Adaptive and anisotropic approximation and the mesh/metric equivalence

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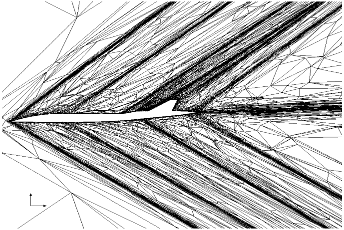
Motivation : anisotropic phenomena

The solutions of many PDE's exhibit a strongly anisotropic behavior.

- Boundary layers in fluid simulation.
- Spikes and edges of metallic objects in electromagnetism.
- Shockwaves in transport equations.



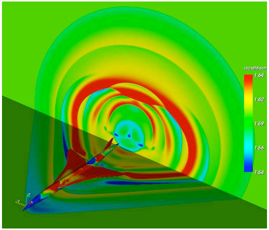


Figure : Fluid simulation around a supersonic plane (F. Alauzet).

Mesh optimization

Objective: improve the trade-off between accuracy and complexity.

- Accuracy: for example, the error between the solution and its approximation in some given norm.
- Complexity: typically tied to the cardinality of the mesh.

An appetizer: given a function $f : \Omega \to \mathbb{R}$ and an integer N, construct a triangulation $\mathcal{T}$ of Ω which minimizes

$$\|\nabla(f - \mathrm{I}_{\mathcal{T}} f)\|_{L^2(\Omega)},$$

over all triangulations such that $\#(\mathcal{T}) \leq N$, with $\mathrm{I}_{\mathcal{T}}$ the piecewise linear interpolant.

In the numerical examples $N = 500$ and

$$f(x, y) := \tanh(10(\sin(5y) - 2x)) + x^2 y + y^3.$$

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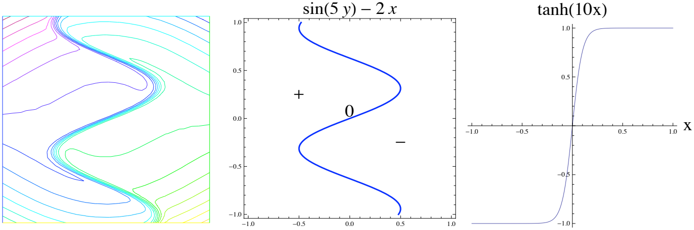

Figure : Sharp transition along the curve $\sin(5y) = 2x$, of width $1/10$.

A classical result

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain, let $f \in H^2(\Omega)$ and let $\mathcal{T}$ be a triangulation.

Ciarlet-Raviart

On each $T \in \mathcal{T}$, the local error satisfies

$$\|\nabla(f - \mathrm{I}_{\mathcal{T}} f)\|_{L^2(T)} \leq C_0 \frac{h_T^2}{r_T} \|d^2 f\|_{L^2(T)},$$

where $h_T := \mathrm{diam}(T)$ and r_T is the radius of the largest disc inscribed in T, and C_0 is an absolute constant.

Consequence: with $h = \max_{T \in \mathcal{T}} h_T$

$$\|\nabla(f - \mathrm{I}_{\mathcal{T}} f)\|_{L^2(\Omega)} \leq C(\mathcal{T})\, h \,\|d^2 f\|_{L^2(\Omega)},$$

with $C(\mathcal{T}) = C_0 \max_{T \in \mathcal{T}} \frac{h_T}{r_T}$ that remains bounded for isotropic triangulations.

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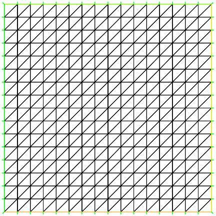
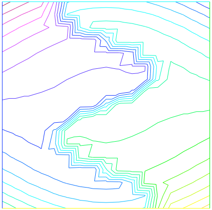
In terms of $N = \#(\mathcal{T})$, this gives

$$\sqrt{N}\,\|\nabla(f - \mathrm{I}_{\mathcal{T}} f)\|_{L^2(\Omega)} \leq C'(\mathcal{T})\,\|d^2 f\|_{L^2(\Omega)},$$

where $C'(\mathcal{T}) = C_0 \sqrt{|\Omega|}\,\frac{\max_{T\in\mathcal{T}} h_T^2/r_T}{\min_{T\in\mathcal{T}} \sqrt{|T|}}$, that remains bounded for uniform triangulations:

$$h \sim h_T \sim r_T \sim \sqrt{|T|} \Rightarrow h \sim N^{-1/2}.$$

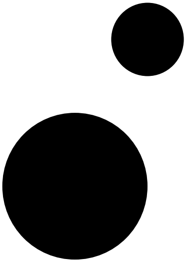



The parameters of a triangle.

Position

Uniform triangulation

Area

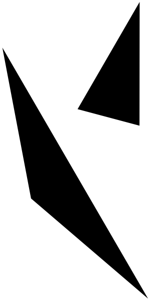


Isotropic triangulation

Aspect ratio and orientation

Anisotropic triangulation

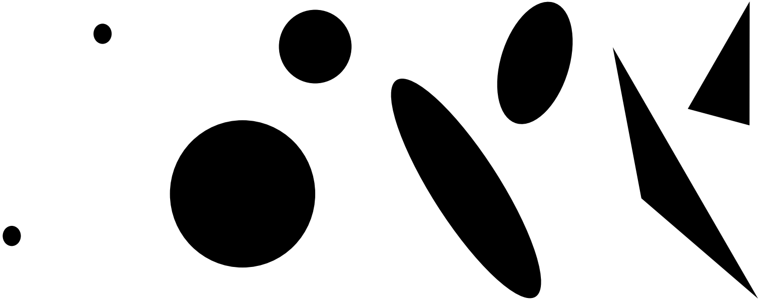
Angles



Optimal anisotropic triangulation

The parameters of a triangle.

Position | Area | Aspect ratio and orientation | Angles



Uniform triangulation | Isotropic triangulation | Anisotropic triangulation | Optimal anisotropic triangulation

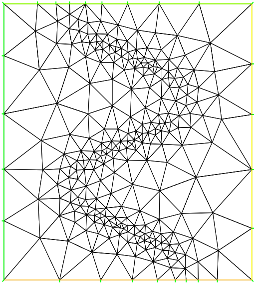
Isotropic meshes: the triangle seen as a disk.

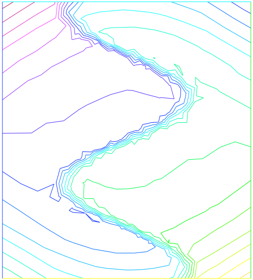
Theorem (Adaptive approximation: DeVore-Yu)

For any $f \in W^{2,1}(\Omega)$, $\Omega =]0,1[^2$, there exists a sequence $(\mathcal{T}_N)_{N \geq 2}$ of (isotropic) triangulations of Ω, $\#(\mathcal{T}_N) \leq N$, such that

$$\sqrt{N}\,\|\nabla(f - \mathrm{I}_{\mathcal{T}_N} f)\|_{L^2(\Omega)} \leq C \|M(d^2 f)\|_{L^1(\Omega)}$$

$M(g)$: Hardy-Littlewood maximal function of g.

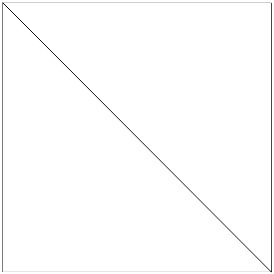




Key principle : error equidistribution

Such sequences of triangulations may be obtained by a hierarchical refinement algorithm, starting from a coarse mesh.

- Refine the triangle with largest local error $\|\nabla(f - \mathrm{I}_{\mathcal{T}} f)\|_{L^2(T)}$.
- Propagate the refinement to preserve conformity.
- Iterate until prescribed number of triangles is met.



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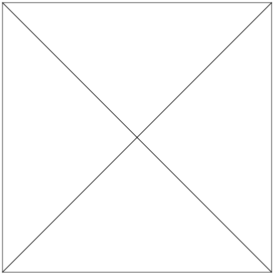

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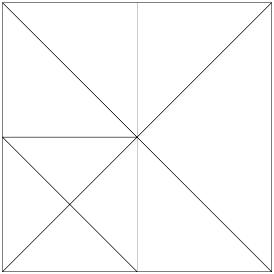

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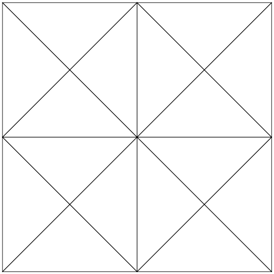

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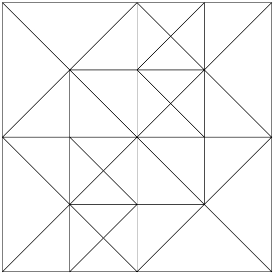

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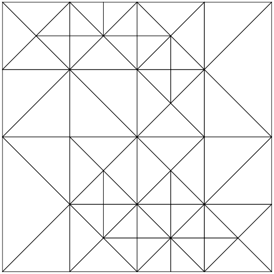

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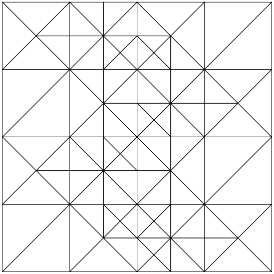

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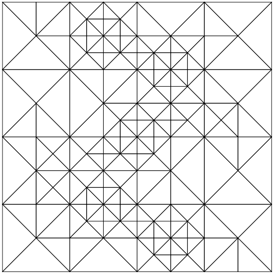

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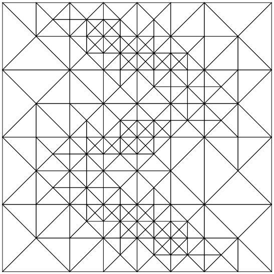

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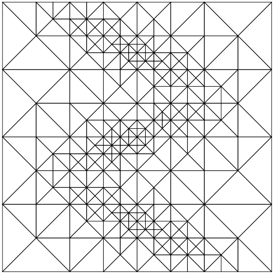

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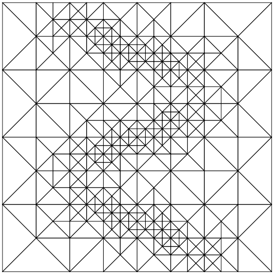

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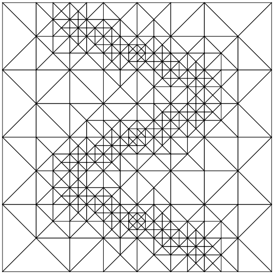

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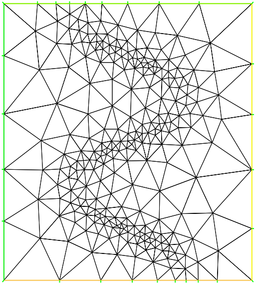

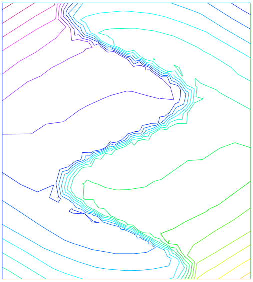

Aspect ratio: the triangle seen as an ellipse.

The ellipse of minimal area containing a triangle T has equation

$$(z - z_T)^{\mathrm{T}} \mathcal{H}_T (z - z_T) \leq 1,$$

where $\mathcal{H}_T$ is a symmetric positive definite matrix and z_T is the barycenter of T.

Anisotropic mesh generation

Given a metric $H : \Omega \to S_2^+$ produce a triangulation $\mathcal{T}$ such that : for any $T \in \mathcal{T}$ and any $z \in T$,

$$H(z) \simeq \mathcal{H}_T$$

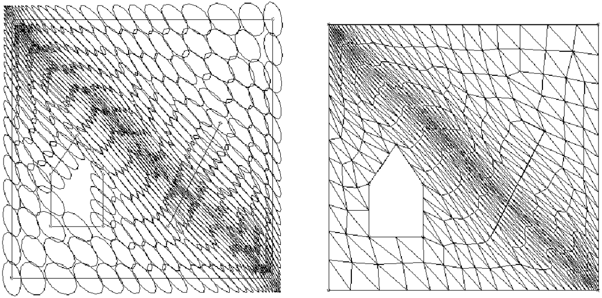

Figure : A metric and an adapted triangulation (credit: J. Schoen)

More detail on this later.

$\pi = ax^2 + 2bxy + cy^2$: homogeneous quadratic polynomial.

$$L_G(\pi) := \inf_{\det H = 1} \sup_{\mathcal{H}_T = H} \|\nabla(\pi - \mathrm{I}_T\, \pi)\|_{L^2(T)}.$$

(Near) Minimizing matrix H

Figure : Level lines of π (red, dashed), ellipse (blue, thick) associated to (near) optimal H which is proportional to the absolute value of the matrix associated to π.

Explicit equivalent of L_G

$$L_G(\pi) \simeq \sqrt{\|\pi\|}\,\sqrt[4]{|\det \pi|},$$

where $\|\pi\|$ and $\det \pi$ are the norm and determinant of the symmetric matrix associated to π.

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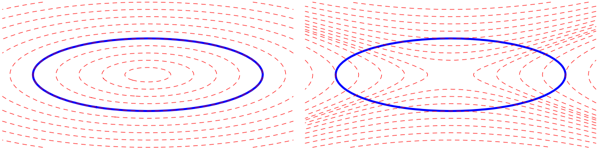

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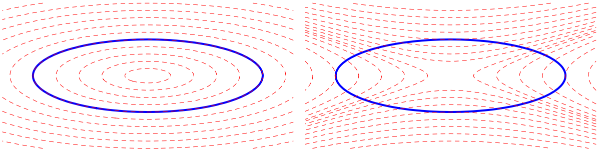

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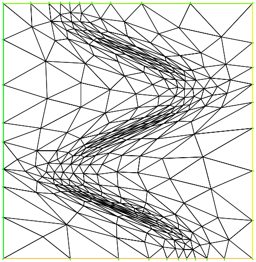
Local model :

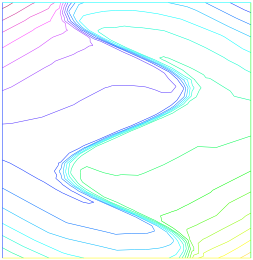
$$f(x_0+x, y_0+y) = \alpha + (\beta x + \gamma y) + \underbrace{(ax^2 + 2bxy + cy^2)}_{\pi = \frac{1}{2} d^2 f(x_0, y_0)} + \mathcal{O}(|x|^3 + |y|^3)$$

Theorem

For any bounded polygonal domain Ω and any $f \in C^2(\overline{\Omega})$ there exists a sequence $(\mathcal{T}_N)_{N \geq N_0}$ of triangulations of Ω, $\#(\mathcal{T}_N) \leq N$, such that

$$\limsup_{N \to \infty} \sqrt{N} \, \|\nabla(f - \mathrm{I}_{\mathcal{T}_N} f)\|_{L^2(\Omega)} \leq C \|L_G(d^2 f)\|_{L^1(\Omega)}$$





An unusual estimate

$$\limsup_{N\to\infty} \sqrt{N}\,\|\nabla(f - \mathrm{I}_{\mathcal{T}_N} f)\|_{L^2(\Omega)} \leq C\|L_G(d^2 f)\|_{L^1(\Omega)}$$

- The quantity $L_G(d^2 f(z)) \simeq \sqrt{\|d^2 f(z)\|\sqrt[4]{|\det(d^2 f(z))|}}$ depends nonlinearly on f.
- Defining $A(f) := \|L_G(d^2 f)\|_{L^1}$ we generally do not have

$$A(f+g) \leq C(A(f) + A(g)).$$

- The estimate holds asymptotically as $N \to \infty$.

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Angles: the triangle seen as a triangle (!)

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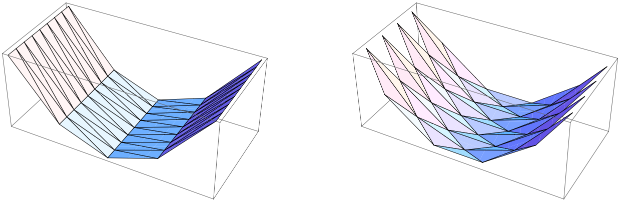

Figure : Interpolation of a parabola on a acute or obtuse mesh.

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(Near) Minimizing triangle

$$L_G \Rightarrow \frac{h_T}{r_T} \simeq \frac{1}{\sqrt{\epsilon}}$$

$$L_A \Rightarrow \frac{h_T}{r_T} \simeq \frac{1}{\epsilon}$$

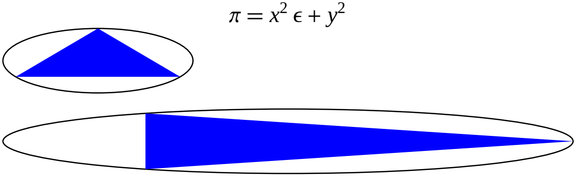

Figure : The minimizing triangle for L_A has acute angles and is more anisotropic than the minimizing ellipse for L_G.

Explicit equivalent of L_A

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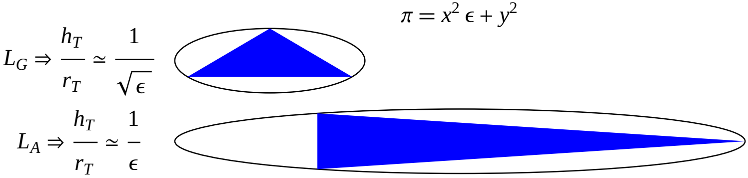

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Furthermore for any admissible sequence $(\mathcal{T}_N)_{N \geq N_0}$ of triangulations, $\#(\mathcal{T}_N) \leq N$, one has

$$\liminf_{N \to \infty} \sqrt{N} \, \|\nabla(f - \mathrm{I}_{\mathcal{T}_N} f)\|_{L^2(\Omega)} \geq \left\| L_A \left(\frac{d^2 f}{2} \right) \right\|_{L^1(\Omega)}.$$

Admissibility :

$$\sup_{N \geq N_0} \left(N^{\frac{1}{2}} \sup_{T \in \mathcal{T}_N} \mathrm{diam}(T) \right) < \infty.$$

Theorem

For any bounded polygonal domain Ω and any $f \in C^2(\overline{\Omega})$ there exists a sequence $(\mathcal{T}_N)_{N \geq N_0}$ of triangulations of Ω, $\#(\mathcal{T}_N) \leq N$, such that

$$\limsup_{N \to \infty} \sqrt{N} \, \|\nabla(f - \mathrm{I}_{\mathcal{T}_N} f)\|_{L^2(\Omega)} \leq \left\| L_A \left(\frac{d^2 f}{2} \right) \right\|_{L^1(\Omega)}.$$

Furthermore for any admissible sequence $(\mathcal{T}_N)_{N \geq N_0}$ of triangulations, $\#(\mathcal{T}_N) \leq N$, one has

$$\liminf_{N \to \infty} \sqrt{N} \, \|\nabla(f - \mathrm{I}_{\mathcal{T}_N} f)\|_{L^2(\Omega)} \geq \left\| L_A \left(\frac{d^2 f}{2} \right) \right\|_{L^1(\Omega)}.$$

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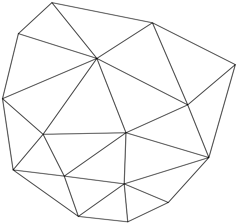
Guideline of the upper estimate (heuristic)

The asymptotically optimal sequence is built using a two-scale local patching strategy. (Not suited for applications)



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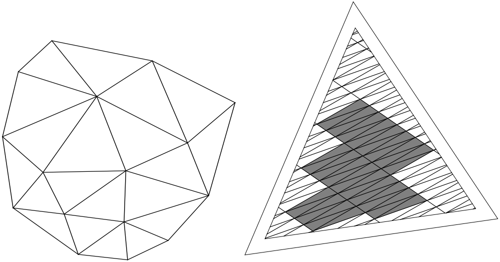

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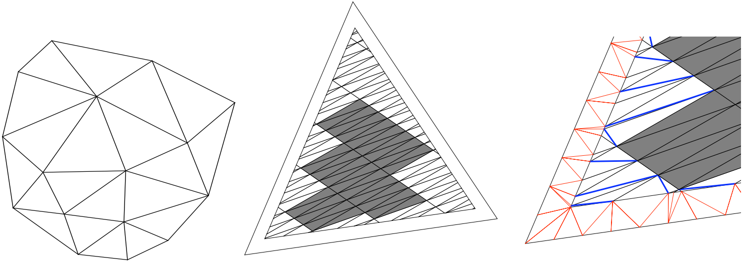

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Finite elements of arbitrary degree $m-1$

- $L_G(d^m f)$ and $L_A(d^m f)$ similarly defined.
- Optimal asymptotic estimate involving L_A.
- Explicit expression of L_A using Hilbert's invariants.
- Explicit near minimizers of L_A, L_G for $\mathbb{P}_2$ elements.

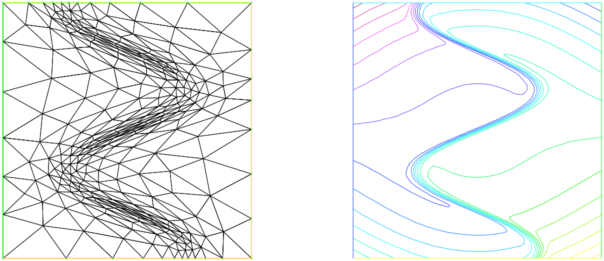

Figure : Interpolation with anisotropic $\mathbf{P}_2$ elements.

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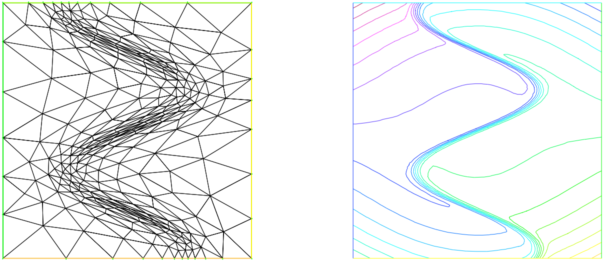

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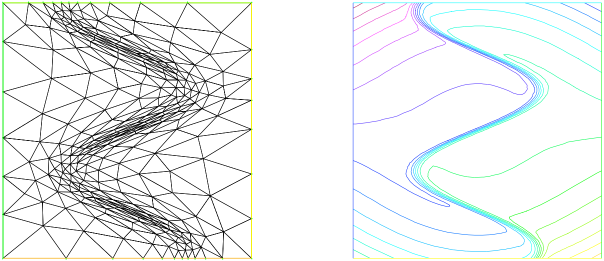

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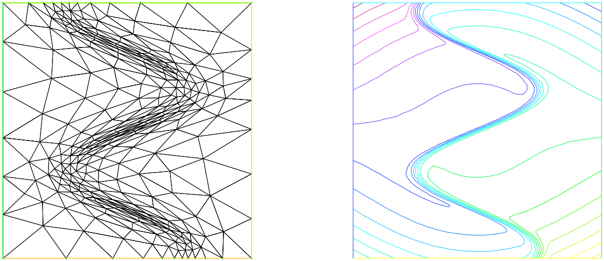

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- Explicit near minimizers of L_A, L_G for $\mathbb{P}_2$ elements. Aspect ratio for $\pi = ax^3 + 3bx^2y + 3cxy^2 + dy^3$:

$$\mathcal{M}_A(\pi) := \sqrt{\begin{pmatrix} a & b \\ b & c \end{pmatrix}^2 + \begin{pmatrix} b & c \\ c & d \end{pmatrix}^2},$$

$$\mathcal{M}_G(\pi) := \mathcal{M}_A(\pi) + \left(\frac{-\operatorname{disc}(\pi)}{\|\pi\|}\right)_+^{\frac{1}{3}} \mathrm{Id},$$

where $\operatorname{disc}\pi = 4(ac-b^2)(bd-c^2) - (ad-bc)^2$.

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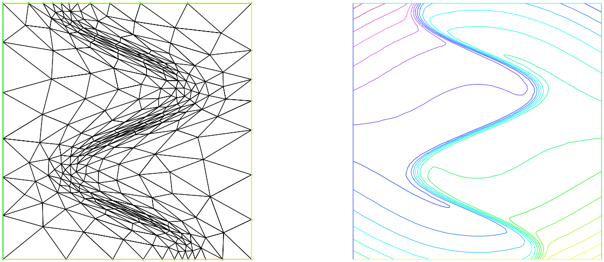

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Numerical experiments : $\|\nabla(f - \mathrm{I}_{\mathcal{T}}^{m-1} f)\|_{L^2}$, with 500 triangles.

	Uniform	Isotropic	Based on L_G	Based on L_A
$\mathbf{P}_1$	110	51	11	?
$\mathbf{P}_2$	79	14	0.88	?

Equivalence of meshes and metrics

Metrics and triangulations on $\mathbb{R}^2$

Definition (Equivalence triangulation/metric)

A (conforming) triangulation $\mathcal{T}$ of $\mathbb{R}^2$ is C-equivalent to a metric $H \in C^0(\mathbb{R}^2, S_2^+)$ if for all $T \in \mathcal{T}$ and $z \in T$ one has

$$C^{-1} H(z) \leq \mathcal{H}_T \leq C H(z).$$

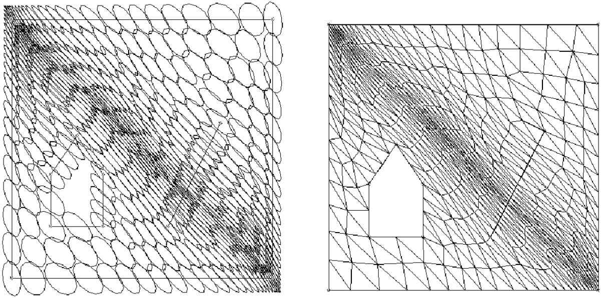

Figure : A metric and an equivalent triangulation, Credit : J. Schoen

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$$C^{-1} H(z) \leq \mathcal{H}_T \leq C H(z).$$

Definition (Equivalence collection of triangulations/ collection of metrics)

A collection $\mathbb{T}$ of triangulations of $\mathbb{R}^2$ is equivalent to a collection $\mathbb{H} \subset C^0(\mathbb{R}^2, S_2^+)$ of metrics if there exists C such that

- $\forall \mathcal{T} \in \mathbb{T}, \exists H \in \mathbb{H}$, *such that $\mathcal{T}$ and H are C-equivalent.*
- $\forall H \in \mathbb{H}, \exists \mathcal{T} \in \mathbb{T}$, *such that $\mathcal{T}$ and H are C-equivalent.*

Isotropic triangulations

Theorem

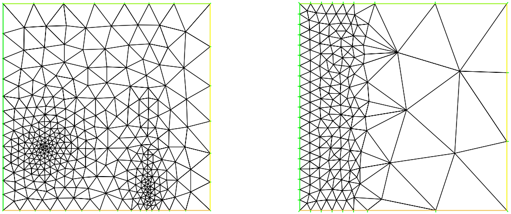
The collection $\mathbb{T}$ *of all triangulations* $\mathcal{T}$ *satisfying for each* $T \in \mathcal{T}$

$$\mathrm{diam}(T)^2 \leq 4|T|$$

is equivalent to the collection $\mathbb{H}$ *of metrics* H *of the form*

$$H(z) = \frac{\mathrm{Id}}{s(z)^2} \quad \textit{where} \quad |s(z) - s(z')| \leq |z - z'|$$

Triangulations produced by FreeFem



From isotropic to anisotropic metrics

Isotropic "Lipschitz" metrics

$H(z) = s(z)^{-2}\,\mathrm{Id}$. Two equivalent properties:

- (d) $\forall z, z' \in \mathbb{R}^2$, $|s(z) - s(z')| \le |z - z'|$
- (r) $\forall z, z' \in \mathbb{R}^2$, $\left|\ln\left(\frac{s(z')}{s(z)}\right)\right| \le d_H(z, z')$

where d_H denotes the Riemannian distance

$$d_H(z, z') := \inf_{\substack{\gamma(0)=z \\ \gamma(1)=z'}} \int_0^1 \sqrt{\gamma'(t)^{\mathrm{T}} H(\gamma(t))\, \gamma'(t)}\, dt.$$

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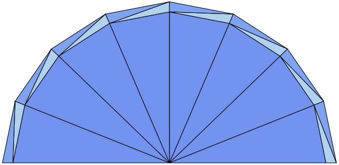
Graded Triangulations

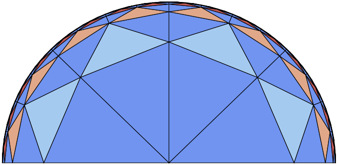
Definition

A triangulation $\mathcal{T}$ of $\mathbb{R}^2$ is K-graded if for all $T, T' \in \mathcal{T}$,

$$T \text{ intersects } T' \quad \Rightarrow \quad K^{-1}\mathcal{H}_T \leq \mathcal{H}_{T'} \leq K\,\mathcal{H}_T.$$



Non Graded



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Theorem

For any $K \geq K_0$ the collection $\mathbb{T}$ of K-graded triangulations is equivalent to the collection $\mathbb{H}$ of metrics satisfying (R).

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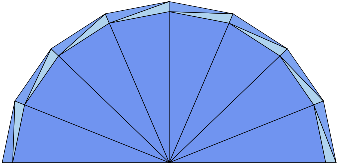

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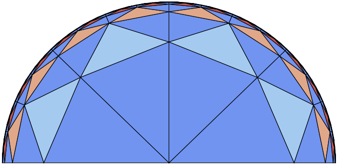

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Heuristic of the construction of $\mathcal{T}$ from H

Construct a collection $\mathcal{V} \subset \mathbb{R}^2$ of sites which satisfies:

covering For all $z \in \mathbb{R}^2$, $d_H(z, \mathcal{V}) := \min_{v \in \mathcal{V}} d(z, v) \le 1$.

separation For all $v \neq w \in \mathcal{V}$, $d_H(v, w) \ge 1$. (or $\ge \delta_0 > 0$).

Connect sites when Anisotropic Voronoi regions intersect.

Euclidean case $\mathrm{Vor}(v) := \{z \,;\, |z - v| = \min_{w \in \mathcal{V}} |z - w|\}$.

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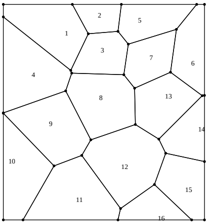

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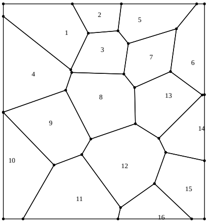

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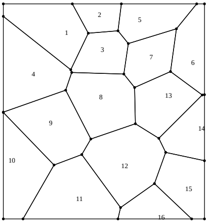
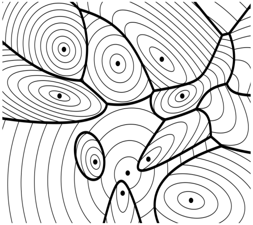




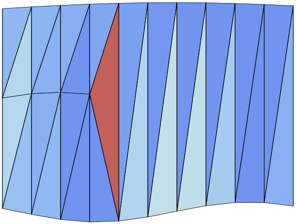
QuasiAcute triangulations

Definition

A triangulation $\mathcal{T}$ is K-QuasiAcute if

- *$\mathcal{T}$ is K-graded.*
- *There exists a K-refinement $\mathcal{T}'$ of $\mathcal{T}$ such that any angle θ of any $T \in \mathcal{T}'$ satisfies*

$$\theta \leq \pi - \frac{1}{K}.$$



$\mathcal{T}$: K-QuasiAcute $\quad$ $\mathcal{T}'$: K-refinement of $\mathcal{T}$.

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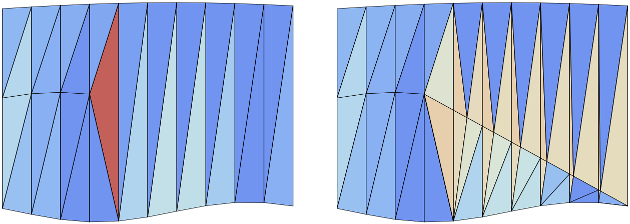

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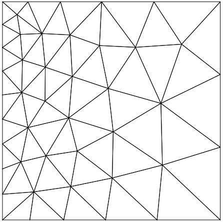
Theorem

For all $K \geq K_0$ the collection $\mathbb{T}$ of K-QuasiAcute triangulations is equivalent to the collection $\mathbb{H}$ of metrics satisfying simultaneously (R) and (D).

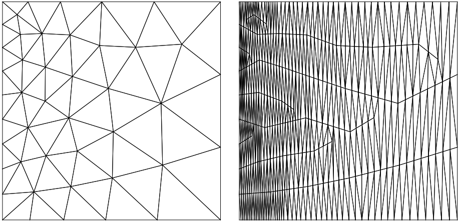
Quasi-Acute mesh generation



Heuristic : the vertices of the anisotropic triangulation must be aligned. A solution :

- Generate an isotropic triangulation adapted to $\|M^{-1}\|^{-1}\,\mathrm{Id}$.
- Sample some part of the edges of the isotropic triangulation, to obtain the vertices of the anisotropic triangulation.
- Intersect with the full edges to refine and eliminate remaining obtuse triangles.

Quasi-Acute mesh generation



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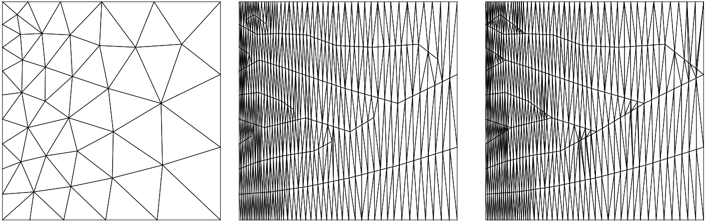

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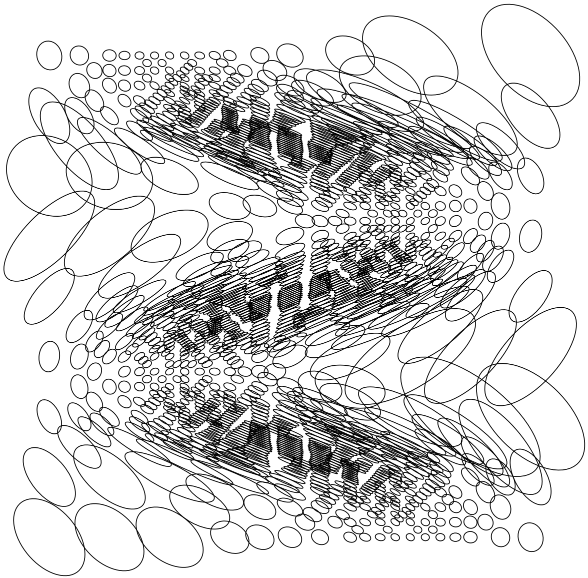
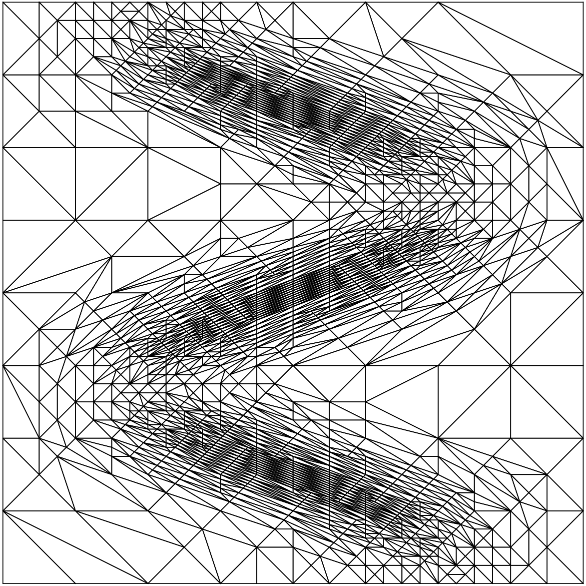


Figure : Ellipse field, quasi-acute triangulation.



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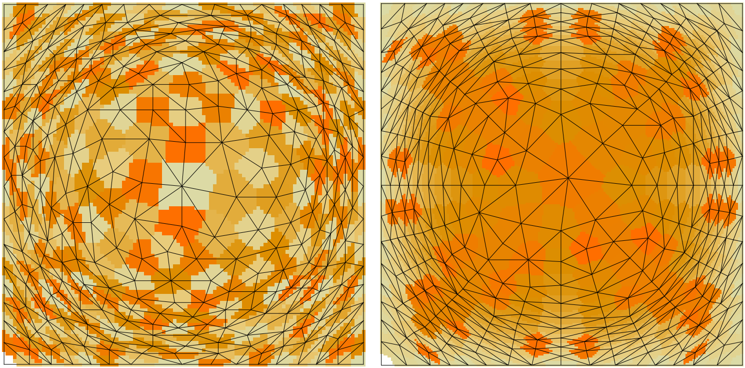
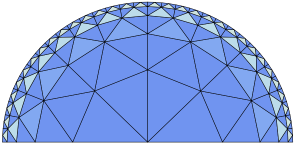


Figure : Voronoi diagram computed via anisotropic fast marching.
Left: Point insertion via farthest point sampling.
Right: point insertion aimed at producing QA-triangulation.

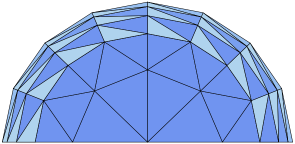
A comparison: how to capture a curvilinear discontinuity.

Objective: layer of width δ of triangles covering a smooth curve, using an Isotropic, QuasiAcute or Graded triangulation.



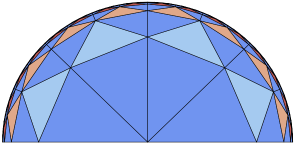
Isotropic

$\#(\mathcal{T}) \simeq \delta^{-1}$



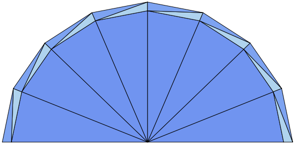
QuasiAcute

$\#(\mathcal{T}) \simeq \delta^{-\frac{1}{2}}|\ln \delta|$



Graded

$\#(\mathcal{T}) \simeq \delta^{-\frac{1}{2}}$



No restriction

$\#(\mathcal{T}) \simeq \delta^{-\frac{1}{2}}$

Anisotropic smoothness classes:

from finite element approximation to image models

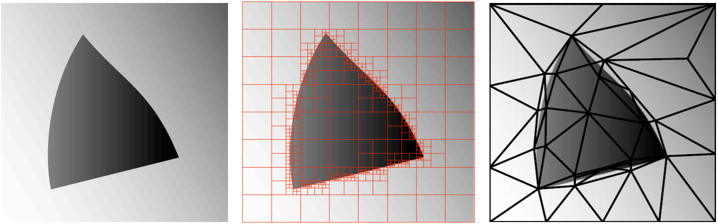


Figure : A cartoon function, and an adapted triangulation. Picture : Gabriel Peyré

A. Cohen, J.-M. Mirebeau, *Anisotropic smoothness classes: from finite element approximation to image models*, Journal of Mathematical Imaging and Vision, 2010.

Approximation of cartoon functions

If $g = \sum_{1\leq i\leq r} g_i \chi_{\Omega_i}$ where $g_i \in C^2(\overline{\Omega}_i)$ and $\partial\Omega_i$ is piecewise C^2, then there exists a sequence $(\mathcal{T}_N)_{N\geq N_0}$ of triangulations such that

$$N\|g - \mathrm{I}_{\mathcal{T}_N} g\|_{L^2(\Omega)} \leq C(g).$$

On the other hand, we have for smooth functions:

Theorem (Chen,Sun Xu; Babenko)

If $f \in C^2(\overline{\Omega})$ and $(\mathcal{T}_N)_{N\geq N_0}$ is an optimally adapted sequence then

$$\limsup_{N\to\infty} N\,\|f - \mathrm{I}_{\mathcal{T}_N} f\|_{L^2(\Omega)} \leq C \left\|\sqrt{|\det d^2 f|}\right\|_{L^{\frac{2}{3}}(\Omega)}$$

How to connect these estimates ?

Does $\left\|\sqrt{|\det d^2 g|}\right\|_{L^{\frac{2}{3}}}$ make sense if g is a cartoon function ?

Approximation of cartoon functions

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Does $\left\| \sqrt{|\det d^2 g|} \right\|_{L^{\frac{2}{3}}}$ make sense if g is a cartoon function ?

For any $f \in C^2(\overline{\Omega})$

$$J(f) := \left\| \sqrt{|\det d^2 f|} \right\|_{L^{\frac{2}{3}}} .$$

If g is a cartoon function with discontinuity set E we define

$$J(g) := \lim_{\delta \to 0} J(g * \varphi_\delta),$$

where $\varphi_\delta := \delta^{-2}\varphi(\delta^{-1}\cdot)$ is a mollifier.

Proposition

$$J(g)^{\frac{2}{3}} = \left\| \sqrt{|\det d^2 g|} \right\|_{L^{\frac{2}{3}}(\Omega \setminus E)}^{\frac{2}{3}} + C(\varphi) \left\| [g] \sqrt{|\kappa|} \right\|_{L^{\frac{2}{3}}(E)}^{\frac{2}{3}}$$

where $[g]$ is the jump of g, and κ the curvature of E.

Compare with

$$TV(g) = \|\nabla g\|_{L^1(\Omega \setminus E)} + \|[g]\|_{L^1(E)}.$$

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Piecewise constant functions

$$TV(g) = \int_\Gamma |[g]|$$

$$J(g)^{\frac{2}{3}} = \int_\Gamma |[g]|^{\frac{2}{3}} |\kappa|^{\frac{1}{3}}$$

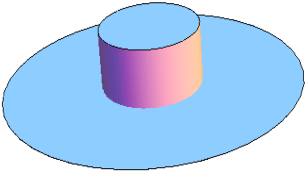

Figure : $TV(g) \simeq J(g)$

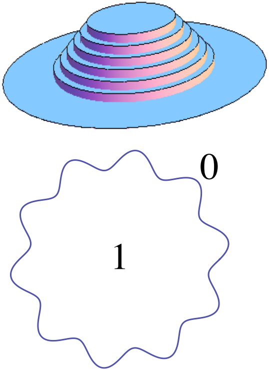

Figure : $TV(g) \ll J(g)$

A model inspired by the virtual cortex first layer V1

- Eye sees an intensity map $u : \mathbb{R}^2 \to \mathbb{R}$.
- Neurons of V1 are specialized in the detection of a stimulus at position $x \in \mathbb{R}^2$ in direction $\theta \in \mathbb{S}^1$.
- Neuron at (x, θ) interacts with $(x, \theta + \delta\theta)$ and $(x + \theta\delta h, \theta)$.

Denote the neural state by $U : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{R}$. We propose a model where the brain constructs U by minimizing

$$\iint |\langle \nabla_x U, \theta \rangle|^p + |\partial_\theta U|^p \, dx \, d\theta.$$

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Question: Non-asymptotic approximation results, based on anisotropic triangulations, for those u which admit a lift U of finite energy with $p = 3/2$?

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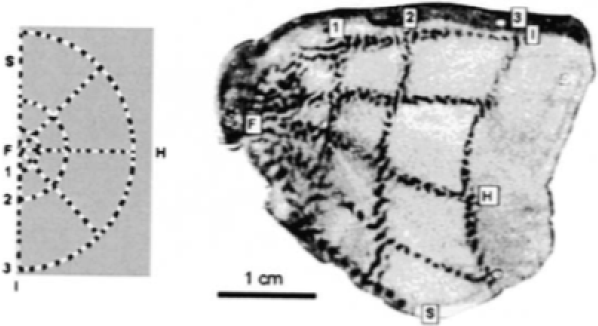

Figure : Bosking et al (97), Petitot (99)

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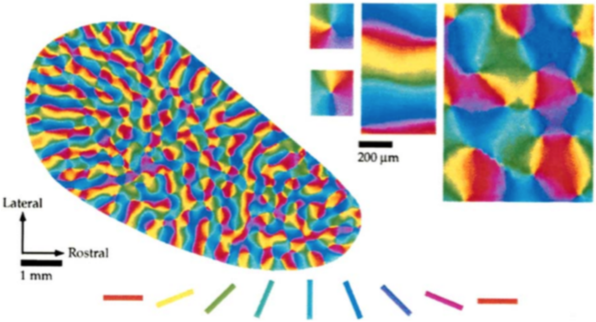

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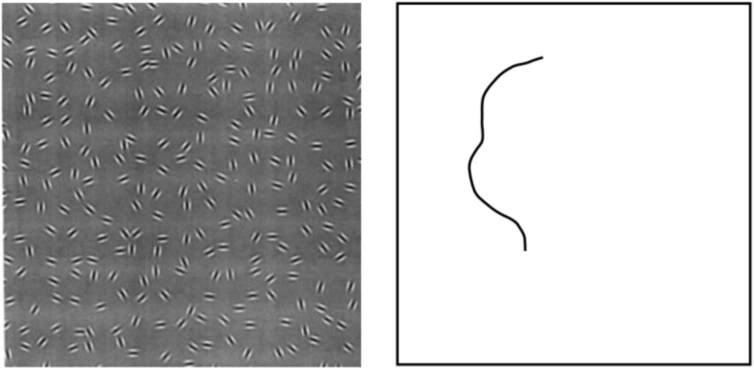

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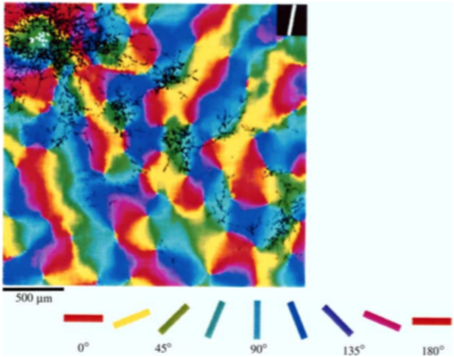

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Conclusion and perspectives

Conclusion:

- A result of algorithmic geometry for QuasiAcute triangulations.
- Sharp asymptotic estimates for $\mathbb{P}_m$ interpolation error on optimal mesh, for H^1 but also L^p and $W^{1,p}$ norms.
- Some quantities remain meaningful for cartoon functions. e.g. $J(f) = \|\sqrt{\det(d^2 f)}\|_{L^{\frac{2}{3}}}$.

Perspectives:

- Quasi-Acute meshes in 3D ?
- Anisotropic Function spaces.
- Non asymptotic error estimates.

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