Semi-Discrete Optimal Transport and its applications

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Semi-Discrete Optimal Transport

Applications with the standard quadratic cost

- Image quantization
- Euler equations of incompressible fluids
- Evolution PDEs via the JKO flow
- Optimization under the constraint of convexity

Applications involving a modified transport cost

- Reflector design
- Unbalanced Optimal Transport
- Optimal mining

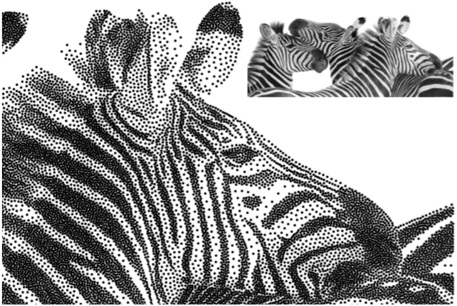
Semi-Discrete Optimal Transport

Image quantization

- Use semi-Discrete Optimal transport to measure the distance from a collection of Dirac masses to a density.

$$W\left(\sum_{1\leq i\leq n} \mu_i \delta_{x_i},\ \rho\,\mathrm{Leb}\right)$$

- Goes, Breeden, Ostromoukhov, Desbrun (2012) take an image intensity for ρ, fix identical weights $\mu_i = \mu_*$, and optimize over the positions x_i, $1 \leq i \leq n$.



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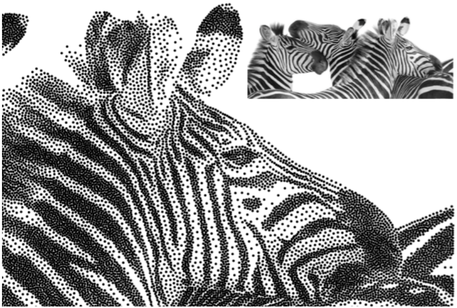

Euler equations of incompressible fluids

- Incompressible fluid in domain X, $|X| = 1$. No viscosity.

$$\partial_t v + (v \cdot \nabla) v = \nabla p, \qquad \operatorname{div} v = 0.$$

- Observe Initial $x_{0,i}$ and final $x_{T,i}$ positions of N particles.
- Reconstruct intermediate positions $x_{t,i}$ by minimizing

$$\underbrace{\frac{T}{N} \sum_{0 \leq t < T} \sum_{1 \leq i \leq N} |x_{t,i} - x_{t,i+1}|^2}_{\text{Kinetic energy}} + \lambda \underbrace{\sum_{1 \leq t < T} W\left(\frac{1}{N} \sum_{1 \leq i \leq N} \delta_{x_{t,i}}, \mathrm{Leb}_X\right)}_{\text{Penalization of compression}}$$

- Motivation: geodesics on $\mathbb{S}\mathrm{Diff} = \{s \in C^\infty(X, X);\ \det \nabla s = 1\}$, w.r.t the L^2 metric, obey Euler equations, in Lagrangian coordinates. Arnold (66)
- Convergence: as $N, T, \lambda \to \infty$ suitably, minimizers converge to a Generalized flow (Brenier 89), i.e. a measure on $C^0([0,1], X)$, solving a relaxation of Euler equations.

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$$\ddot{s}_t = \nabla p \circ s_t. \tag{1}$$

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Evolution PDEs via the JKO scheme

- JKO $\Leftrightarrow$ implicit gradient descent w.r.t Wasserstein metric

$$\mu_{n+1} := \operatorname*{argmin}_{\mu \in \mathrm{Prob}(\Omega)} \frac{1}{2\tau} W(\mu_n, \mu) + \mathcal{F}(\mu).$$

- Formally converges to an evolution PDE as $\tau \to 0$

$$\mathcal{F}(\rho\, \mathrm{Leb}_\Omega) = \int_\Omega F(x, \rho(x))dx \quad \Rightarrow \quad \partial_t \rho = \mathrm{div}(\rho \nabla \partial_\rho F(\cdot, \rho)).$$

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Figure : Simulation of crowd motion under a congestion constraint. Benamou, Carlier, Mérigot, Oudet (2014).

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Optimization under the constraint of convexity

Finite set $X \subset \mathbb{R}^d$, map $u : X \to \mathbb{R}$.

- u has a convex extension iff its subgradients are non-empty

$$\partial u(x) := \{p \in \mathbb{R}^d; \forall y \in X, u(x) + \langle p, y - x\rangle \leq u(y)\}$$

- $\partial u(x) = \mathrm{Lag}_\psi(x)$ with $\psi(x) = \frac{1}{2}|x|^2 - u(x)$.
- u has a strictly convex extension with gradients in Ω, where Ω is a convex bounded domain, iff

$$\infty > -\sum_{x \in X} \ln |\partial u(x) \cap \Omega| = \mathrm{Ent}(\nabla u^*_\# \mathrm{Leb}_\Omega)$$

  This barrier function is convex in u.
- Optimal transport interpretation: the gradient of the Legendre Fenchel conjugate u^* defines an optimal transport $\nabla u^* : (\Omega, \mathrm{Leb}_\Omega) \to (X, \nabla u^*_\# \mathrm{Leb}_\Omega)$, product $\to$ customer. We penalize the image measure entropy.

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The Monopolist problem (Rochet, Chone, 98)

- Monopolist produces goods $q \in \mathbb{R}_+^d$ at cost $c(q)$, $c(0) = 0$.
- Monopolist unilaterally sets prices $\pi(q)$, except $\pi(0) = 0$.
- Customer of type $z \in \mathbb{R}_+^d$ maximizes his net utility

$$U(z) = \sup_{q \in \mathbb{R}_+^d} \underbrace{\langle q, z\rangle}_{\text{Utility}} - \underbrace{\pi(q)}_{\text{Price}}$$

Customer z buys product $q(z)$, monopolist gains

$$\pi(q(z)) - c(q(z)) = \langle q(z), z\rangle - U(z) - c(q(z)).$$

- Customer density μ on $\mathbb{R}_+^d$, is known to the monopolist

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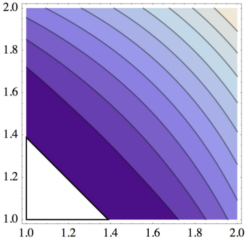

Figure : Customer density μ uniform on $\Omega = [1,2]^2$, monopolist production cost $c(q) = \frac{1}{2}|q|^2$. Left: solution u. Center: product sales density $\nabla u \# \mu$. Right: $\Omega_k := \{\mathrm{rank}(\nabla^2 u) = k\}$.

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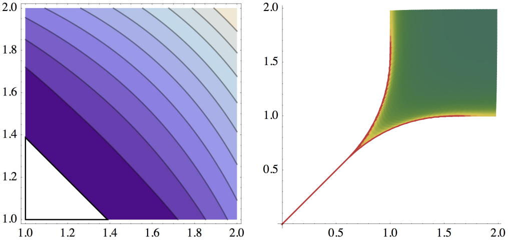

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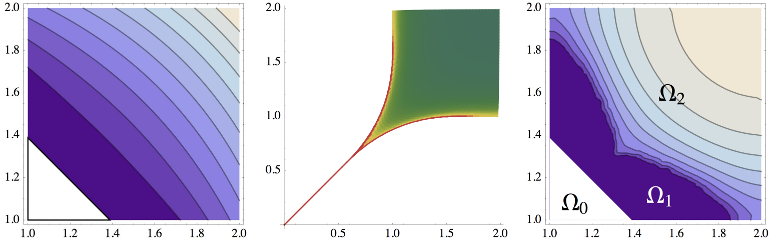

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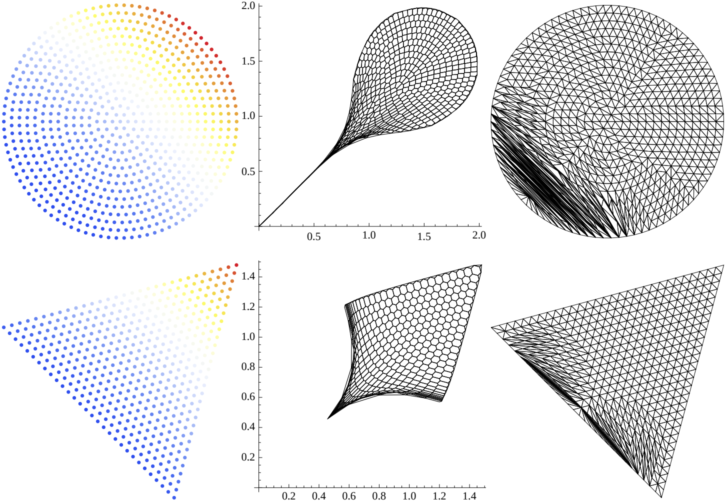

Figure : Customer density μ uniform on disk $D((3/2,3/2),1/2)$ or triangle centered at $(3/2,3/2)$. Left: solution u. Center: product line (=Subgradient cells=Laguerre diagram). Right: dual triangulation. No one buys the null product in the triangle case.

Joint work with Q. Merigot

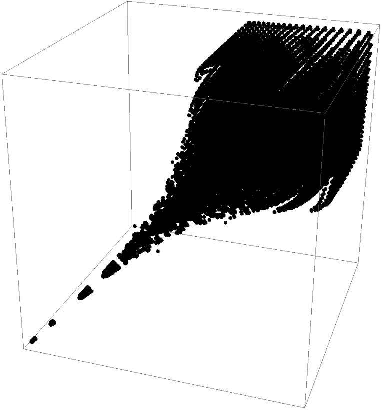

Figure : Product line for a uniform density of customers on $[1,2]^3$

$$\mathrm{Profit} = \max_{u \text{ convex}} \int_{\mathbb{R}^d} \langle \nabla u(z), z \rangle - u(z) - c(\nabla u(z)) \, d\mu(z).$$

Minimizing over convex bodies

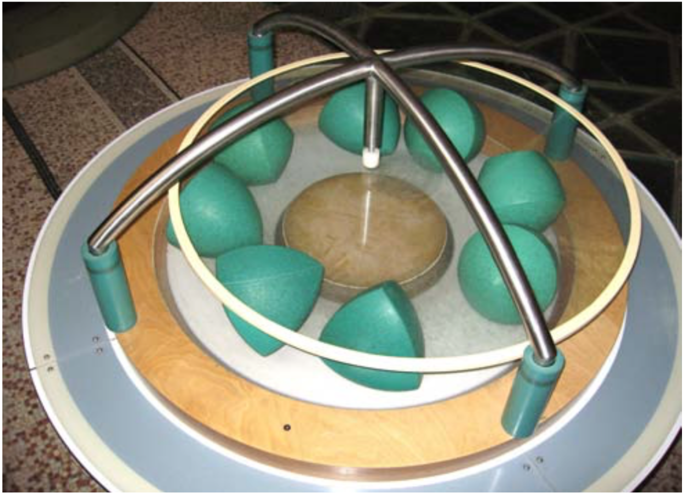

Figure : Several bodies of width 1. The glass surface remains flat and at height 1 (Palais de la découverte, Paris).

Minimizing over convex bodies

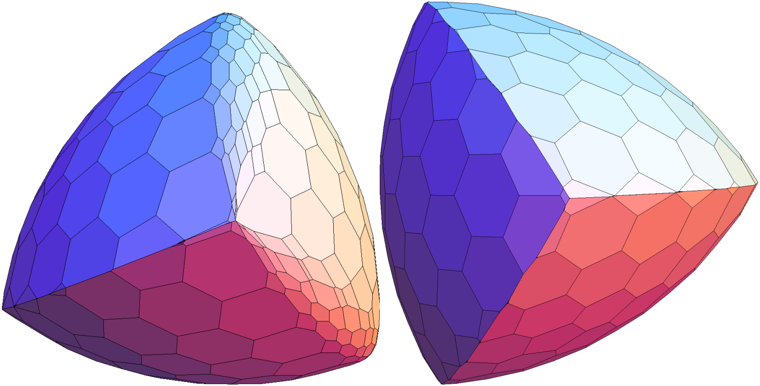

Figure : Numerical minimization of volume among convex bodies of width 1. Our experiments support Meissner's conjecture.

Reflector design

- Objective: Project a point-source light onto a uniform density, a non-blinding headlight, a logo...
- Masks loose energy $\Rightarrow$ transport instead light by reflection
- Light rays only follow shortest paths $\Rightarrow$ optimal transport.
- Non-quadratic cost function e.g. $c(x,y) = -\ln(1-\langle x,y\rangle)$, for $x, y \in \mathbb{S}^2$. (concave reflector and point source.)
- Machado, Merigot, Thibert (2015) implement CGAL® exact geometric predicates for these Laguerre diagrams.

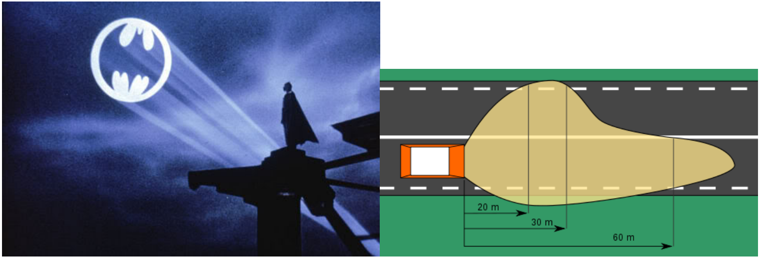

Figure : by Warner®, Mercedes®, Benamou, Merigot, Thibert (15)

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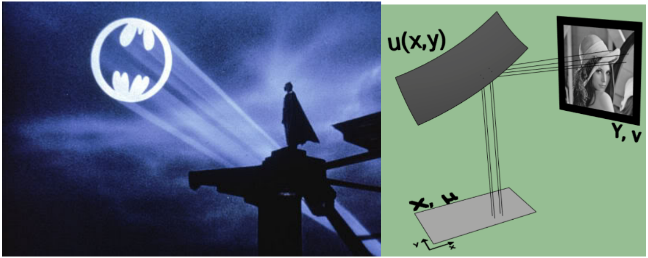

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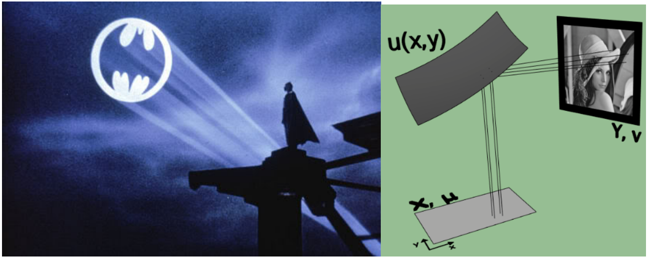

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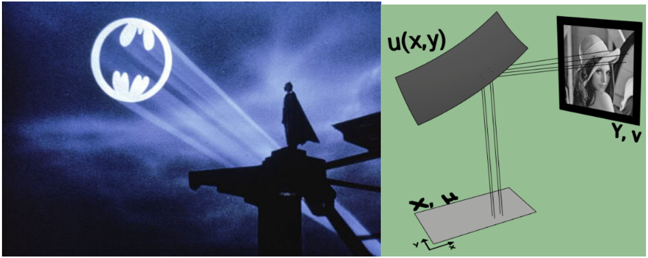

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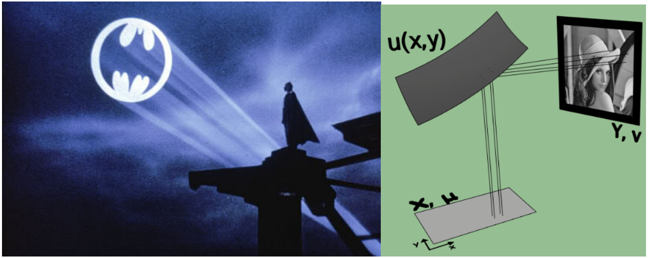

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Optimal transport with source terms

$$W(\rho_0, \rho_1) = \inf_{\rho, m} \int_0^1 \int_\Omega \frac{|m|^2}{\rho} \quad \text{s.t.} \quad \begin{cases} \partial_t \rho + \operatorname{div} m = 0 \\ \rho(0) = \rho_0, \rho(0) = \rho_1 \end{cases}$$

- Optimal transport cost, with $c(x, y) = \frac{1}{2}|x - y|^2$, is the kinetic energy required to move a pressureless fluid from density ρ_0 to ρ_1. Benamou, Brenier (2002).
- ρ: fluid density. m: fluid momentum.
- Possibility to add a source term σ.
  Peyre et al, Savaré et al, Kondratyev et al (2015)
- $\tilde{W}$ defines a distance between measures of distinct masses.
- Static semi-discrete formulation requires constructing cells

$$\operatorname{Lag}_\psi(x) := \left\{ p \in \mathbb{R}^d; \forall y \in X, \frac{\cos_+ |p - x|}{1 - \psi(x)} \geq \frac{\cos_+ |p - y|}{1 - \psi(y)} \right\}$$

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Optimal mining

- Open pit mining requires excavating topsoil above ore.
- Ore at x is extracted if it pays for removal of cone $C(x)$ above, left in place otherwise. (Risk of landslide)
- Optimal pit given by an Optimal Transport problem, from ore to topsoil distributions, and with cost

$$c(x,y) = \begin{cases} 0 & \text{if } y \in C(x), \\ +\infty & \text{otherwise,} \end{cases}$$

+ two sinks. Ekeland, Queyranne (14)

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