PDE discretizations based on local adaptive stencils.

Applications to image processing.

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CNRS, University Paris Sud

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Journées de Géométrie Algorithmique

S_d^+: positive definite matrices. $\|v\|_D := \sqrt{v^{\mathrm{T}} D v}$.

Problem: tensor decomposition based on close neighbors

Let $X \subset \mathbb{R}^d$ be a discrete point set. Given $x \in X$ and $D \in S_d^+$.
Find $Y \subset X$ finite, and non-negative weights $(\nu_y)_{y \in Y}$ such that

$$x = \sum_{y \in Y} \nu_y y, \qquad D = \sum_{y \in Y} \nu_y \, (y - x) \otimes (y - x).$$

- Select weighted neighbors $(\nu_y, y)_{y \in Y}$ which average to x and have prescribed covariance D.
- False lead: resembles decomposition $D = \sum_{v \in V} \lambda_v v \otimes v$ given by eigenvalues $\lambda_v \geq 0$ and eigenvectors $v \in \mathbb{S}^{d-1}$.
- The set Y should be small in cardinality and diameter.
- Procedure must be scalable: $D = \mathbf{D}(x)$, for each $x \in X$.

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Stencil construction when $X = \mathbb{Z}^d$, $d \in \{2, 3\}$

Definition (Basis of $\mathbb{Z}^d$)

A d-plet $(e_i)_{i=1}^d \in (\mathbb{Z}^d)^d$ such that $|\det(e_1, \cdots, e_d)| = 1$.

Definition (Superbase of $\mathbb{Z}^d$)

A $(d+1)$-plet $(e_0, \cdots, e_d) \in (\mathbb{Z}^d)^{d+1}$ such that $(e_1, \cdots, e_d)_{i=1}^d$ is a basis and $e_0 + \cdots + e_d = 0$.

- Superbases allow for tensor decompositions: if $D \in S_d^+$

$$D = - \sum_{0 \leq i < j \leq d} \langle e_i, De_j \rangle v_{ij} \otimes v_{ij},$$

integer neighbors $v_{ij} = e_k^\perp$ when $\{i, j, k\} = \{0, 1, 2\}$ (d=2).

Definition (D-obtuse superbase, where $D \in S_d^+$)

A superbase such that $\langle e_i, De_j \rangle \leq 0$ for all $0 \leq i < j \leq d$.

- Used to classify 3D lattices. Conway, Sloane (1992)
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Selling's algorithm (1874)

Produces a D-obtuse superbase, for $D \in S_d^+$, $d \leq 3$

Initialize $(e_1, \cdots, e_d)$ as canonical basis, $e_0 = -(e_1 + \cdots + e_d)$.

While $(e_0, \cdots, e_d)$ is not D-obtuse

$d = 2$: $(e_0, e_1, e_2) \leftarrow (-e_i,\ e_j,\ e_i - e_j)$

- $\sum_{I \subset \{0, \cdots, d\}} \| \sum_{i \in I} e_i \|_D^2$ strictly decreases at each iteration.

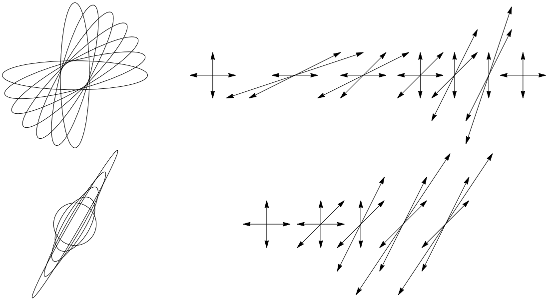

Figure : Left: ellipse $\{u \in \mathbb{R}^d; \|u\|_D \leq 1\}$.
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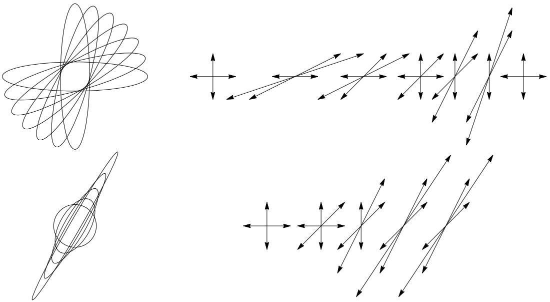

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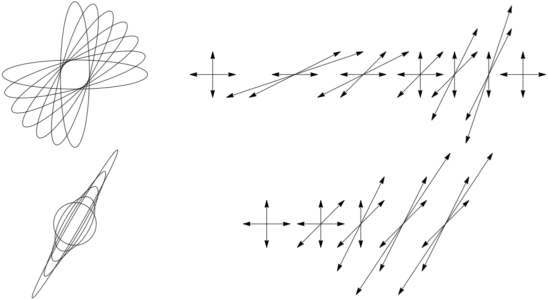

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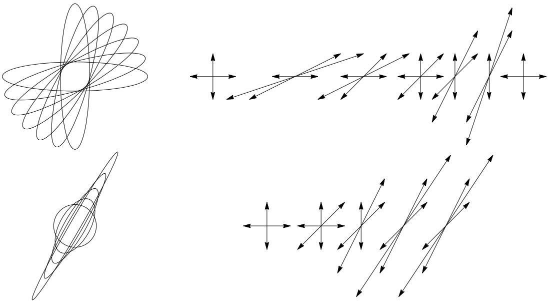

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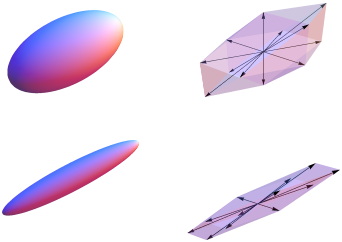

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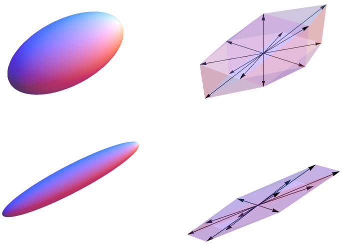

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Theorem (Optimality of tensor decomposition with obt sb)

Let $D \in S_2^+$, and let $(e_i)_{i=0}^2$ be a D-obtuse superbase. Assume also $D = \sum_{v \in V} \nu_v\, v \otimes v$ with $V \subset \mathbb{Z}^2$, $\nu_v \geq 0$. Then for a.e. D

$$\mathrm{Hull}(\pm e_i^\perp; 0 \leq i \leq 3) \subset \mathrm{Hull}(\pm v; v \in V).$$

J.-M. M., Minimal Stencil for Monotony or Causality Preserving Discretizations of Anisotropic PDEs, Preprint.

Assume that

$$x = \sum_{y \in Y} \nu_y y, \qquad D = \sum_{y \in Y} \nu_y \, (y-x) \otimes (y-x).$$

Monotone finite difference schemes

If $u \in C^2(\mathbb{R}^d)$, and Y is sufficiently close to x, then at first order

$$\sum_{y \in Y} \nu_y (u(y) - u(x))^2 \approx \sum_{y \in Y} \nu_y \langle \nabla u(x), y - x \rangle^2 = \|\nabla u(x)\|_D^2,$$

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Application: Anisotropic diffusion in divergence form

$$\partial_t u = \mathrm{div}(\,\mathbf{D}(x)\nabla u(x)).$$

- Gradient flow of $\int_X \|\nabla u(x)\|^2_{D(x)} dx$ w.r.t. L^2 metric.
- Non-linear smooth dependence $\mathbf{D} = \mathbf{D}_u$ is not a problem.
- Weickert's diffusion tensors $\mathbf{D}_u(x)$ are strongly anisotropic, so as to smooth tangentially to image discontinuities.

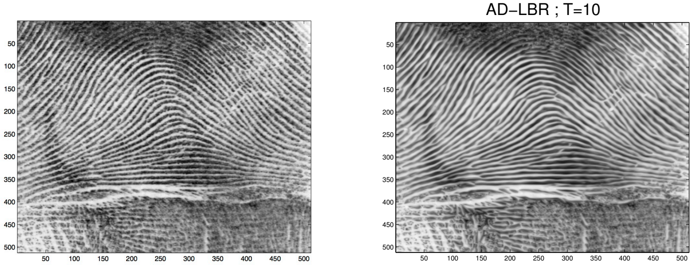

Figure : Left: original. Right: smoothed. J. Fehrenbach, J.-M. M., Sparse non-negative stencils for anisotropic diffusion, JMIV 2013

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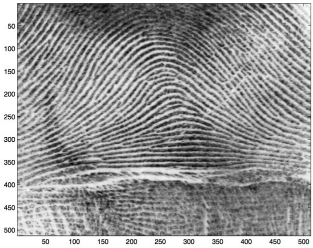

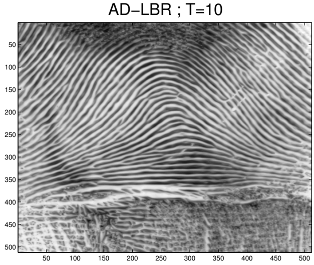

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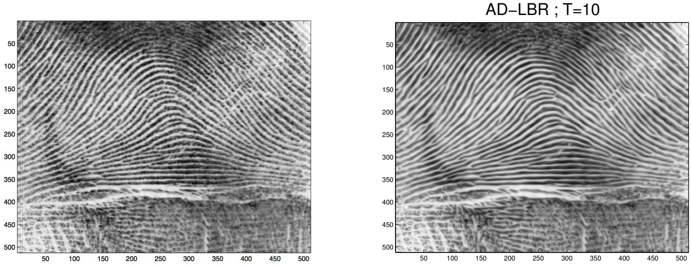

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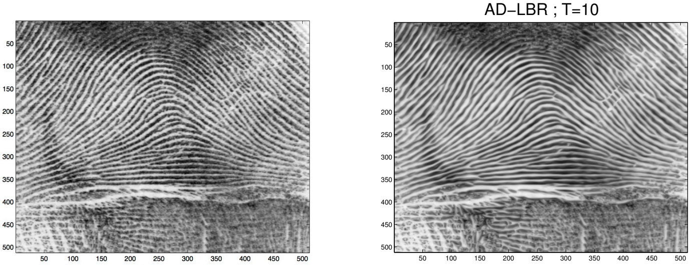

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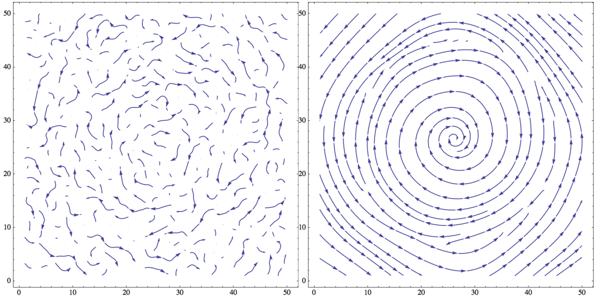

Figure : Left: original. Right: smoothed. Fehrenback, Risser, Tobji, M, Anisotropic Diffusion in ITK, The Insight Journal 2015

Application: Anisotropic diffusion in divergence form

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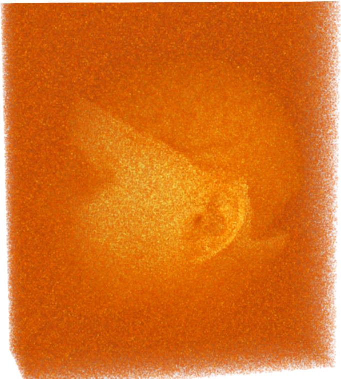

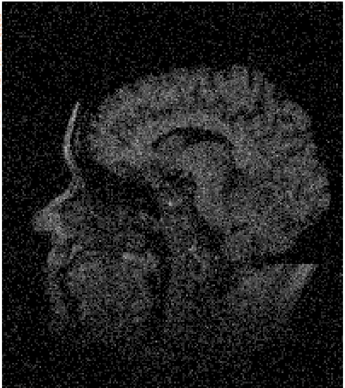

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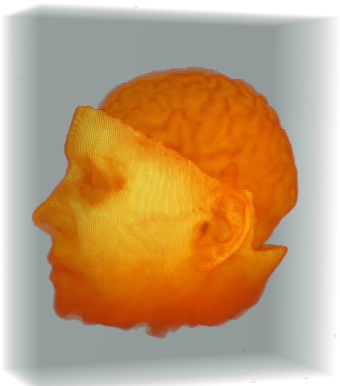
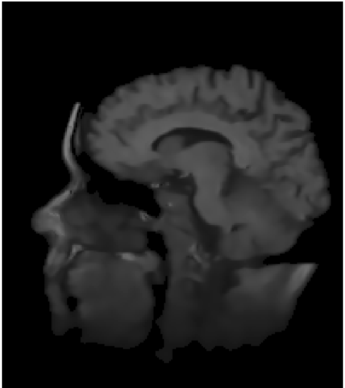

Application: Hamilton Jacobi Bellman equations

Given $D_{\alpha,\beta} : \Omega \to S_d^+$, ..., find $u : \Omega \to \mathbb{R}$ solving $\forall x \in \Omega$

$$0 = \sup_{\alpha \in A} \inf_{\beta \in B} \mathrm{Tr}(D_{\alpha,\beta} \nabla^2 u) + \langle b_{\alpha,\beta}, \nabla u \rangle + c_{\alpha,\beta} u + d_{\alpha,\beta},$$

plus some boundary conditions on $\partial\Omega$. (In viscosity sense.)

- Extremely general, encompasses PDEs involving

$$\det(\nabla^2 u), \qquad \lambda_{\max}(\nabla^2 u), \qquad \frac{(\nabla u)^T \nabla^2 u (\nabla u)}{|\nabla u|^2}.$$

- Discretization of $\mathrm{Tr}(D\nabla^2 u)$ is a key.
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Distance maps and Shortest Paths

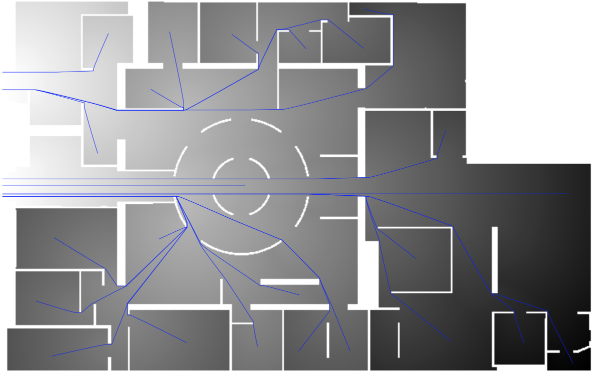

Figure : Distance from the exit of centre Pompidou, and associated shortest paths.

Distance maps and Shortest Paths

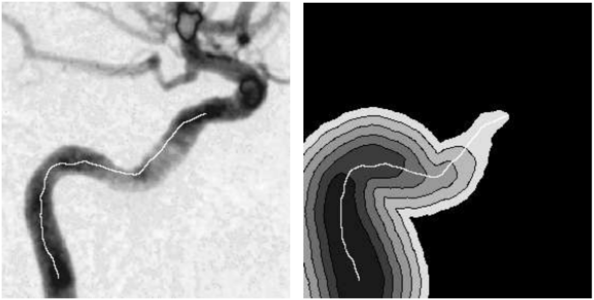

Figure : Distance with respect to a metric constructed from a vessel image. Credit L.Cohen.

Asymmetric norms and Finsler metrics

Definition (Asymmetric norm)

A map $F : \mathbb{R}^d \to \mathbb{R}_+$ such that

- (Definiteness) $F(u) = 0$ implies $u = 0$.
- (Homogeneity) $F(\lambda u) = \lambda F(u)$ for $\lambda \geq 0$.
- (Triangle inequality) $F(u + v) \leq F(u) + F(v)$.

Definition (Finsler metric on a domain $\Omega \subset \mathbb{R}^d$)

A continuous map $\mathcal{F} : \overline{\Omega} \times \mathbb{R}^d \to \mathbb{R}_+,\ (z, u) \to \mathcal{F}_z(u)$, such that $\mathcal{F}_z$ is an asymmetric norm for all $z \in \overline{\Omega}$.

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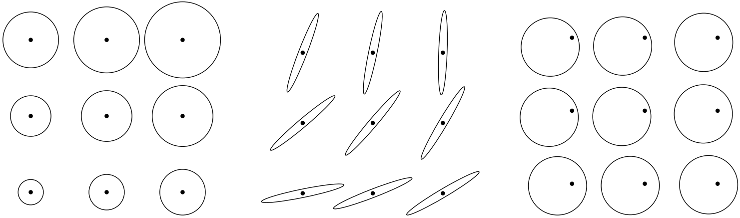

Isotropic Riemannian Finsler

Path Length and asymmetric Distance

Connected domain $\Omega \subset \mathbb{R}^d$ equipped with a Finsler Metric $\mathcal{F}$.

Definition (Length of a path $\gamma \in C^1([0,1], \overline{\Omega})$)

$$\mathrm{length}(\gamma) := \int_0^1 \mathcal{F}_{\gamma(t)}(\gamma'(t))\, dt$$

Definition (Asymmetric Distance on $\overline{\Omega}$)

$$D(x,y) := \inf\{\mathrm{length}(\gamma);\, \gamma(0) = x,\, \gamma(1) = y\}.$$

Addressed Problem.
Input: $\mathcal{F}$, z_0.

Output: $D(z_0, \cdot)$, paths of minimal length form z_0.

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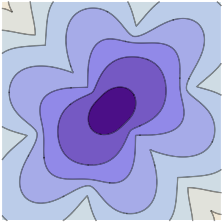

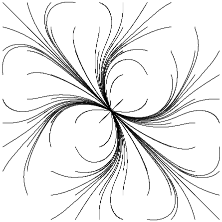

The Eikonal Equation

Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Let $z_0 \in \Omega$, let $\mathcal{F}$ be a Finsler metric. Then $u(x) := D(z_0, x)$ is the unique viscosity solution of:

$$\begin{cases} \mathcal{F}_x^*(\nabla u(x)) = 1 & \forall x \in \Omega \setminus \{z_0\}, \\ u(z_0) = 0, & \\ \langle \nabla u(x), n(x) \rangle \geq 0 & \forall x \in \partial\Omega. \end{cases} \tag{1}$$

- Dual norm $\mathcal{F}_x^*(e) := \max\{\langle e, f\rangle;\, \mathcal{F}_x(f) = 1\}$.
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Escape time from $x \in \Omega$

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Bellman's optimality principle

If $x \in V \subset \Omega$, then to escape Ω one must cross ∂V

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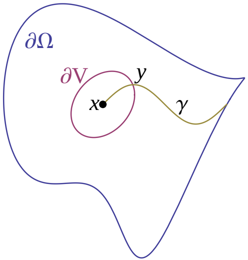

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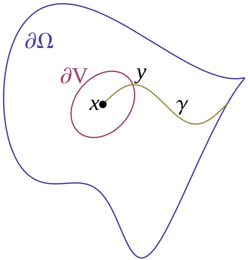

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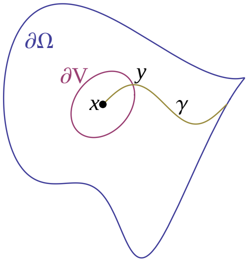


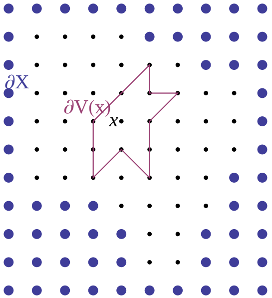
Let X and ∂X be finite point sets discretizing Ω, $\partial\Omega$.

Definition (Hopf-Lax update operator)

For $u : X \cup \partial X \to \mathbb{R}$, $x \in X$ with polygonal stencil $V(x)$.

$$\Lambda u(x) := \min_{y \in \partial V(x)} \mathcal{F}_x(y-x) + u(y),$$

where u is piecewise-linearly interpolated on the faces of $\partial V(x)$.



Discrete fixed point problem

$$\begin{cases} u(x) = \Lambda u(x) & \text{for all } x \in X, \\ u(x) = 0 & \text{for all } x \in \partial X. \end{cases}$$

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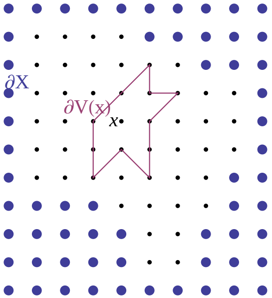

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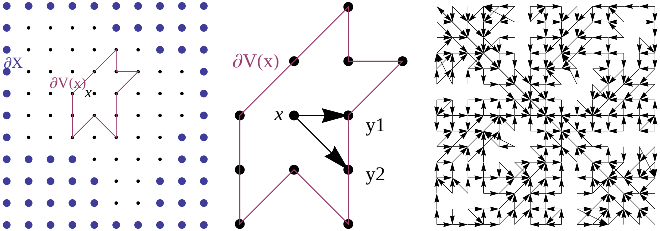


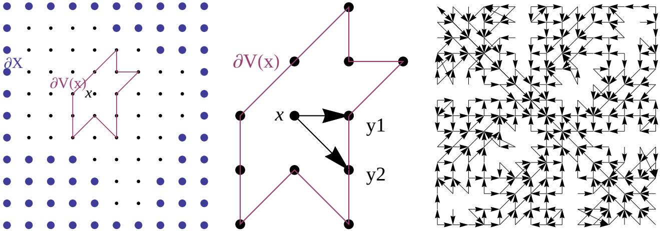
Definition (Causality property)

Operator Λ is causal if for any $u : X \to \mathbb{R}$, $x \in X$, denoting by $[y_0, \cdots, y_k]$ of the minimal facet of $V(x)$ where the minimum defining $\Lambda u(x)$ is attained, one has

$$\forall i \in [\![0, k]\!],\ \Lambda u(x) > u(y_i).$$

The fast marching algorithm. Tsitsilikis, 95.

If causality holds, then the discrete system can be solved in a single pass using a variant of Dijkstra's algorithm.



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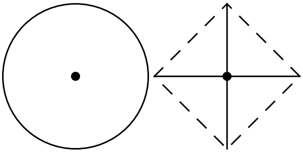

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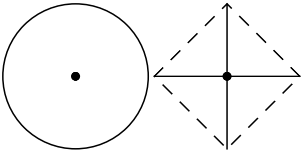

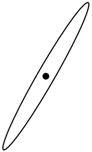

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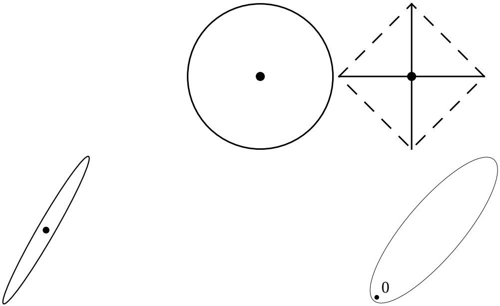

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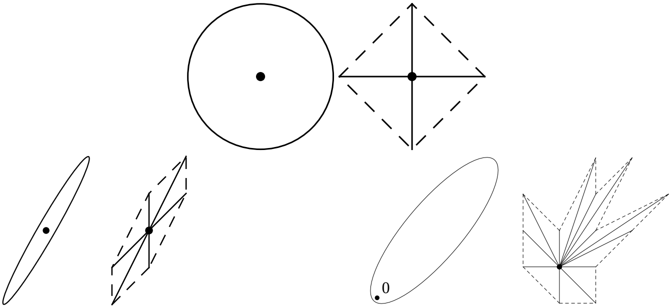


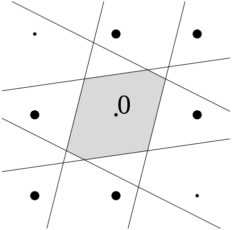
Let $\|e\|_M := \sqrt{\langle e, Me\rangle}$, for $e \in \mathbb{R}^d$, $M \in S_d^+$.

Definition (Voronoi cell and facets)

For each matrix $M \in S_d^+$, introduce the Voronoi cell and facet

$$\mathrm{Vor}(M) := \{g \in \mathbb{R}^d;\ \forall e \in \mathbb{Z}^d,\ \|g\|_M \leq \|g - e\|_M\},$$
$$\mathrm{Vor}(M; e) := \{g \in \mathrm{Vor}(M);\ \|g\|_M = \|g - e\|_M\}.$$

e is a Voronoi Vector $\Leftrightarrow \mathrm{Vor}(M, e) \neq \emptyset$.



Let $\|e\|_M := \sqrt{\langle e, Me\rangle}$, for $e \in \mathbb{R}^d$, $M \in S_d^+$.

Definition (Voronoi cell and facets)

For each matrix $M \in S_d^+$, introduce the Voronoi cell and facet

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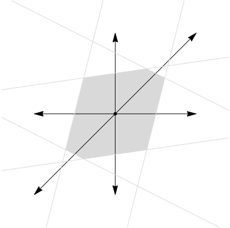

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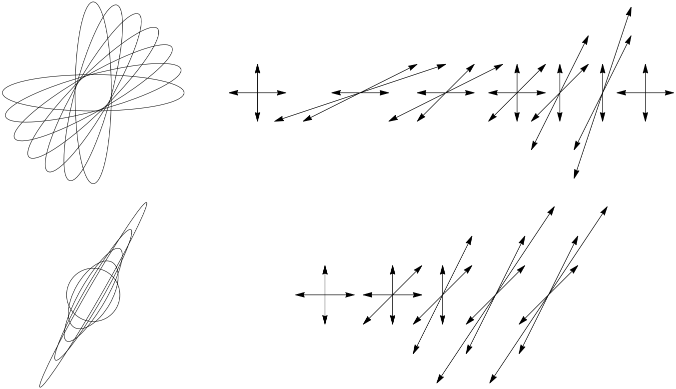

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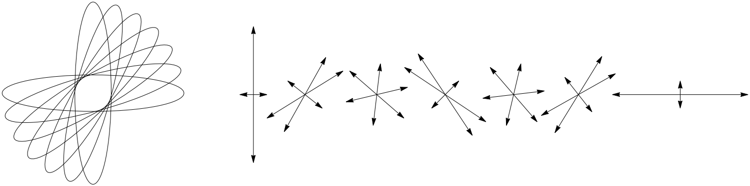
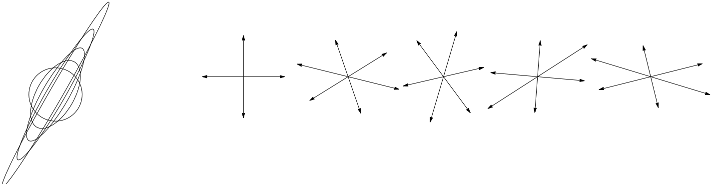
Coordinates transformed by $M^{\frac{1}{2}}$.





Proposition (Connected Voronoi vertices form acute angles)

If $\mathrm{Vor}(M;e) \cap \mathrm{Vor}(M;f) \neq \emptyset$, *then* $\langle e, Mf\rangle \geq 0$.

Indeed, let $p \in \mathrm{Vor}(M;e) \cap \mathrm{Vor}(M;f)$. Then

$$\|p\|_M = \|p-e\|_M = \|p-f\|_M \leq \|p-(e+f)\|_M.$$

$$0 \leq \|p-(e+f)\|_M^2 - \|p-e\|_M^2 - \|p-f\|_M^2 + \|p\|_M^2 = 2\langle e, Mf\rangle.$$

Causal stencils for Riemannian metrics, on a cartesian grid

$\mathcal{F}_x = \|\cdot\|_{\mathbf{M}(x)}$, and $X = \Omega \cap \mathbb{Z}^d$. Define $V(x)$ as the collection of $\mathbf{M}(x)$-Voronoi vectors, with the topology dual to $\mathrm{Vor}(\mathbf{M}(x))$.

Theorem (Optimality of Voronoi based stencils, 2D)

Voronoi based stencils are the smallest, in the sense of convex hull inclusion, satisfying the acuteness condition.

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Size of the stencils

For each $\kappa \geq 1$, $\theta \in [0, \pi]$, introduce the symmetric matrix

$$M_\kappa(\theta) := e_\theta \otimes e_\theta + \kappa^2 e_\theta^\perp \otimes e_\theta^\perp$$

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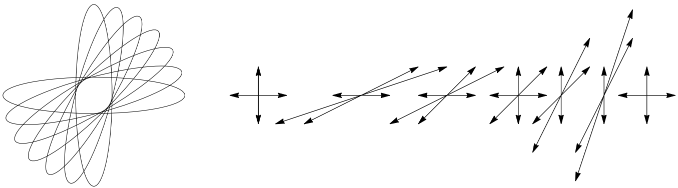

Figure : Stencil size strongly depends on orientation. $\times M^{\frac{1}{2}}$

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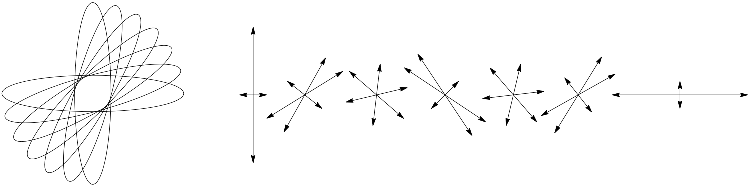

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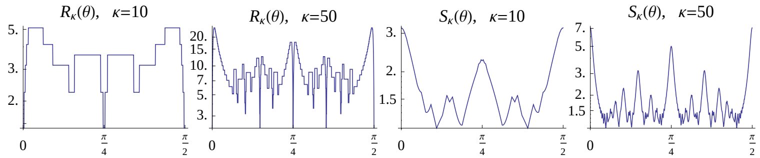
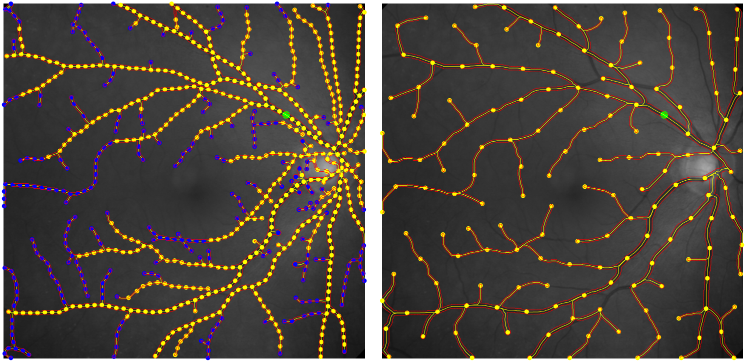
Theorem (Euclidean and intrinsic stencil radius, as $\kappa \to \infty$)

$$\|R_\kappa\|_{L^p} \approx \kappa^{\frac{1}{2}} \|S_\kappa\|_{L^p}. \qquad \|S_\kappa\|_{L^p} \approx \begin{cases} \kappa^{\frac{1}{2}-\frac{1}{p}} & \text{if } p > 2, \\ (\ln \kappa)^{\frac{1}{2}} & \text{if } p = 2, \\ 1 & \text{if } p < 2. \end{cases}$$



Taking advantage of Anisotropy

Anisotropic fast marching (left) allows to take smaller steps in the iterative extraction of retinal vessel trees.



Da Chen, Laurent Cohen, J.-M. M, Vessel Extraction Using Anisotropic Minimal Paths and Path Score, ICIP 2014

Petitot's model: curvature penalized length

$\gamma : [0,1] \to \mathbb{R}^2$, s: curvilinear abcissa, κ: curvature.

$$\mathcal{E}(\gamma) := \int_\gamma \sqrt{1+\kappa^2} ds$$

Orientation lifting and Sub-Riemannian reformulation

For $(\gamma, \theta) : [0,1] \to \mathbb{R}^2 \times \mathbb{S}^1$ consider, with $e_\theta := (\cos\theta, \sin\theta)$

$$\mathcal{E}(\gamma, \theta) := \int_0^1 \sqrt{\|\gamma'\|^2 + |\theta'|^2} dt$$

if $\det(\gamma', e_\theta) = 0$ identically. Otherwise $\mathcal{E}(\gamma, \theta) = +\infty$.

Riemannian approximation by constraint penalization.

Choose $\lambda \gg 1$ and consider

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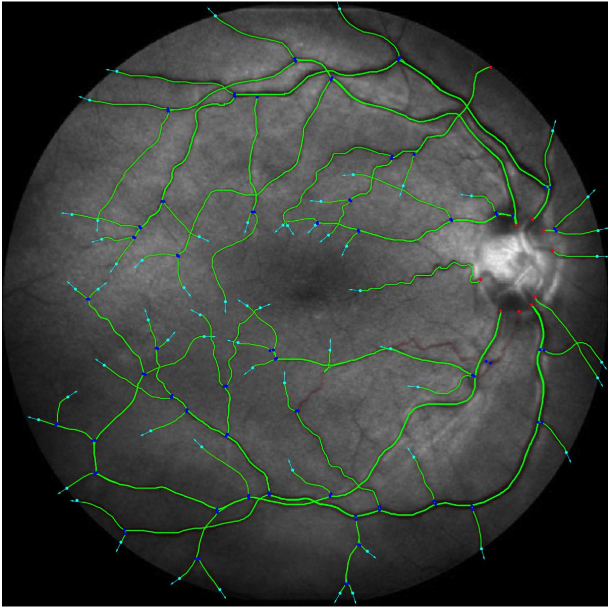
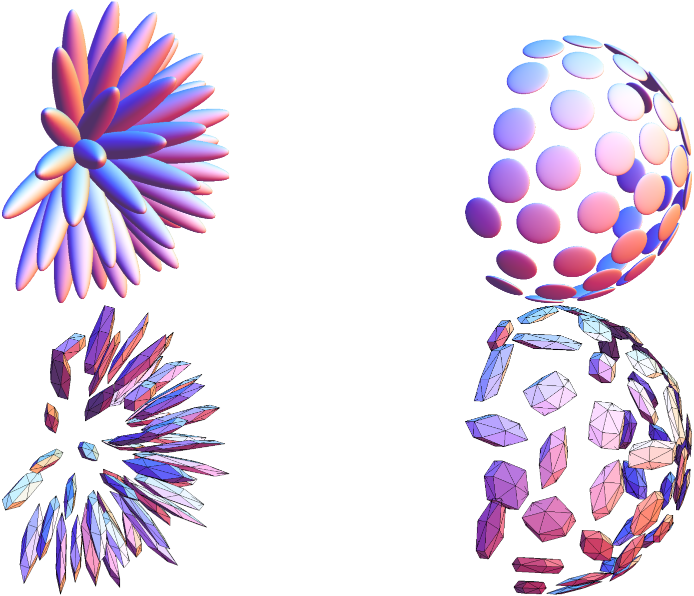
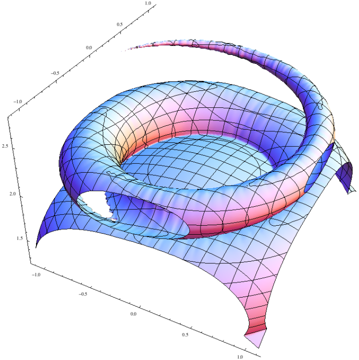


Figure : Extraction of the retina vessels, with R. Duits, G.Sanguinetti

Voronoi-based stencils for three dimensional Riemannian Shortest paths



Applications in progress



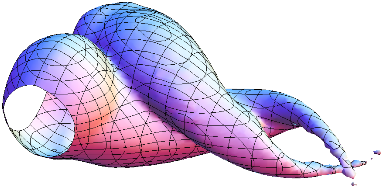


Figure : Front propagations with respect to anisotropic metrics

Applications in progress

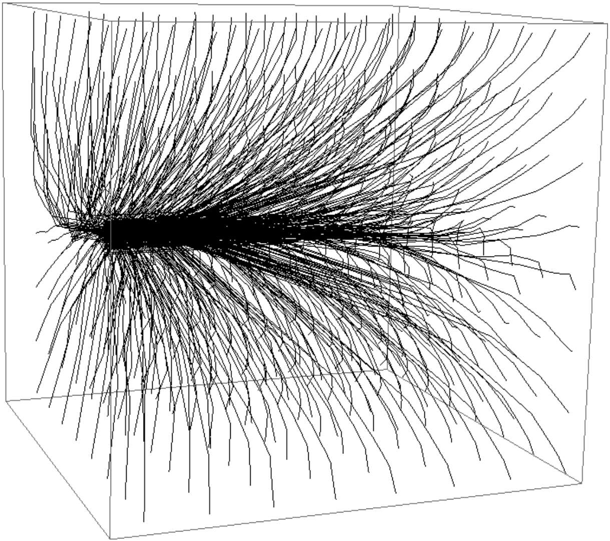
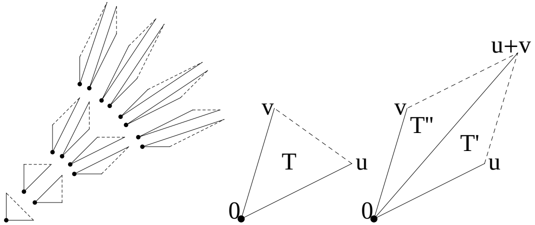


Figure : Curvature penalized geodesics via 5D sub-riemannian fast marching on $\mathbb{R}^3 \times \mathbb{S}^2$. Work in progress with R.Duits, G.Sanguinetti

Definition (The Stern-Brocot tree of triangles)

Root: $T_0 = [(0,0),(1,0),(0,1)]$ Children of $T = [0,u,v]$:
$T' = [0,u,u+v]$, $T'' = [0,u+v,v]$.



The map $T = [0,(a,b),(a',b')] \mapsto q = \frac{a+a'}{b+b'}$ induces a bijection between the triangles and the positive rationals.

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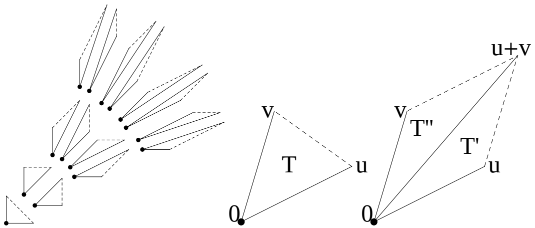

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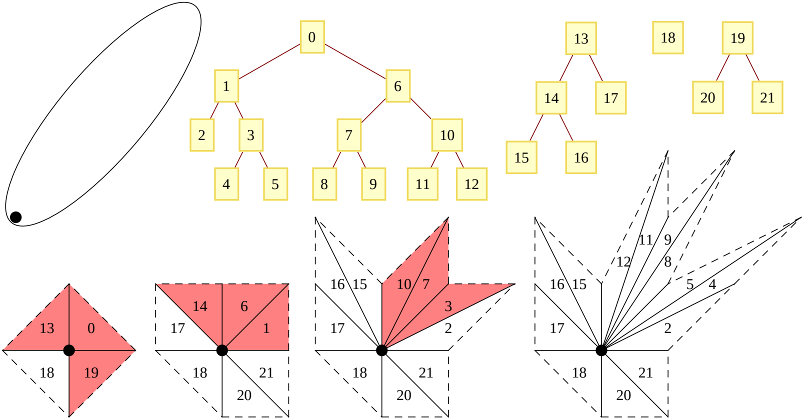
Stencil and tree structure

$V(F)$: mesh obtained by recursively refining the 4 element mesh $\mathcal{T}_0$ (bottom left), until all triangles are F-acute.



J.-M. M, Efficient Fast Marching with Finsler Metrics, Numerische Mathematic, 2013

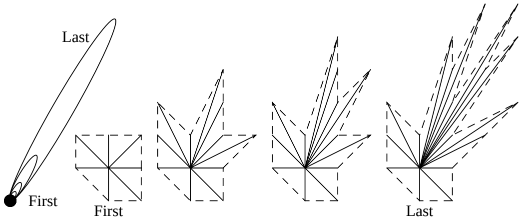
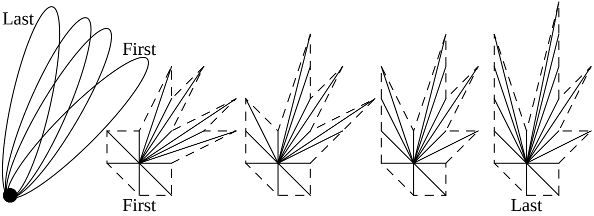


Figure : Stencils are adaptive and depend on both the orientation and the anisotropy of the asymmetric norm

Theorem

Let F be an asymmetric norm on $\mathbb{R}^2$, and let $n_F(\theta)$ be the cardinality of the stencil associated to $F \circ R_\theta$. Then

$$\int_0^{2\pi} n_F(\theta) d\theta \leq C(1 + \ln^2 \kappa), \quad \textit{where } \kappa := \max_{|u|=|v|=1} \frac{F(u)}{F(v)}.$$



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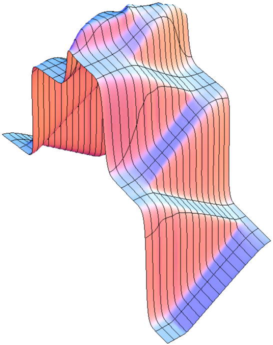
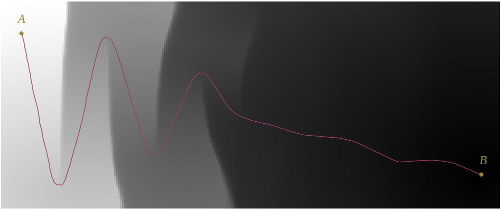
Applications of Finsler shortest paths

- Models in which ascent is harder than descent.
- Navigation at unit speed + drift due to currents.
- Segmentation with black on right, white on left. (Zach, Chan, Niethammer, 09)





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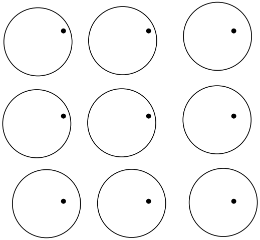
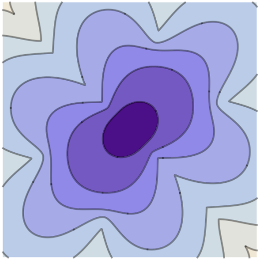
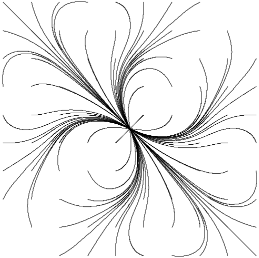

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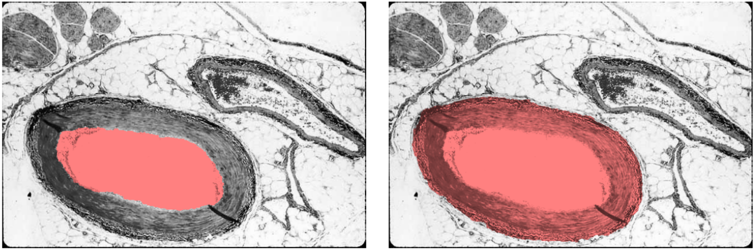

(a) Geodesic active contour (b) Finsler active contour

Euler elastica: squared curvature penalized length

$\gamma : [0,1] \to \mathbb{R}^2$, s: curvilinear abcissa, κ: curvature

$$\mathcal{E}(\gamma) := \int_\gamma (1+\kappa^2) ds$$

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Choose $\lambda \gg 1$ and consider

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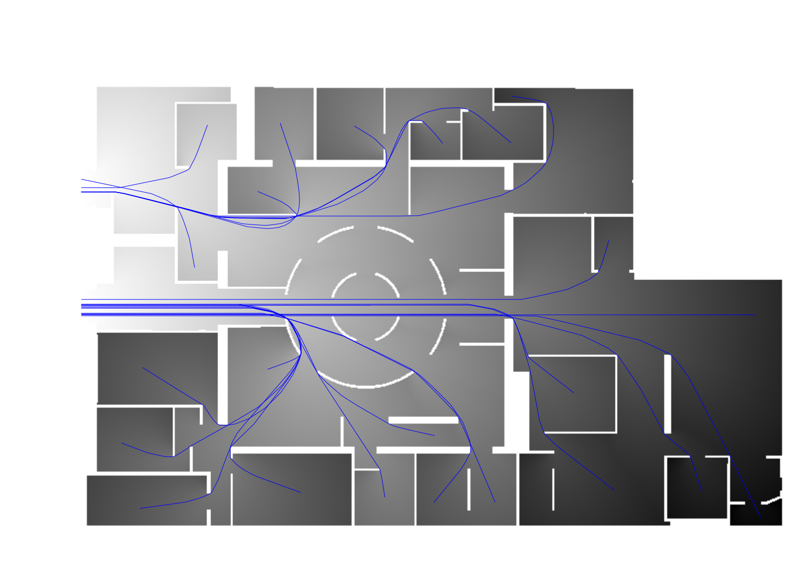
$$\mathcal{E}_\lambda(\gamma,\theta) := \int_0^1 \|\gamma'\| + \frac{\|\theta'\|^2}{\|\gamma'\|} + (\lambda-1)(\|\gamma'\| - \langle \gamma', e_\theta\rangle) + \mathcal{O}(1/\lambda).$$

Leaving an expo of centre Pompidou



Conclusion

- Image processing requires robust, structure preserving discretizations of strongly anisotropic PDEs.

Realisations

- Adaptive numerical schemes, relying on sparse stencils, of limited extension, without restrictions on anisotropy.
- Quantitative results on minimal stencil cardinality and size.
- New applications, e.g. curvature penalized shortest paths.

Tools and techniques

- The geometry of lattices of $\mathbb{R}^2$, $\mathbb{R}^3$.
- The Stern-Brocot tree of triangles.
- Open questions: Unstructured point sets. 3D asymmetric norms.

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