Multi-marginal optimal transport and applications

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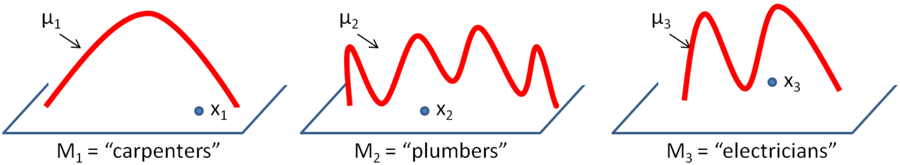
- Different combinations of people work more of less well together.
    - Different potential teams have different efficiencies.
- Want to construct teams to make overall process as efficient as possible.

Introduction



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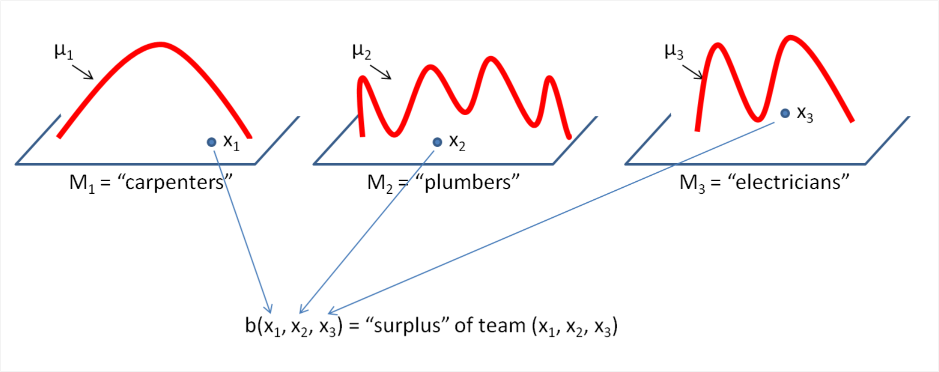

Multi-marginal problem: Monge formulation

- $M_i \subseteq \mathbb{R}^n$, open and bounded, $i = 1, 2.....m$.
- μ_i Borel probability measures on M_i.
- $b : M_1 \times M_2 \times ... \times M_m \to \mathbb{R}$ smooth *surplus* function.

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maximize:

$$\int_{M_1} b(x_1, F_2(x_1), F_3(x_1), ..., F_m(x_1)) d\mu_1(x_1)$$

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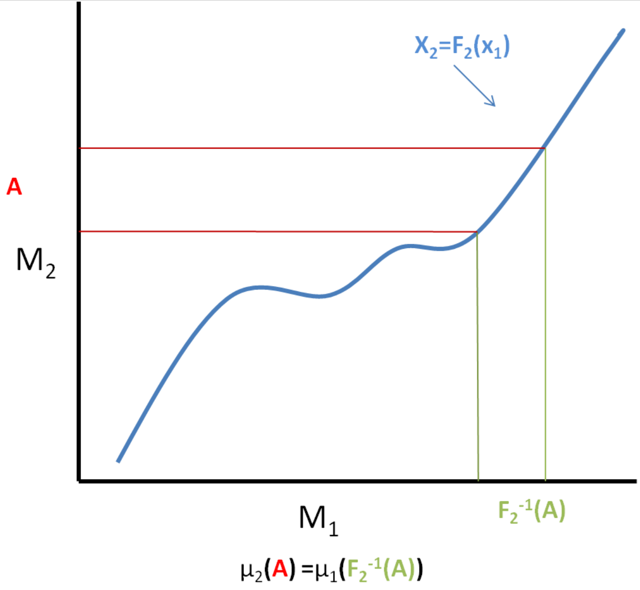

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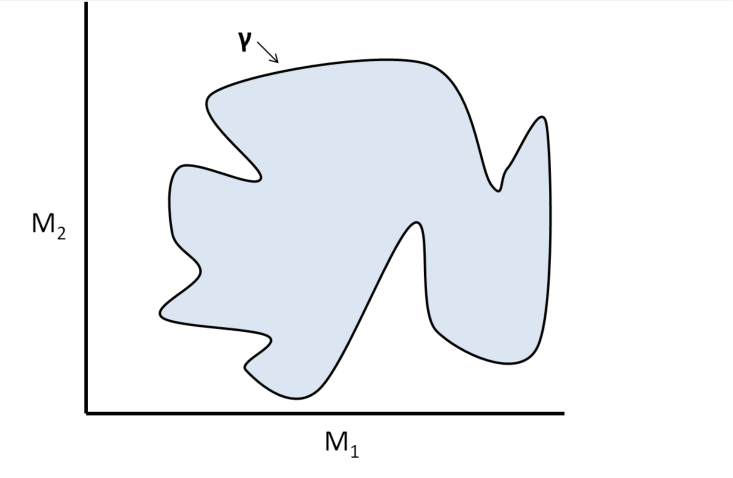

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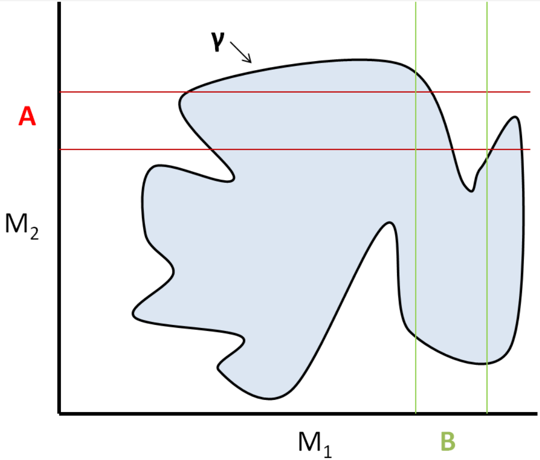

$\mu_2(A) = \gamma(M_1 \times A)$
$\mu_1(B) = \gamma(B \times M_2)$

Background on (two marginal) optimal transportation

- Optimal transportation with two marginals ($m = 2$) is an active and well established area of research.
- Many diverse applications, including: fluid mechanics, cosmology, interacting gases, meteorology, image processing, economics, etc.
- Brenier '87, Gangbo '95, Caffarelli '96, Gangbo-McCann '96, Levin '96: Assume $\mu_1 << dx_1$ and that b is twisted, ie:

$$x_2 \mapsto D_{x_1} b(x_1, x_2) \text{ is injective.}$$

  Then γ is concentrated on the graph of a function over x_1 and is unique.
- Example: $b(x_1, x_2) = -|x_1 - x_2|^2$.

Background on multi-marginal problems: good surpluses

- Multi-marginal problems have many emerging applications, in economics, physics, m-monotonicity, image processing, financial math, statistics, etc., but are not well understood.
- For certain *special* surplus functions, the optimal γ is unique and is concentrated on a graph over x_1: $\{(x_1, F_2(x_1), ..., F_m(x_1)\}$.
- Gangbo-Swiech '98: $b(x_1, x_2, ..., x_m) = -\sum_{i \neq j} |x_i - x_j|^2$
- Heinich '02: $b(x_1, x_2, ..., x_m) = h(x_1 + x_2 + ... + x_m)$ where $h : \mathbb{R}^n \to \mathbb{R}$ is strictly convex.
- P '11: Strong second order conditions on b. For example, when $m = 3$, we require, for all $x_1, \overline{x_1} \in M_1$, $x_2 \in M_2$, $x_3, \overline{x_3} \in M_3$, we have:

$$D^2_{x_2x_3} b[D^2_{x_1x_3} b]^{-1} D^2_{x_1x_2} b(x_1, x_2, x_3) - D^2_{x_2x_2} b(x_1, x_2, x_3) + D^2_{x_2x_2} b(\overline{x_1}, x_2, \overline{x_3}) > 0.$$

- Kim-P '13: $b(x_1, x_2, ..., x_m) = -\inf_{y \in M}[\sum_{i=1}^m d^2(x_i, y)]$ on a Riemannian manifold M.

Background on multi-marginal problems: bad surpluses

- For other surplus functions, solutions can be non-unique and have high dimensional support. Examples:
- $b(x_1, x_2, ..., x_m) = -\sum_{i \neq j}^{m} \frac{1}{|x_i - x_j|}$, arises in density functional theory for Coulombic electronic interactions in quantum physics (Cotar-Friesecke-Kluppelberg '11 and Buttazzo-De Pascale-Gori-Giorgi '12).
- $b(x_1, x_2, ..., x_m) = \sum_{i \neq j}^{m} |x_i - x_j|^2$, arises when Coulombic interactions are replaced by repulsive, harmonic oscillator interactions.
- $b(x_1, x_2, ..., x_m) = \det(x_1, x_2, ..., x_m)$ (when $n = m$) Carlier-Nazaret '06 .
- $b(x_1, x_2, ..., x_m) = h(x_1 + x_2 + ... + x_m)$, h strictly concave .

Example application 1: multi-agent matching

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- Buyer x_1 has a preference $f_1(x_1, z)$ for a house type $z \in Z \subseteq \mathbb{R}^n$; worker x_i $(i \geq 2)$ has a preference $f_i(x_i, z)$ to build house of type z.

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- Finding an equilibrium in this market is equivalent to solving an optimal transport problem with surplus

$$b(x_1, x_2, ..., x_m) = \sup_{z \in Z} \sum_{i=1}^{m} f_i(x_i, z)$$

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- Electrons are indistinguishable $\rightarrow \mu_i = \mu$.
- Given μ, the single particle density, want to find the m-particle density (a measure on $\mathbb{R}^{nm}$) minimizing the total interaction energy.
- Leads to an optimal transport problem with $b(x_1, x_2, ..., x_m) = -\sum_{i \neq j} \frac{1}{|x_i - x_j|}$.

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- For $m = 3$, if $\frac{\partial^2 b}{\partial x_i \partial x_j} > 0$, for all $i \neq j$ $\exists!$ optimal maps $x_2 = F_2(x_1), x_3 = F_3(x_1)$, both increasing (Carlier '03)

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- Coordinate invariant condition: $\frac{\partial^2 b}{\partial x_1 \partial x_2}[\frac{\partial^2 b}{\partial x_3 \partial x_2}]^{-1}\frac{\partial^2 b}{\partial x_3 \partial x_1} > 0$.

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Structure of solutions

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- $mn - \lambda_- = n$ iff $(D^2_{x_1x_2}b)[(D^2_{x_3x_2}b)]^{-1}(D^2_{x_3x_1}b) > 0$.

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- Some rotations fix x but not y, assuming x and y are not co-linear (get non Monge solutions).
- These rotational directions are extra spacelike directions for G.

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- Can easily calculate $(D^2_{x_1x_2}b)[(D^2_{x_3x_2}b)]^{-1}(D^2_{x_3x_1}b) > 0$ (under mild conditions on the f_i).

Monge solution and uniqueness results

- For which surplus functions is the optimizer concentrated on the graph of a function over x_1?
- For $m=2$, the twist, injectivity of $x_2 \mapsto D_{x_1}b(x_1,x_2)$, suffices.
- For $m=3$, these type of results hold for

$$b(x_1,x_2,x_3) = \sup_{z\in Z}\sum_{i=1}^{3} f_i(x_i,z)$$

- This class includes $-\sum_{i=1}^{3}|x_i-x_j|^2$ (Gangbo-Swiech surplus), $h(x_1+x_2+x_3)$, for strictly convex h, (Heinich surplus).
- Optimal maps factor through a measure on Z (the *generalized barycenter*) Agueh-Carlier '10.
- Can easily calculate $(D^2_{x_1x_2}b)[(D^2_{x_3x_2}b)]^{-1}(D^2_{x_3x_1}b) > 0$ (under mild conditions on the f_i).
- One can also prove Monge solutions and uniqueness under strong differential conditions on b (P '11), or under a twist like condition on special sets (Kim-P (in preparation)).

The limit $m \to \infty$

- In the limit as $m \to \infty$, the differences become even more pronounced.
- For the surplus $-\int_0^1 \int_0^1 |x_s - x_t|^2 dsdt$, we get unique Monge type solutions (P '13).
- For $-\lim_{m\to\infty} \frac{1}{\binom{m}{2}} \sum_{i \neq j}^{m} \frac{1}{|x_i - x_j|}$; the (unique) optimal measure is product measure (Cotar-Friesecke-P (in preparation)).