, q_2, q_3, q_4) \rightarrow g(x, q_1, q_2, q_3, q_4) \text{ is convex}$$

Coupling

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- For uniqueness and stability, the following assumption will be useful:

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- Same thing for $\Phi_{0,h}$

The approximation of the Fokker-Planck equation

$$\frac{\partial m}{\partial t} + \nu \Delta m + \operatorname{div}\left(m \frac{\partial H}{\partial p}(x, \nabla v) \right) = 0. \qquad (\dagger)$$

It is chosen so that

- each time step leads to a linear system for m with a matrix
  - whose diagonal coefficients are negative
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- The argument for uniqueness should hold in the discrete case, so **the discrete Hamiltonian g should be used for $(\dagger)$ as well**

Principle

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by $$h^2 \sum_{i,j} m_{i,j} \nabla_q g(x_{i,j}, [D_h u]_{i,j}) \cdot [D_h w]_{i,j}$$

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by $$-h^2\sum_{i,j}\mathcal{T}_{i,j}(u,m)w_{i,j} \equiv h^2\sum_{i,j} m_{i,j}\nabla_q g(x_{i,j},[D_h u]_{i,j})\cdot [D_h w]_{i,j}$$

Discrete version of $\operatorname{div}(mH_p(x,\nabla u))$:

$$\begin{aligned}
&\mathcal{T}_{i,j}(u,m)\\
=&\frac{1}{h}\left(\begin{array}{c}\left(\begin{array}{c} m_{i,j}\dfrac{\partial g}{\partial q_1}(x_{i,j},[D_h u]_{i,j}) - m_{i-1,j}\dfrac{\partial g}{\partial q_1}(x_{i-1,j},[D_h u]_{i-1,j})\\ +m_{i+1,j}\dfrac{\partial g}{\partial q_2}(x_{i+1,j},[D_h u]_{i+1,j}) - m_{i,j}\dfrac{\partial g}{\partial q_2}(x_{i,j},[D_h u]_{i,j})\end{array}\right)\\ +\left(\begin{array}{c} m_{i,j}\dfrac{\partial g}{\partial q_3}(x_{i,j},[D_h u]_{i,j}) - m_{i,j-1}\dfrac{\partial g}{\partial q_3}(x_{i,j-1},[D_h u]_{i,j-1})\\ +m_{i,j+1}\dfrac{\partial g}{\partial q_4}(x_{i,j+1},[D_h u]_{i,j+1}) - m_{i,j}\dfrac{\partial g}{\partial q_4}(x_{i,j},[D_h u]_{i,j})\end{array}\right)\end{array}\right)
\end{aligned}$$

Semi-implicit scheme

$$\begin{cases} \dfrac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} - \nu(\Delta_h u^{n+1})_{i,j} + g(x_{i,j}, [D_h u^{n+1}]_{i,j}) = (\Phi_h[m^n])_{i,j} \\ \\ \dfrac{m_{i,j}^{n+1} - m_{i,j}^n}{\Delta t} + \nu(\Delta_h m^n)_{i,j} + \mathcal{T}_{i,j}(u^{n+1}, m^n) = 0 \end{cases}$$

The operator $m \mapsto \nu(\Delta_h m)_{i,j} + \mathcal{T}_{i,j}(u,m)$ is the adjoint of the linearized version of $u \mapsto \nu(\Delta_h u)_{i,j} - g(x_{i,j}, [D_h u]_{i,j})$.

The discrete MFG system has the same structure as the continuous one.

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Well known discrete Hamiltonians g can be chosen.

For example, if the Hamiltonian is of the form $H(x, \nabla u) = \psi(x, |\nabla u|)$, a possible choice is the **upwind scheme**:

$$g(x, q_1, q_2, q_3, q_4) = \psi\left(x, \sqrt{(q_1^-)^2 + (q_2^+)^2 + (q_3^-)^2 + (q_4^+)^2}\right).$$

Existence and bounds

Define the set of discrete probability densities

$$\mathcal{K} = \left\{ (m_{i,j})_{0 \le i,j < N} : h^2 \sum_{i,j} m_{i,j} = 1, m_{i,j} \ge 0 \right\}.$$

If

- g is of class $\mathcal{C}^1$, and monotone w.r.t. q
- Φ_h and $\Phi_{0,h}$ are continuous operators on $\mathcal{K}$

then **the discrete problem has a solution such that** $m^n \in \mathcal{K}$, $\forall n$.

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Discrete Lipschitz estimates on u can be obtained if Φ_h is a suitable approximation of a nonlocal smoothing operator and if g satisfies additional properties, for example

$$\left| \frac{\partial g}{\partial x}\Big(x, (q_1, q_2, q_3, q_4)\Big) \right| \le C(1 + |q_1| + |q_2| + |q_3| + |q_4|).$$

Strategy of proof

- Brouwer fixed point theorem in $\mathcal{K}^{N_T}$ taking advantage of the structure of the system
- estimates on u uniform w.r.t m, but possibly depending on h and Δt (using the monotonicity of g)
- if Φ is a nonlocal smoothing operator, discrete Lipchitz bounds on $\Phi_h[m]$ yield estimates on the discrete Lipschitz norm of u, uniform in m, h and Δt

A key identity for uniqueness and stability

A perturbed system

$$\begin{cases} \dfrac{\tilde{u}_{i,j}^{n+1} - \tilde{u}_{i,j}^{n}}{\Delta t} - \nu(\Delta_h \tilde{u}^{n+1})_{i,j} + g(x_{i,j}, [D_h \tilde{u}^{n+1}]_{i,j}) = (\Phi_h[\tilde{m}^n])_{i,j} + a_{i,j}^n \\ \dfrac{\tilde{m}_{i,j}^{n+1} - \tilde{m}_{i,j}^{n}}{\Delta t} + \nu(\Delta_h \tilde{m}^{n})_{i,j} + \mathcal{T}_{i,j}(\tilde{u}^{n+1}, \tilde{m}^n) = b_{i,j}^n \end{cases}$$

- Multiply the 2 discrete HJB equations by $m_{i,j}^n - \tilde{m}_{i,j}^n$, sum on n, i, j, and subtract the results
- Multiply the 2 discrete FP equations by $u_{i,j}^{n+1} - \tilde{u}_{i,j}^{n+1}$, sum on n, i, j, and subtract the results
- Add the 2 resulting identities

One gets

$$-\frac{1}{\Delta t}\left(m^{N_T}-\tilde m^{N_T},u^{N_T}-\tilde u^{N_T}\right)_2+\frac{1}{\Delta t}\left(m^0-\tilde m^0,u^0-\tilde u^0\right)_2$$

$$+\mathcal{E}(m,u,\tilde u)+\mathcal{E}(\tilde m,\tilde u,u)+\sum_{n=0}^{N_T-1}\left(\Phi_h[m^n]-\Phi_h[\tilde m^n],m^n-\tilde m^n\right)_2$$

$$=\sum_{n=0}^{N_T-1}\left(a^n,m^n-\tilde m^n\right)_2+\sum_{n=1}^{N_T}\left(b^{n-1},u^n-\tilde u^n\right)_2$$

where

$$\mathcal{E}(m,u,\tilde u)=\sum_{i,j,n} m_{i,j}^{n-1}\left(\begin{array}{l} g(x_{i,j},[D\tilde u^n]_{i,j})-g(x_{i,j},[Du^n]_{i,j})- \\ -g_q(x_{i,j},[Du^n]_{i,j})\cdot([D\tilde u^n]_{i,j}-[Du^n]_{i,j}) \end{array}\right)$$

One gets

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$$+\mathcal{E}(m,u,\tilde{u})+\mathcal{E}(\tilde{m},\tilde{u},u)+\sum_{n=0}^{N_T-1}\left(\Phi_h[m^n]-\Phi_h[\tilde{m}^n],m^n-\tilde{m}^n\right)_2$$

$$=\sum_{n=0}^{N_T-1}\left(a^n,m^n-\tilde{m}^n\right)_2+\sum_{n=1}^{N_T}\left(b^{n-1},u^n-\tilde{u}^n\right)_2$$

where

$$\mathcal{E}(m,u,\tilde{u})=\sum_{i,j,n}m_{i,j}^{n-1}\left(\begin{array}{l} g(x_{i,j},[D\tilde{u}^n]_{i,j})-g(x_{i,j},[Du^n]_{i,j})- \\ -g_q(x_{i,j},[Du^n]_{i,j})\cdot([D\tilde{u}^n]_{i,j}-[Du^n]_{i,j}) \end{array}\right)$$

- Convexity of $g \Rightarrow \mathcal{E}(m,u,\tilde{u})\geq 0$ if $m\geq 0$
- If Φ_h is monotone, $\left(\Phi_h[m^n]-\Phi_h[\tilde{m}^n],m^n-\tilde{m}^n\right)_2\geq 0$

First consequence: uniqueness

If

- g is convex
- Φ_h is monotone

$$(\Phi_h[m] - \Phi_h[\tilde{m}], m - \tilde{m})_2 \leq 0 \quad \Rightarrow \quad \Phi_h[m] = \Phi_h[\tilde{m}]$$

- If $u^0 = \Phi_{0,h}[m^0]$ and

$$(\Phi_{0,h}[m] - \Phi_{0,h}[\tilde{m}], m - \tilde{m})_2 \leq 0 \quad \Rightarrow \quad \Phi_{0,h}[m] = \Phi_{0,h}[\tilde{m}]$$

then

the discrete version of the MFG system has a unique solution.

A convergence result with local coupling

Assumptions (1/3)

- $\nu > 0$
- $d = 2$ (only for example)
- periodicity (but everything would work with Neumann boundary conditions, or suitable Dirichlet conditions)
- $u|_{t=0} = u_0$ and the data u_0 and m_T are smooth
- $$0 < \underline{\mathrm{m}}_T \leq m_T(x) \leq \overline{m}_T$$

Assumptions (2/3)

- The Hamiltonian is of the form

$$H(x, \nabla u) = \mathcal{H}(x) + |\nabla u|^\beta$$

where $\beta > 1$ and $\mathcal{H}$ is a smooth function

- The discrete Hamiltonian is of the form $g(x_{i,j}, [D_h u]_{i,j})$. The function $g : \mathbb{T} \times \mathbb{R}^4 \to \mathbb{R}$ is defined by

$$g(x, q) = \mathcal{H}(x) + \left((q_1^-)^2 + (q_2^+)^2 + (q_3^-)^2 + (q_4^+)^2\right)^{\frac{\beta}{2}}$$

where $r^+ = \max(r, 0)$ and $r^- = \max(-r, 0)$

Assumptions (3/3)

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$$\Phi[m](x) = F(m(x))$$

where F is $\mathcal{C}^1$ on $\mathbb{R}_+$

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$$mF(m) \geq c_1 |F(m)|^\gamma - c_2 \qquad \forall m$$

- There exist three positive constants c_3, η_1 and $\eta_2 < 1$ s.t.

$$F'(m) \geq c_3 \min(m^{\eta_1}, m^{-\eta_2}) \qquad \forall m$$

Convergence

Assume that the MFG system of pdes has a unique smooth solution (u, m) s.t.

$$m \geq \underline{m} > 0.$$

Let u_h (resp. m_h) be the piecewise trilinear function in $\mathcal{C}([0,T] \times \mathbb{T})$ obtained by interpolating the values $u^n_{i,j}$ (resp $m^n_{i,j}$) at the nodes of the space-time grid.

$$\lim_{h,\Delta t \to 0} \left(\|u - u_h\|_{L^\beta(0,T;W^{1,\beta}(\mathbb{T}))} + \|m - m_h\|_{L^{2-\eta_2}((0,T)\times\mathbb{T})} \right) = 0$$

Main steps of the proof

1. Obtain a priori bounds on the solution of the discrete problem, in particular on $\|F(m_h)\|_{L^\gamma((0,T)\times\mathbb{T})}$

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   $\|\nabla u - \nabla u_h\|_{L^\beta((0,T)\times\mathbb{T})}$ and $\|m - m_h\|_{L^{2-\eta_2}((0,T)\times\mathbb{T})}$
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   converge to 0
3. To get the full convergence for u, one has to pass to the limit in the Bellman equation. To pass to the limit in the term $F(m_h)$, use the equiintegrability of $F(m_h)$ and Vitali's theorem

A convergence result with nonlocal coupling

Assumptions

- Same assumptions on H and g
- Φ is non local, smoothing and monotone:

$$(\Phi(m_1) - \Phi(m_2), m_1 - m_2) \leq 0 \quad \Rightarrow \quad m_1 = m_2$$

- The discrete cost operator Φ_h continuously maps $\mathcal{K}$ to a set of grid functions bounded in the discrete Lipschitz norm
- The discrete cost operator Φ_h is monotone
- Consistency: for all probability density m and discrete probability density m',

$$\left\| \, \Phi[m] - \Phi_h[m'] \, \right\|_{L^\infty(\mathbb{T}_h)} \leq \omega \left(\|m - m'_h\|_{L^1(\mathbb{T})} \right)$$

where m'_h is a bilinear interpolation of m'

Convergence

When h and Δt tend to 0,

- (u_h) converges to u

  $$\text{uniformly and in } L^{\max(\beta,2)}(0,T;W^{1,\max(\beta,2)}(\mathbb{T}))$$

- If $\beta \geq 2$, (m_h) converges to m

  $$\text{in } C^0([0,T];L^2(\mathbb{T})) \cap L^2(0,T;H^1(\mathbb{T}))$$

- If $1 < \beta < 2$, (m_h) converges to m

  $$\text{in } L^2((0,T) \times \mathbb{T})$$

Solvers for the discrete systems

Due to the **forward-backward structure, marching in time is not possible.** One has to solve the system for u and m as a whole. This leads to large systems of nonlinear equations with $\sim 2N^{d+1}$ unknowns.

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Our choice: Newton methods

- linearized discrete MFG systems : well-posed if $m > 0$, which is not sure. Hence, breakdowns of the Newton method may occur
- Careful initial guess avoids breakdown
- **Initial guesses: continuation method, by decreasing ν progressively**
- In practice, can be applied even if Φ is not monotone

Solvers for linearized discrete MFG systems

- Due to the forward-backward structure, marching in time is not possible
- Preconditioned iterative method for the whole system in (u, m)
- A good understanding of the PDE system and multigrid lead to solvers with optimal linear complexity
- We have developed several optimal solvers based on multigrid methods

A possible strategy for solving the linearized discrete MFG systems

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4. Plug m back in the HJB equation and solve marching in time

PDE interpretation of the preconditioned operator

The preconditioned operator is of the form $I - K$ where

$$K(n) = (\text{linear-FP}_m)^{-1} \circ \text{div}\,(mH_{pp}(Du)D\cdot) \circ (\text{linear-HJB}_u)^{-1} \big(\Phi'(m)n\big)$$

If $\nu > 0$ and if m and u are smooth, K is a **compact operator in L^2.**

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If $\nu > 0$ and if m and u are smooth, K is a **compact operator in L^2.**

Thus, the convergence of a (bi)conjugate gradient like method should not depend on h and Δt.

Table: solving the linearized MFG system: average (on the Newton loop) number of iterations of BiCGstab to decrease the residual by a factor 10^{-3}

grid	$32 \times 32 \times 32$	$64 \times 64 \times 64$	$128 \times 128 \times 128$
$\nu = 0.6$	1	1	1
$\nu = 0.36$	1.75	1.75	1.8
$\nu = 0.2$	2	2	2
$\nu = 0.12$	3	3	3
$\nu = 0.046$	4.9	5.1	5.1

Multigrid methods can be used for solving the linearized HJ and FP eqs $\Rightarrow$ optimal complexity.

Second strategy for solving the linear systems when Φ is strictly monotone

The idea is to apply directly a multigrid method to the full system of pdes.

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The multigrid method must be special:

indeed, eliminating m from the linearized HJB equation, (this is possible since Φ is strictly monotone), we get a degenerate elliptic pde, with the operator

$$\mathrm{div}\left(m\frac{\partial^2 H(Du)}{\partial p^2}D\cdot\right)-(\text{linear- FP})\circ((\Phi'(m))^{-1}\cdot)\circ(\text{linear- HJB}).$$

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Operator: order 4 w.r.t. x and 2 w.r.t. t.

Principal part: $$(\Phi'(m))^{-1}\left(-\frac{\partial^2}{\partial t^2}+\nu^2\Delta^2\right).$$

Hence, when ν is large enough, we use a **multigrid method** with a hierarchy of grids obtained by **coarsening the grids only in the x variable.**

Table: average (on the Newton loop) number of iterations of the BiCGstab method to decrease the residual by a factor 10^{-3}

$\nu\backslash$ grid	$32 \times 32 \times 32$	$64 \times 64 \times 64$	$128 \times 128 \times 128$
0.6	1.75	1.5	1.25
0.36	2.2	2	2
0.2	4.9	3.5	2.9
0.12	14.4	11.4	6.8

Some numerical results

A. Exit from a hall with obstacles

$$\frac{\partial u}{\partial t} + \nu \Delta u - H(x, m, \nabla u) = -F(m), \qquad \text{in } (0,T)\times\Omega$$

$$\frac{\partial m}{\partial t} - \nu \Delta m - \text{div}\left(m \frac{\partial H}{\partial p}(\cdot, m, \nabla u)\right) = 0, \qquad \text{in } (0,T)\times\Omega$$

$$\frac{\partial u}{\partial n} = \frac{\partial m}{\partial n} = 0 \qquad \text{on walls}$$

$$u = k, \qquad m = 0 \qquad \text{at exits}$$

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$$u = k, \qquad m = 0 \qquad \text{at exits}$$

Congestion

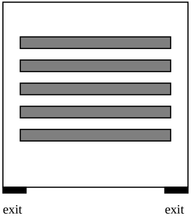
$$H(x, m, p) = \mathcal{H}(x) + \frac{|p|^\beta}{(c_0 + c_1 m)^\gamma}$$

with $c_0 > 0$, $c_1 \geq 0$, $\beta > 1$ and $0 \leq \gamma < 4(\beta - 1)/\beta$. Existence and uniqueness was proven by P-L. Lions, and hold in the discrete case.

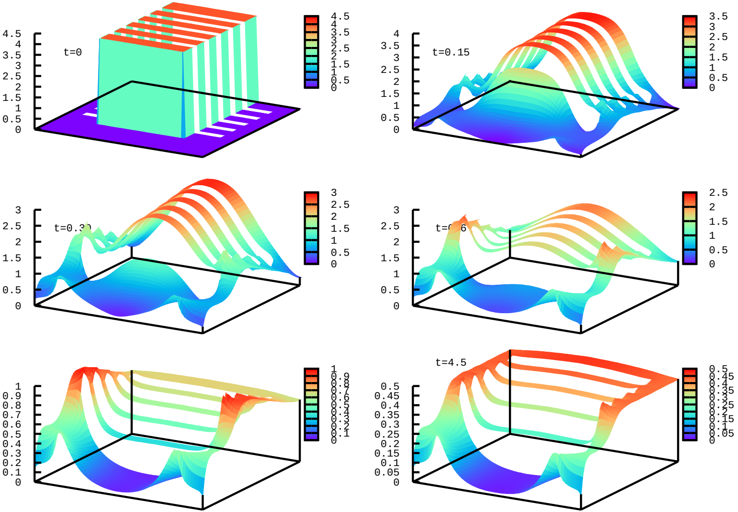
A. Exit from a hall with obstacles

- $T = 6$
- $\nu = 0.015$
- $u(t = T) = 0$
- $F(m) = m$
- Hamiltonian

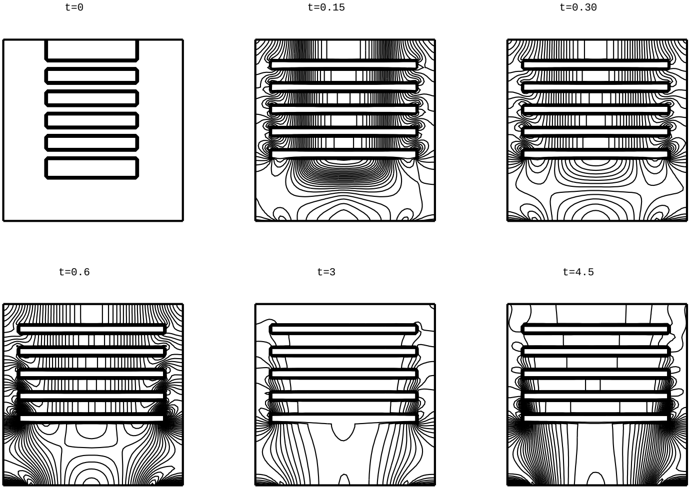
$$H(x, m, p) = -0.1 + \frac{|p|^2}{(1 + 4m)^{1.5}}$$



Evolution of the density

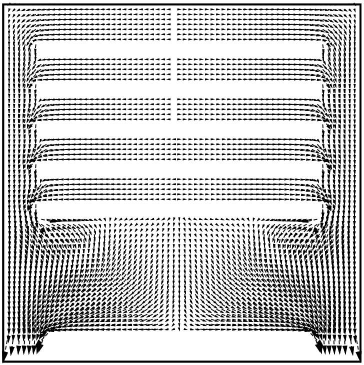


Evolution of the density

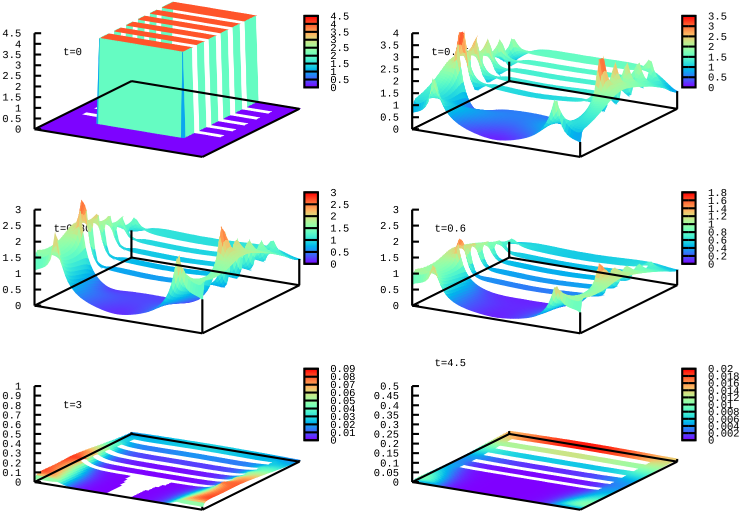


Velocity

t=0.6



Same thing without congestion : $H(x,p) = -0.1 + |p|^2$



B. Two populations

$$
\begin{aligned}
\frac{\partial u_1}{\partial t} + \nu \Delta u_1 - H_1(t, x, m_1 + m_2, \nabla u_1) &= -F_1(m_1, m_2) \\
\frac{\partial m_1}{\partial t} - \nu \Delta m_1 - \operatorname{div}\left(m_1 \frac{\partial H_1}{\partial p}(t, x, m_1 + m_2, \nabla u_1) \right) &= 0 \\
\frac{\partial u_2}{\partial t} + \nu \Delta u_2 - H_2(t, x, m_1 + m_2, \nabla u_2) &= -F_2(m_1, m_2) \\
\frac{\partial m_2}{\partial t} - \nu \Delta m_2 - \operatorname{div}\left(m_2 \frac{\partial H_2}{\partial p}(t, x, m_1 + m_2, \nabla u_2) \right) &= 0 \\
\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} &= 0 \\
\frac{\partial m_1}{\partial n} = \frac{\partial m_2}{\partial n} &= 0
\end{aligned}
$$

A model for segregation proposed by M. Bardi

- The Hamiltonians are uniform in space and the same for the two populations

$$H_i(x, m_i, m_j, p) = 0.1|p|^2$$

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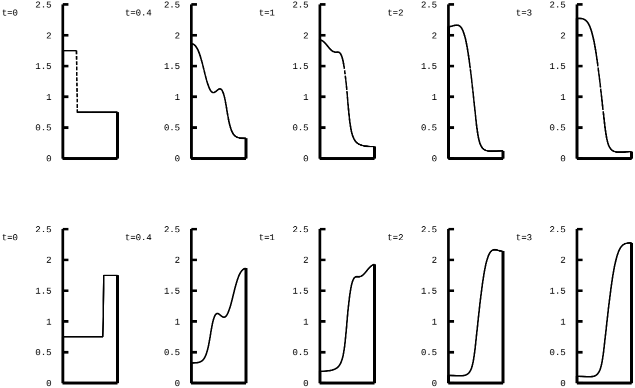
Xenophobia

The cost operators $F_1(m_1, m_2)$ and $F_2(m_1, m_2)$ are given by

$$F_i(m_i, m_j) = 5m_i \left(\frac{m_i}{m_i + m_j} - 0.45 \right)_- + (m_i + m_j - 4)_+$$

Evolution of the densities



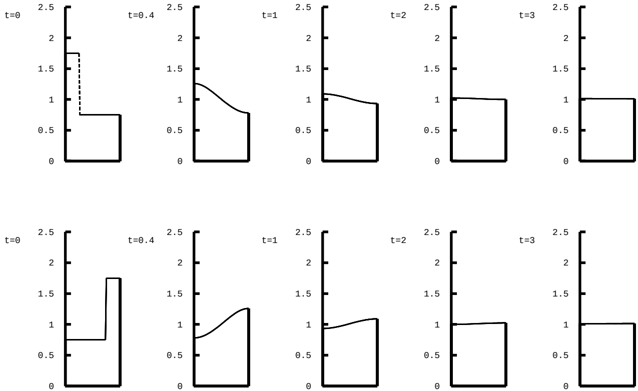
$$\nu = 0.015$$

$$m_1(\cdot, t=0) = 0.75 + 1_{[0,0.25]}, \quad m_2(\cdot, t=0) = 0.75 + 1_{[0.75,1]},$$

A stiffer coupling term

$$F_i(m_i, m_j) = 5\left(\frac{m_i}{m_i+m_j} - 0.45\right)_- + (m_i + m_j - 4)_+$$

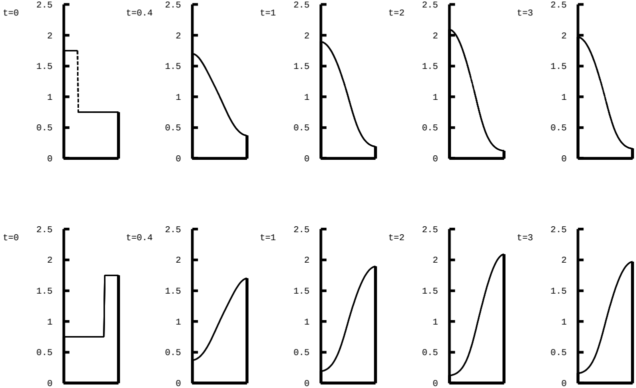
Evolution of the densities



$$\nu = 0.2$$

$$m_1(\cdot, t=0) = 0.75 + 1_{[0,0.25]}, \quad m_2(\cdot, t=0) = 0.75 + 1_{[0.75,1]},$$

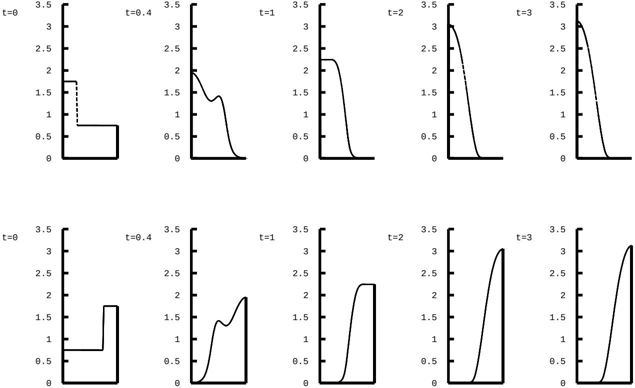
Evolution of the densities



$\nu = 0.1$

$m_1(\cdot, t=0) = 0.75 + 1_{[0,0.25]}, \quad m_2(\cdot, t=0) = 0.75 + 1_{[0.75,1]},$

Evolution of the densities



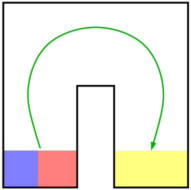
$\nu = 0.025$

$m_1(\cdot, t=0) = 0.75 + 1_{[0,0.25]}, \quad m_2(\cdot, t=0) = 0.75 + 1_{[0.75,1]},$

Who will reach the goal?

$$F_1(m_1, m_2) = m_1 + m_2, \quad F_2(m_1, m_2) = 20\, m_1 + m_2$$

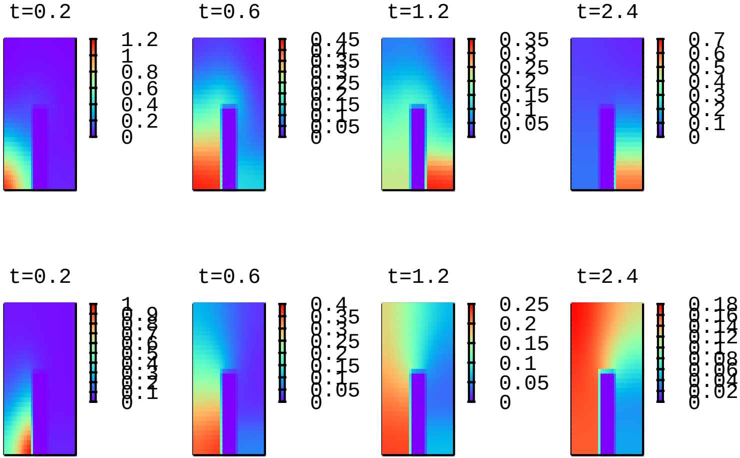
- $\Omega = (0,1)^2 \setminus ([0.4, 0.6] \times [0, 0.55])$
- $T = 4$, $\nu = 0.125$
- $u_1(t = T) = u_2(t = T) = 0$
- Same Hamiltonian for the two populations:



$$H_1(x, m, p) = H_2(x, m, p) = \mathcal{H}(x) + 0.1 \frac{|p|^2}{(1 + 4m)^{1.3}}$$

$$\mathcal{H}(x) = -10 \times 1_{x \notin ([0.6,1] \times [0,0.2])}$$

Evolution of the densities (bottom: the xenophobic pop.; top: the other pop.)



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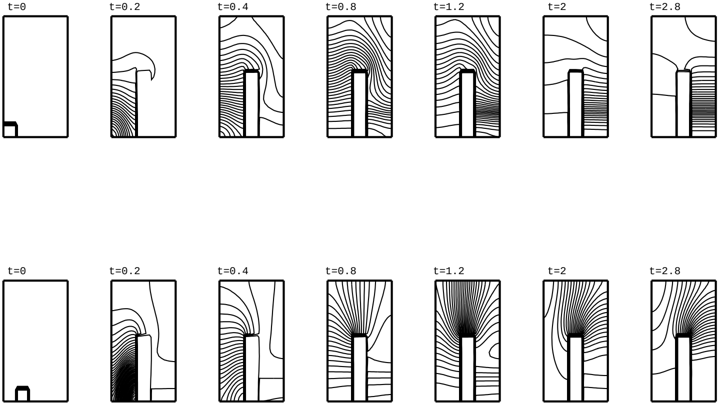
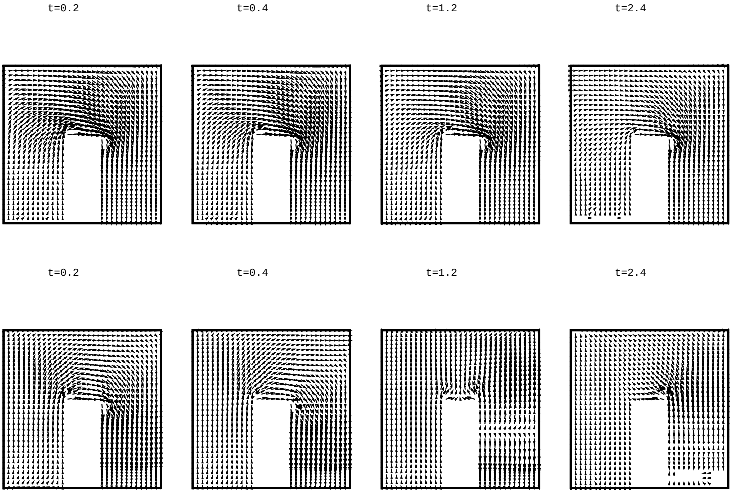


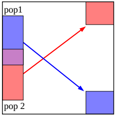
Evolution of the velocities (bottom: the xenophobic pop.; top: the other pop.)



Two populations cross each other

$$F_1(m_1, m_2) = m_1 + m_2, \quad F_2(m_1, m_2) = 20\, m_1 + m_2.$$

- $\Omega = (0, 1)^2$
- $T = 4$, $\nu = 0.015$
- $u_1(t = T) = u_2(t = T) = 0$
- Hamiltonians:



$$H_i(x, m, p) = \mathcal{H}_i(x) + 0.1 \frac{|p|^2}{(1 + 4m)^{1.3}}$$

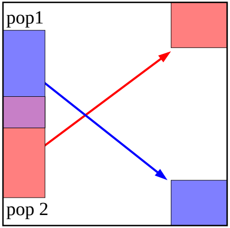
$$\mathcal{H}_1(x) = -10 \times 1_{x \notin ([0.7,1] \times [0,0.2])}$$

$$\mathcal{H}_2(x) = -10 \times 1_{x \notin ([0.7,1] \times [0.8,1])}$$

The two populations pay the same cost for moving and have the same sensitivity to congestion effects, but they aim at different corners

- Finally, at time $t=0$, the densities of the two populations are given by

$$m_1(x, t=0) = 4 \times 1_{[0,0.2]\times[0.4,0.9]}(x)$$
$$m_2(x, t=0) = 4 \times 1_{[0,0.2]\times[0.1,0.6]}(x)$$



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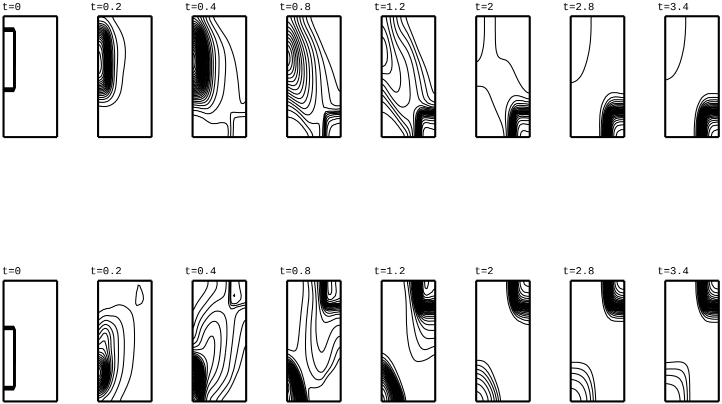

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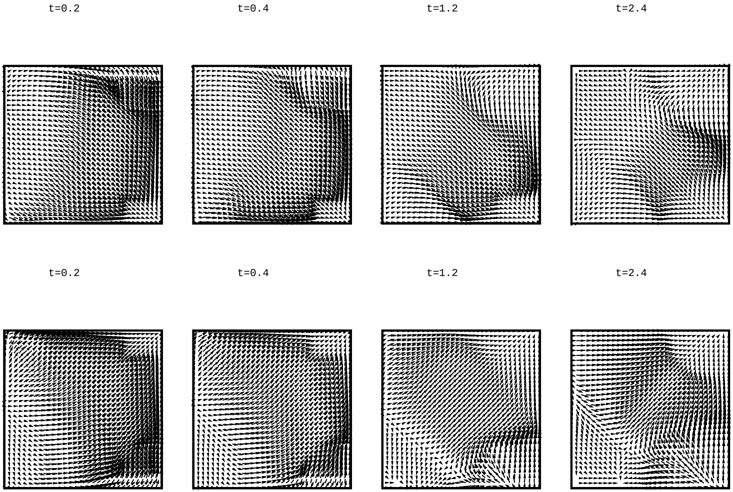
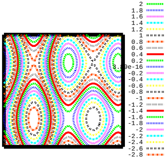


C. Long time behavior (a single population)

$$\nu = 1, \quad T = 1, \quad m(T) = 1$$

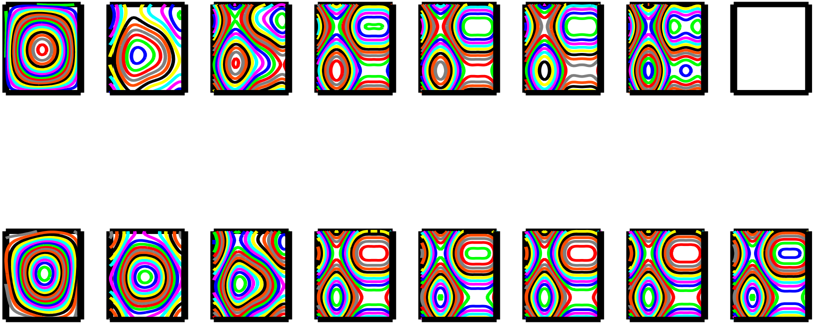
$$H(x,p) = \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1) + |p|^2,$$

$$F(x,m) = m^2, \qquad F_0(x,m) = m^2 + \cos(\pi x_1)\cos(\pi x_2).$$



The potential $H(x,0) = \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1)$.

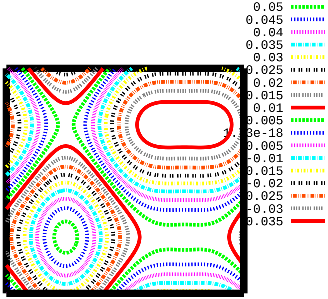
Evolution of m(top) and u(bottom)

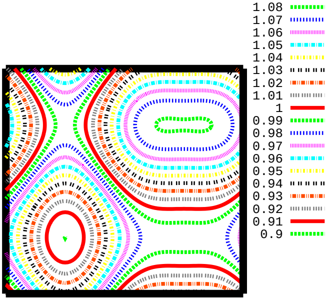


Snapshots at $t = (0, 4, 8, 100, 180, 190, 196, 200)/200$

Comparison with the solution of the infinite horizon MFG system

The solution around $t = T/2$ is very close to the solution of the infinite horizon MFG system





The infinite horizon MFG system

Find $(u, m, \lambda \in \mathbb{R})$ such that

$$\begin{cases} -\nu\Delta u + H(x, \nabla u) + \lambda = F(m), \\ -\nu\Delta m - \operatorname{div}\left(m\dfrac{\partial H}{\partial p}(x, \nabla u)\right) = 0, \\ \displaystyle\int_{\mathbb{T}} u dx = 0, \qquad \int_{\mathbb{T}} m dx = 1, \quad \text{and} \quad m > 0 \quad \text{in } \mathbb{T}. \end{cases}$$

Quadratic Hamiltonian

The Hamiltonian is of the form $H(x,p) = |p|^2 + g(x)$.

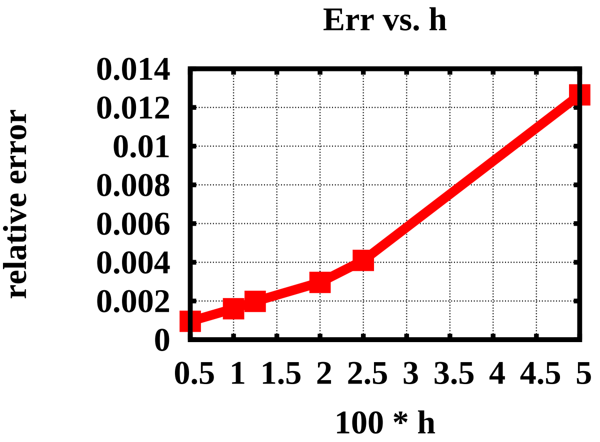
The infinite horizon MFG system is equivalent to a generalized Hartree equation:

$$-\nu^2 \Delta \phi - g\phi + \phi F(\phi^2) = \lambda \phi, \quad \text{in } \mathbb{T}, \quad \text{and} \quad \int_{\mathbb{T}} \phi^2 = 1$$

where $\phi(x) = \phi_0 \exp\left(-u(x)/\nu\right)$ and $m = \phi^2$.
The constant ϕ_0 is fixed by the equation $\int_{\mathbb{T}} \log(\phi/\phi_0) = 0$.

As a consequence, m can be written as a function of u.
This gives a way to test the accuracy of the scheme.

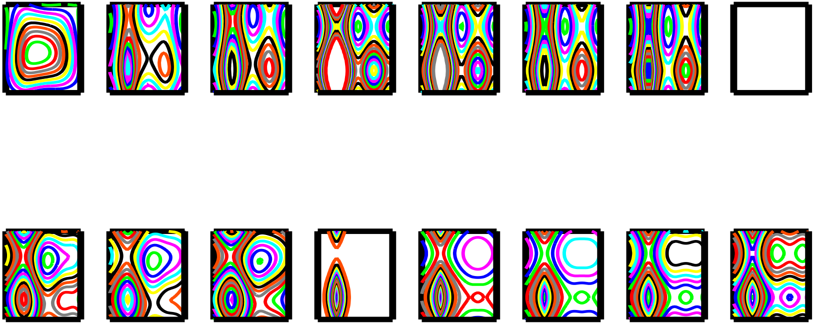
Order of the scheme



Same test except

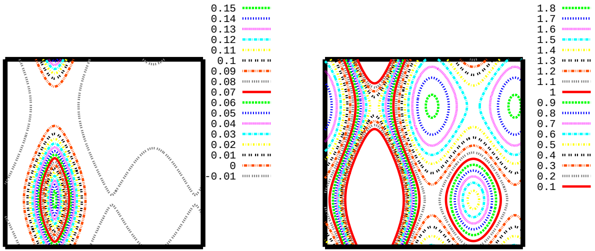
$$\nu = 0.01, \quad \Delta t = 1/200.$$

Evolution of m(top) and u(bottom)



Snapshots at $t = (0, 4, 8, 100, 180, 190, 196, 200)/200$

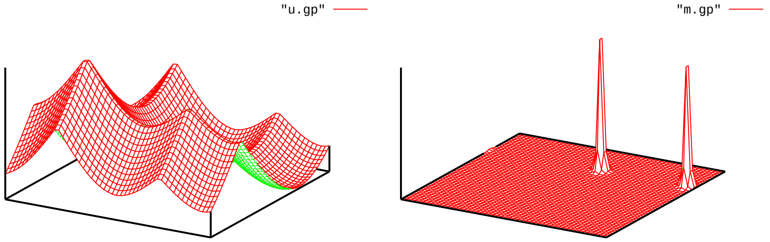
The solution of the infinite horizon problem



$\nu = 0.01$, left: u, right m.

Note that the supports of ∇u and of m tend to be disjoint as $\nu \to 0$.

D. Deterministic infinite horizon MFG with nonlocal coupling



$$\nu = 0.001,$$
$$H(x,p) = \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1) + |p|^2,$$
$$V[m] = (1-\Delta)^{-1}(1-\Delta)^{-1}m$$

left: u, right m.

E. Optimal planning with MFG

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + H(x, \nabla u) = F(m(x)) & \text{in } (0,T) \times \mathbb{T}, \\ \dfrac{\partial m}{\partial t} + \nu \Delta m + \operatorname{div}\left(m \dfrac{\partial H}{\partial p}(x, \nabla u) \right) = 0 & \text{in } (0,T) \times \mathbb{T}, \end{cases}$$

with the initial and terminal conditions

$$m(0,x) = m_0(x), \quad m(T,x) = m_T(x), \quad \text{in } \mathbb{T},$$

and

$$m \geq 0, \qquad \int_{\mathbb{T}} m(t,x) dx = 1.$$

Existence results (P-L. Lions)

- **Ok if $\nu = 0$,** if H coercive, if F is a strictly increasing function and if m_0 and m_T are smooth positive functions. Principle of the (difficult) proof: eliminate m from the Bellman equation and get a boundary value problem for u with a strictly elliptic quasilinear second order PDE, and nonlinear boundary conditions

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- **If $\nu > 0$ and more general Hamiltonians ?**
- Non-existence if H is sublinear, $m_0 \neq m_T$ and T small enough

Optimal control (on PDEs) approach

Assumption:

- $F = W'$ where $W : \mathbb{R} \to \mathbb{R}$ is a strictly convex function
- $H(x,p) = \sup_{\gamma \in \mathbb{R}^d} (p \cdot \gamma - L(x,\gamma))$
- L is strictly convex, $\lim_{|\gamma| \to \infty} \inf_x L(x,\gamma)/|\gamma| = +\infty$

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A weak form of the MFG system can be found by considering the problem of optimal control on PDE:

minimize $\quad (m,\gamma) \to \int_0^T \int_{\mathbb{T}} m(t,x)\, L(x,\gamma(t,x)) + W(m(t,x))$

subject to the constraints

$$\begin{cases} \partial_t m + \nu \Delta m + \mathrm{div}(m\,\gamma) = 0, & \text{in } (0,T) \times \mathbb{T}, \\ m(T,x) = m_T(x) & \text{in } \mathbb{T}, \\ m(0,x) = m_0(x) & \text{in } \mathbb{T}. \end{cases}$$

Convex programming and Fenchel-Rockafeller duality theorem

It is possible to make the constraints linear by the change of variables $z = m\gamma$
$\rightarrow$ optimization problem with a convex cost and linear constraints.

There exists a saddle point of the primal-dual problem, and writing the optimality conditions:

- In the continuous setting, not easy to recover the system of pdes
- Discrete problem: same program, but it is possible to prove that $m > 0 \Rightarrow$ existence and uniqueness for the discrete pb.

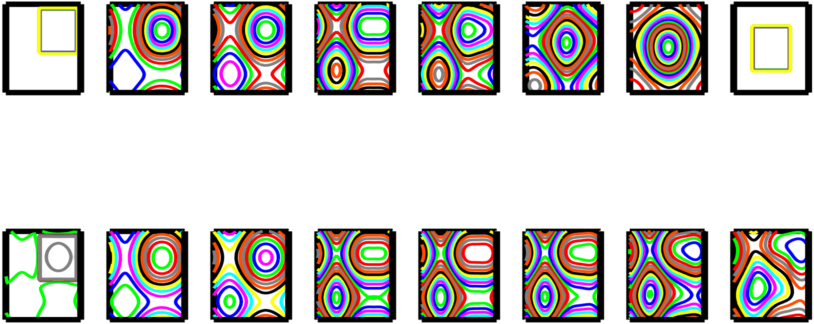
A penalized scheme

$$\begin{cases} \dfrac{u_{i,j}^{\epsilon,n+1} - u_{i,j}^{\epsilon,n}}{\Delta t} - \nu(\Delta_h u^{\epsilon,n+1})_{i,j} + g(x_{i,j}, [D_h u^{\epsilon,n+1}]_{i,j}) = F(m_{i,j}^{\epsilon,n}) \\ \dfrac{m_{i,j}^{\epsilon,n+1} - m_{i,j}^{\epsilon,n}}{\Delta t} + \nu(\Delta_h m^{\epsilon,n})_{i,j} + \mathcal{T}_{i,j}(u^{\epsilon,n+1}, m^{\epsilon,n}) = 0 \\ m^{\epsilon,n} \in \mathcal{K} \end{cases}$$

with the final time and initial time conditions

$$u_{i,j}^{\epsilon,0} = \frac{1}{\epsilon}(m_{i,j}^{\epsilon,0} - (m_0)_{i,j}), \qquad m_{i,j}^{\epsilon,N_T} = (m_T)_{i,j}, \quad \forall i,j$$

Convergence As $\epsilon \to 0$, $m^\epsilon \to m$ solution of the discrete MFG system.



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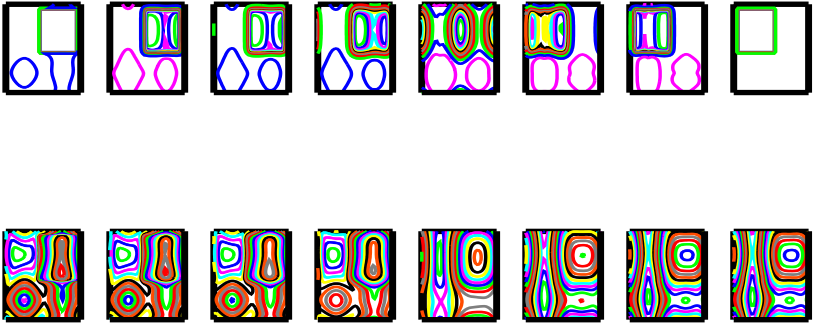

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