Topological tools for stable sets and coloring of graphs.

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The goal of these lectures is to present three tools and their applications to graph coloring problems: the Borsuk-Ulam theorem, the connectivity of the stable set complex, and the Vapnik-Cervonenkis dimension. We will illustrate our purpose on various examples, with some focus on the two following questions on respectively sparse and dense graphs.

A cycle plus triangle graph is a (4-regular) graph on $3k$ vertices which is the edge disjoint union of a hamiltonian cycle with a collection of k disjoint triangles. Erdős asked if such a graph has chromatic number 3, and this was proved by Fleischner and Stiebitz.

The Erdős-Simonovits question asks for the minimum c for which every triangle-free graph on n vertices and minimum degree cn has bounded chromatic number.

Surprisingly, getting insight on these combinatorial questions involves using the aforementioned topological tools. Let us briefly overview them:

1) A restatement of the Borsuk-Ulam theorem asserts that the graph G consisting of the d-dimensional unit sphere as vertex set, and where edges connect near-antipodal pairs of points (i.e. with dot-product close to -1) has chromatic number $\chi(G) = d+2$. This is a major result in graph theory since lower bounds on χ are very seldom. In his celebrated proof, Lovász applied the Borsuk-Ulam theorem to compute the chromatic number of Kneser graphs. He also proposed a general lower bound on the chromatic number of a graph based on Borsuk-Ulam.

2) The stable set complex $S(G)$ of a graph G is the hypergraph which hyperedges are the stable sets of G. A natural invariant to consider is the connectivity η of $S(G)$ (the index of the smallest non zero Betti number). As shown by Aharoni and Berger, the η function can be used as a rank function when intersecting the stable set complex with some arbitrary matroid. More specifically, when given a graph G with a fixed vertex partition $V_1, \dots, V_k$, the existence of a stable set of G intersecting all the parts V_i can be deduced from Hall type conditions based on η. This is an application of Brouwer fixed point theorem.

3) The set of (open) neighborhoods of the vertices of a graph G also forms a natural hypergraph H_G. In particular, the "complexity" of G can be interpreted as the "complexity" of the hyperedges of H_G. One of the crucial measure of complexity of hypergraphs is the Vapnik-Cervonenkis dimension (which is roughly speaking the size of the largest Venn diagram in H_G). When VC-dimension is bounded, the integrality gap between fractional relaxation of various graph parameters is often bounded. In particular, the existence of bounded size dominating set, or colorings, can be deduced.

Let us mention some illustrations of these tools in the two examples above:

i) The proof that a "cycle plus triangle" graph G has chromatic number 3 is algebraic (based on Alon's Combinatorial Nullstellensätz). However, the existence of a stable set of size k can be derived from the fact that $\eta(H)$ is at least $|V(H)|/3$ when H is a disjoint collection of cycles with length 0 or 2 modulo 3. To show this, we will introduce Meshulam Game, which provides a lower bound on η. Few months ago, Noga Alon showed that the existence of two disjoint stable sets of size k can be (very elegantly) derived from the chromatic number of Schrijver graphs (subgraphs of Kneser graphs). To sum-up, one stable set of size k comes from Brouwer, two disjoint ones come from Borsuk-Ulam, and three come from Nullstellensätz (+ lot of work).

ii) Hajnal constructed triangle-free graphs on n vertices with minimum degree $(1/3-\varepsilon)n$ and arbitrarily large chromatic number. His construction is based on Kneser graphs. The nice fact is that $1/3$ is indeed the threshold, and we showed with T. Luczak how VC-dimension can be directly applied to prove that triangle-free graphs with minimum degree $n/3$ have bounded chromatic number. The general picture here is that triangle-free graphs with linear minimum degree have high chromatic number for topological reasons (Borsuk-Ulam). The cost of this is high dimensional subgraphs, and thus high VC-dimension. However, when the threshold $1/3$ is attained, S. Brandt proved that the three-dimensional cube cannot be a subbgraph, making the VC-dimension collapse to three, and in turn bounding the chromatic number.

We will conclude by mentioning open questions on the complexity of effectively constructing a stable set (even in very restricted classes of graphs). An interesting question is also to extend these existence results to weighted versions.