Projection of a Smooth Space Curve:

Numeric and Certified Topology Computation

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Joint work with

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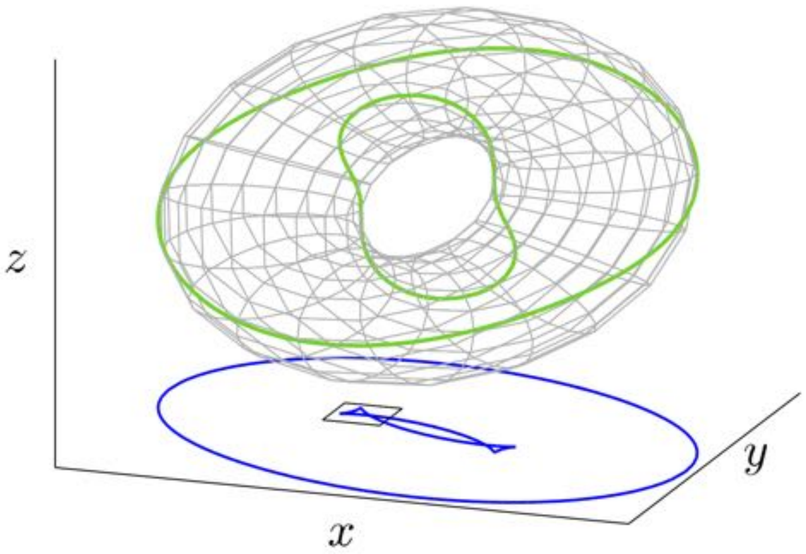

Loria
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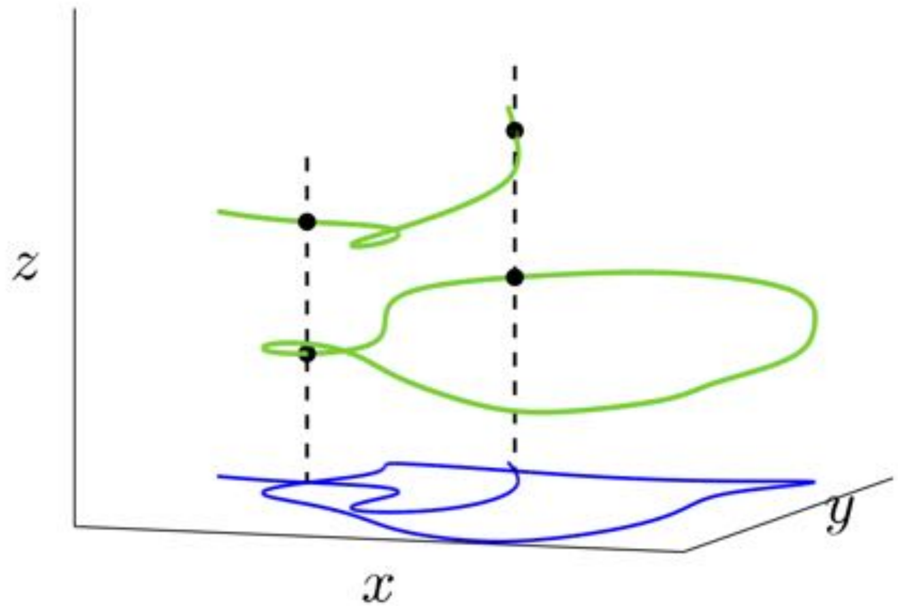
Inria
informatics mathematics

Projection and apparent contour

3D curve = intersection of 2 implicit surfaces

$$\mathcal{C} : \begin{cases} p(x,y,z) = 0 \\ q(x,y,z) = 0 \end{cases}, (x,y,z) \in \mathbb{R}^3$$

$$\mathcal{B} = \pi_{(x,y)}(\mathcal{C})$$



Projection and apparent contour

3D curve = intersection of 2 implicit surfaces

$$\mathcal{C} : \begin{cases} p(x,y,z) = 0 \\ \frac{\partial p}{\partial z}(x,y,z) = 0 \end{cases} , (x,y,z) \in \mathbb{R}^3$$

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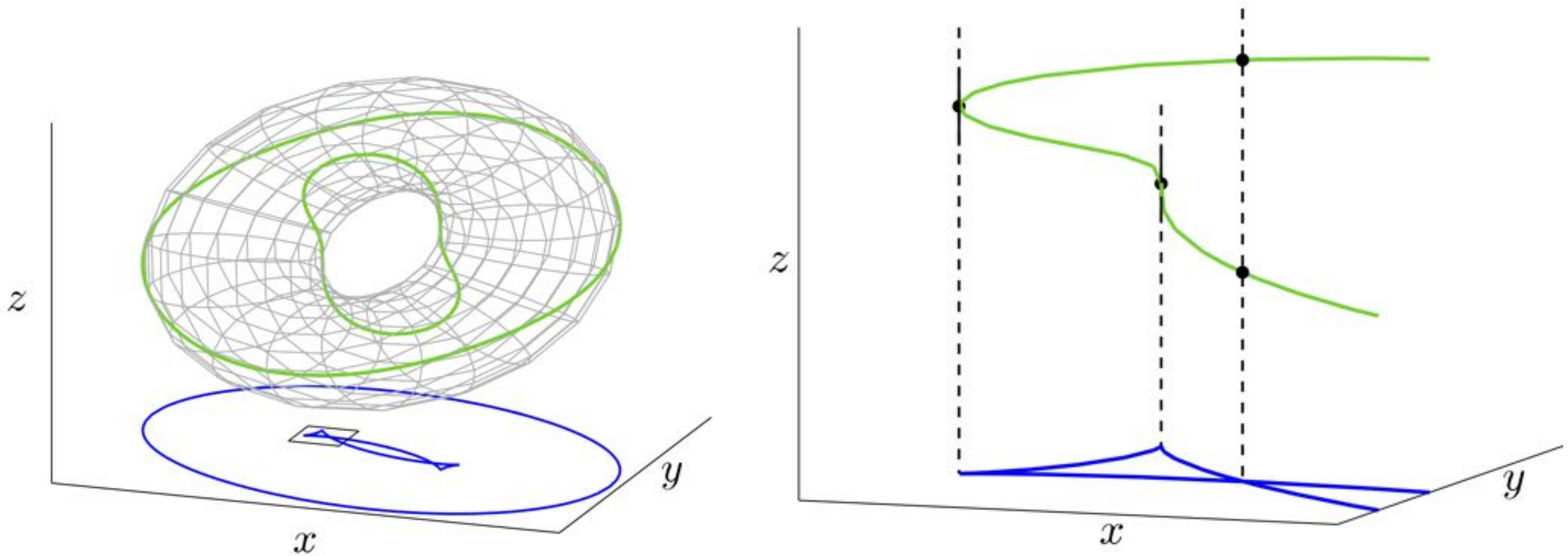


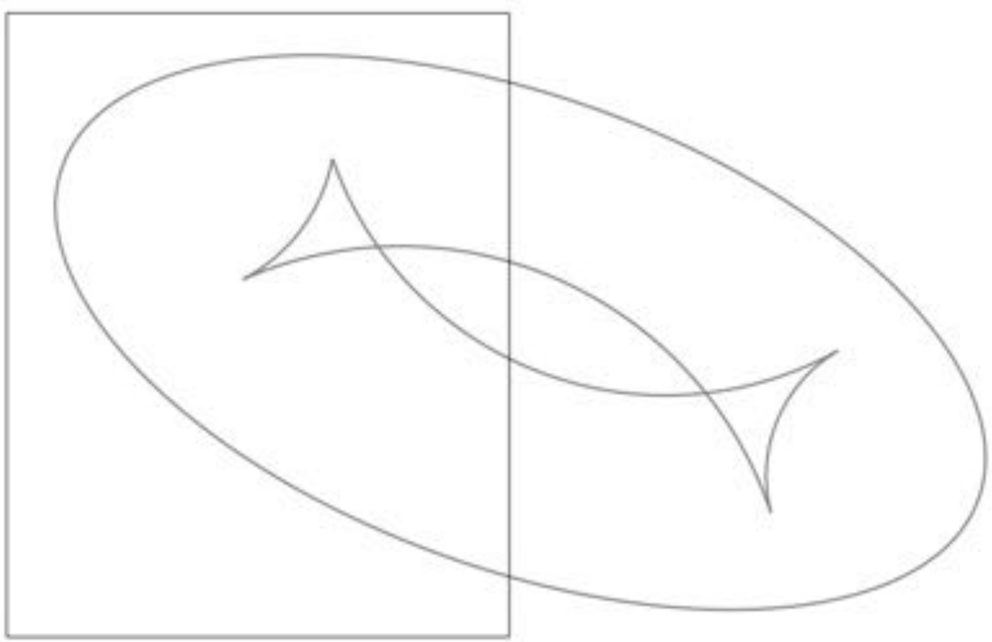
Topology of a plane curve

$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | f(x, y) = 0\}$

Singularities:

$$\{(x, y) \in \mathbb{R}^2 | f(x, y) = \frac{\partial f}{\partial x}(x, y) = \frac{\partial f}{\partial y}(x, y) = 0\}$$

- Restrict to box B_0



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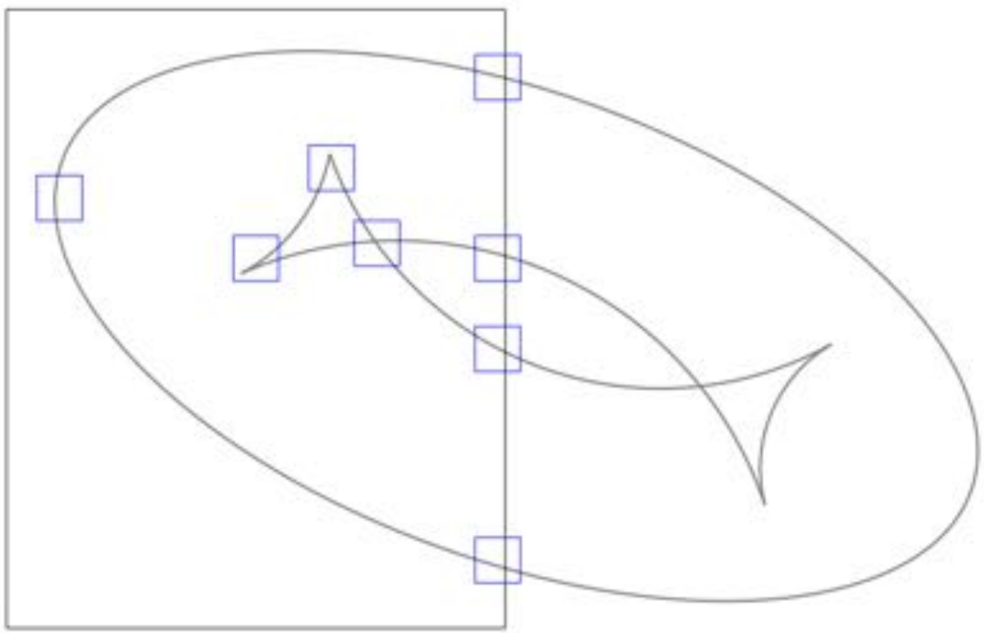
- Restrict to box B_0
- Isolate in boxes:
  - singularities
  - boundary points
  - at least 1 point per cc



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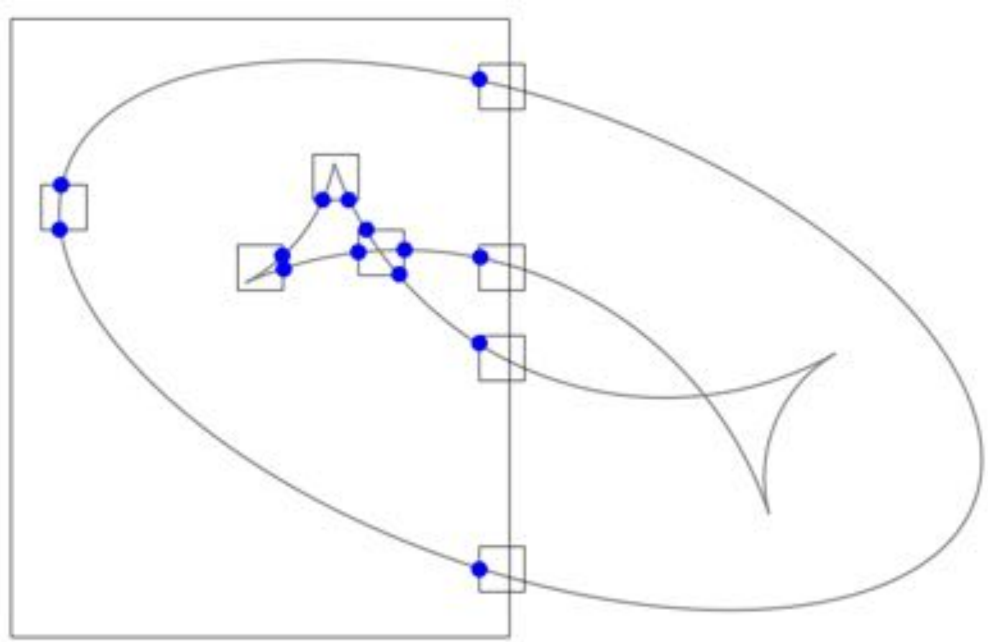

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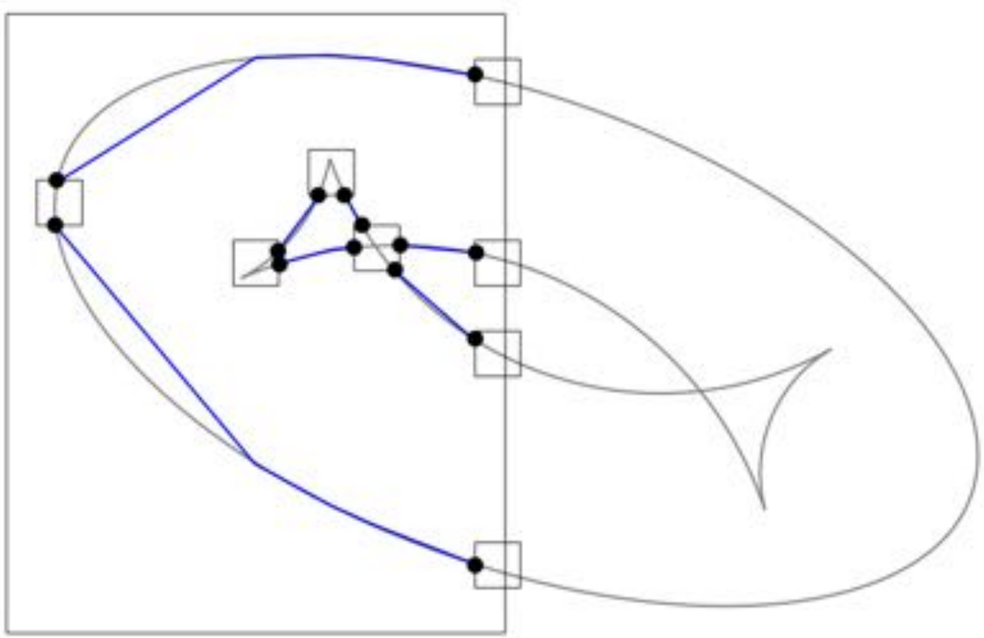

- Restrict to box B_0
- Isolate in boxes:
  - singularities
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- Compute local topology at singularities
- Connect boxes : Graph

Symbolic tool box

- Gröbner basis
- Triangular decomposition
- Rational univariate representation
- ...

+ Handle any types of singularities
+ Bit complexity analysis

– Global analysis
– High complexity
– Restricted to polynomial functions
– Generic case is typically the worst case

Numerical tool box

- Subdivision
- Homotopy

+ Local: analysis restricted to a box
+ Adaptative: running time sensitive to the local geometry
+ Fast limited precision computation
+ Certification via interval analysis
+ Not restricted to polynomials: only evaluation required

– Difficult to analyse the complexity
– Need generic assumptions: regular solutions

Example: counting solutions of $f(x) = 0$

- Symbolic: f polynomial, squarefree part $= f/\gcd(f, f')$
- Numeric:
  - Newton iteration: $x_{n+1} = x_n - f(x_n)/f'(x_n)$
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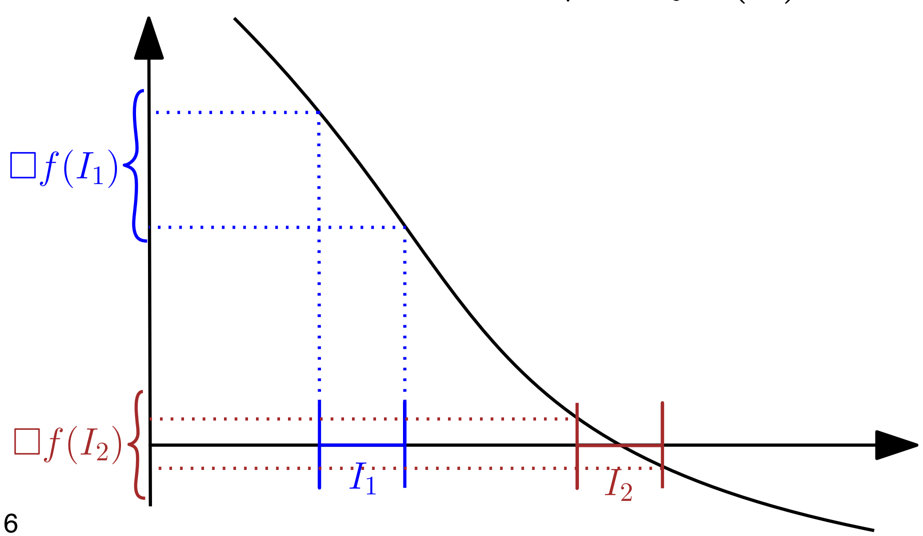
    Subdivide interval I_0 until
    - $0 \notin \Box f(I) \longrightarrow$ no solution in I
    - or $0 \notin \Box f'(I) \longrightarrow$ check the sign of f at endpoints



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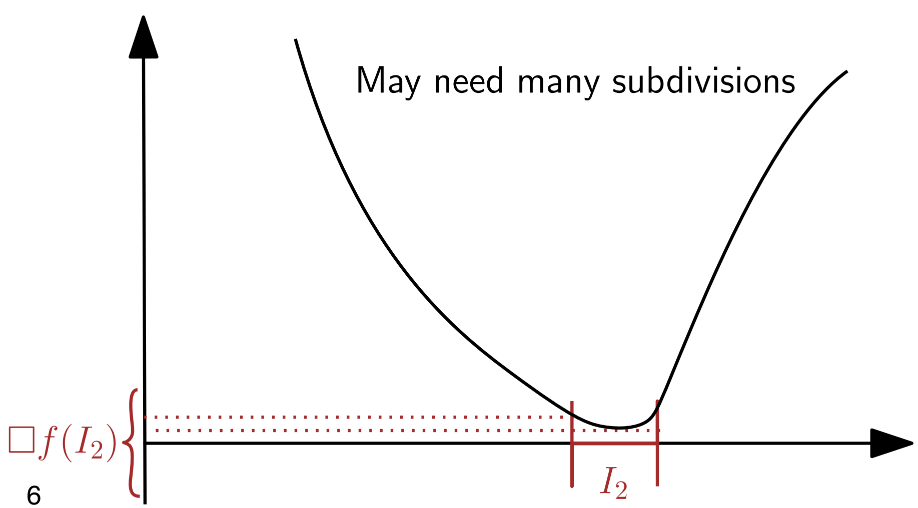

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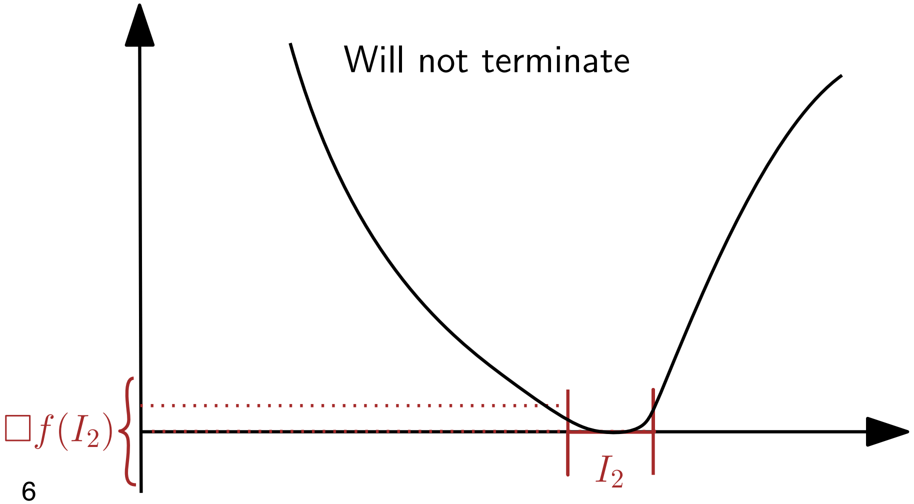


Use instead $g = f + \lambda f'$, $\lambda \in \mathbb{R}$

- works only for solutions of multiplicity 2 $(g' = f' + \lambda f'' \neq 0\ )$
- adds spurious solutions

Interval analysis

Arithmetic operations

$[a, b] \oplus [c, d] = [a + c, b + d]$

$[a, b] \otimes [c, d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)]$

Interval function extension of $f : \mathbb{R} \longrightarrow \mathbb{R}$

- $\Box f(I) \supseteq \{f(x)|x \in I\}$
- convergence: $w(\Box f(I)) \longrightarrow 0$ as $w(I = [a, b]) = b - a \longrightarrow 0$

Examples

- f polynomial, use interval arithmetic operations
- Mean value evaluation $\Box f(I) := f(mid(I)) + \Box f'(I)(I - mid(I))$

Exclusion criterion:

$0 \notin \Box f(I) \Longrightarrow f$ has no solution in I

... But $0 \in \Box f(I)$ does NOT imply that f has no solution in I

Newton/Krawczyk operator

$F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$F(X) = 0$: system of n equations, n unknowns.

Assume the solutions are regular: the determinant of the jacobian J_F does not vanish at the solutions.

Interval Newton operator: $m \in X \subset \mathbb{R}^n, N(X) = m - J_F^{-1}(m)\Box F(X)$

Krawczyk operator = mean value evaluation of N

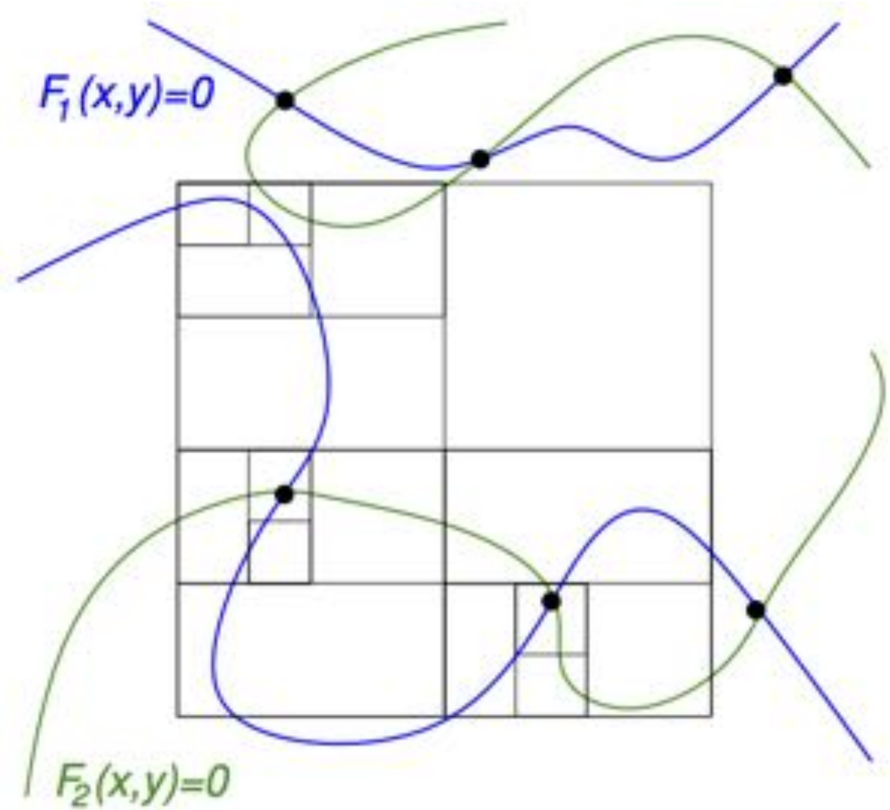
$$K(X) = N(m) + \Box J_N(X)(X - m)$$

Lemma.

- $K(X) \subset X \Longrightarrow \exists!$ solution in X
- $K(X) \cap X = \emptyset \Longrightarrow$ no solution in X
- Quadratic convergence

- Neumaier, Interval methods for systems of equations, 1990
- Dedieu, Points fixes, zeros et la methode de Newton, 2006
- Rump, Verification methods: Rigorous results using floating point arithmetic, Acta Numerica, 2010

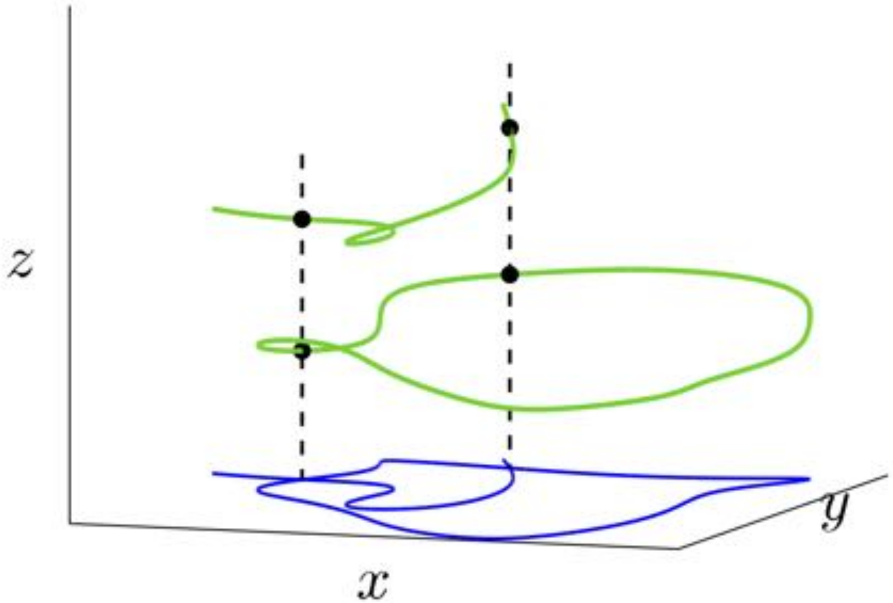
Subdivision algorithm

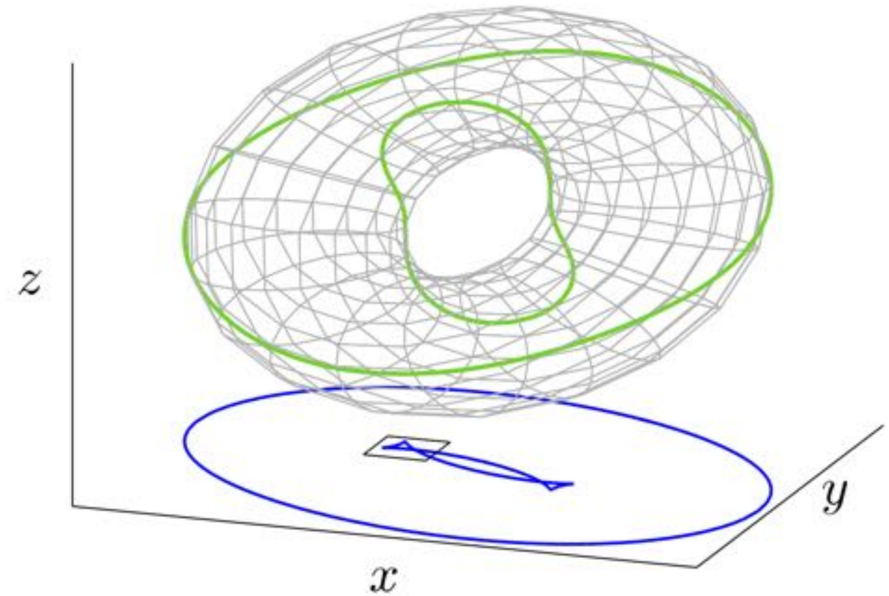


Input: $F : \mathbb{R}^n \to \mathbb{R}^n$, X_0 box of $\mathbb{R}^n$
Output: A list R of boxes containing solutions in X_0 of $F = 0$
$L := \{X_0\}$
Repeat:
  $X := L.pop$
  If $0 \in F(X)$ **then**
    If $K_F(X) \subset Int(X)$ **then**
      insert X in R
    Else If $K_F(X) \cap X \neq \emptyset$ **then**
      bisect X and insert its sub-boxes in L
    End if
  End if
Until $L = \emptyset$
Return R

WARNING:
- Square system: as many equations as unknowns
- Regular solutions: $\det(J_F(s)) \neq 0$

Our problem: Isolate singularities





Projection of a 3D smooth curve: Generic singularities are Nodes = transverse intersection of 2 branches

Apparent contour: Generic singularities are Nodes and Cusps

Subresultant approach in 2D

$\mathcal{B} = \{(x,y) \in \mathbb{R}^2 | r(x,y) = 0\}$, where $r(x,y) = Res(p, p_z, z)(x,y)$

Singularities of $\mathcal{B}$ are the solutions of

$$r(x,y) = \frac{\partial r}{\partial x}(x,y) = \frac{\partial r}{\partial y}(x,y) = 0$$

- Over-determined
- Cusps are solutions of multiplicity 2

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Let s_{10}, s_{11}, s_{22} be the coefficients of the subresultant sequence of p and p_z wrt z

$$(\mathcal{S}_2) \quad s_{10}(x,y) = s_{11}(x,y) = 0 \text{ and } s_{22}(x,y) \neq 0$$

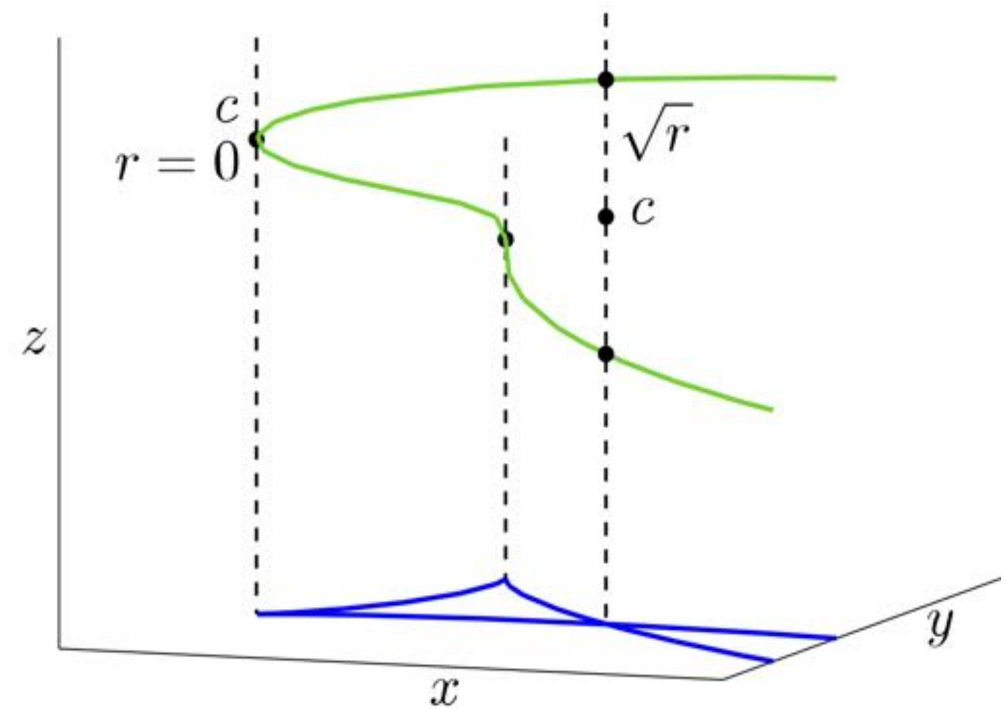
Lemma.

- Square system
- Nodes and cusps are regular solutions

[IMP15] R. Imbach, G. Moroz, and M. Pouget. Numeric certified algorithm for the topology of resultant and discriminant curves. Research Report RR-8653, Inria, April 2015.

4D approach

$\mathcal{C} = \{(x, y, z) \in \mathbb{R}^3 | p(x, y, z) = p_z(x, y, z) = 0\}$
$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | \exists z \in \mathbb{R} \text{ s.t. } (x, y, z) \in \mathcal{C}\}$



- Node: $(x, y, z_1), (x, y, z_2) \in \mathcal{C}$, with $z_1 \neq z_2$
- Cusp: $(x, y, z_1), (x, y, z_2) \in \mathcal{C}$, with $z_1 = z_2$

Set $z_1 = c - \sqrt{r}, z_2 = c + \sqrt{r}$, with c center and $\sqrt{r}$ radius of $[z_1 z_2]$

4D approach

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$$(\mathcal{S}_4) \begin{cases} \frac{1}{2}(p(x,y,c+\sqrt{r}) & + & p(x,y,c-\sqrt{r})) & = 0 \\ \frac{1}{2\sqrt{r}}(p(x,y,c+\sqrt{r}) & - & p(x,y,c-\sqrt{r})) & = 0 \\ \frac{1}{2}(p_z(x,y,c+\sqrt{r}) & + & p_z(x,y,c-\sqrt{r})) & = 0 \\ \frac{1}{2\sqrt{r}}(p_z(x,y,c+\sqrt{r}) & - & p_z(x,y,c-\sqrt{r})) & = 0 \end{cases}$$

Lemma. $\mathcal{S}_4$ is regular, its solutions project to cusps and nodes of $\mathcal{B}$

R. Imbach, G. Moroz, M. Pouget. Numeric and Certified Isolation of the Singularities of the Projection of a Smooth Space Curve, MACIS 2015

- Node: $(x,y,z_1),(x,y,z_2) \in \mathcal{C}$, with $z_1 \neq z_2$
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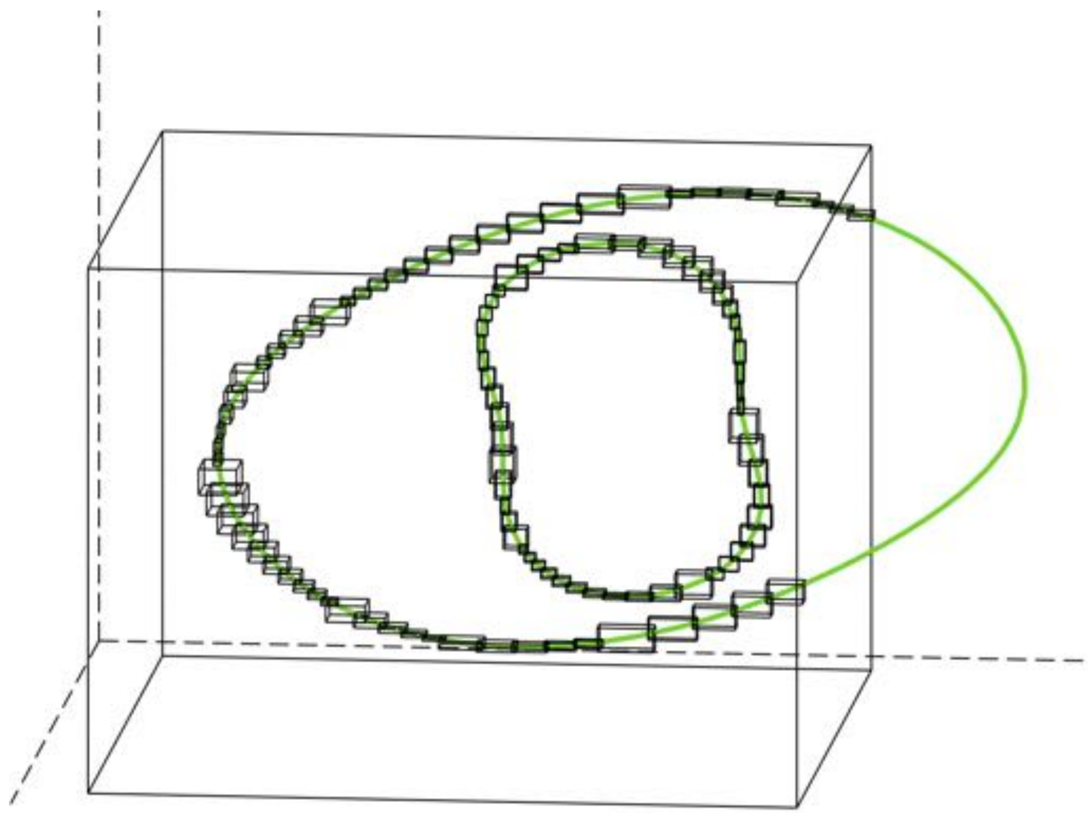
Certified numerical tracking in 3D

- Enclose $\mathcal{C} \subset \cup C_k$, with $C_k = (x_k, y_k, z_k)$ 3D box
  - More efficient than a classic 3D subdivision
  - Correct topology
  - Certification via a parametric interval Krawczyk test

[MGGJ13] B. Martin, A. Goldsztejn, L. Granvilliers, and C. Jermann. Certified parallelotope continuation for one-manifolds. SIAM Journal on Numerical Analysis, 2013.



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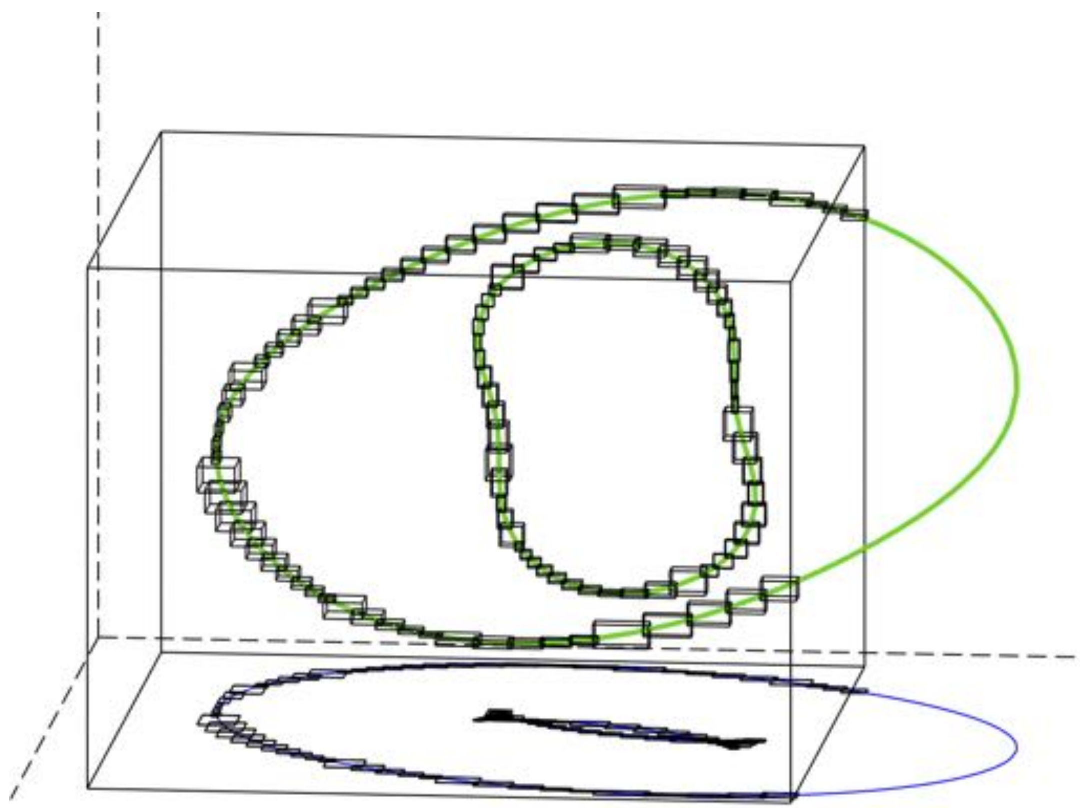

Certified numerical tracking in 3D

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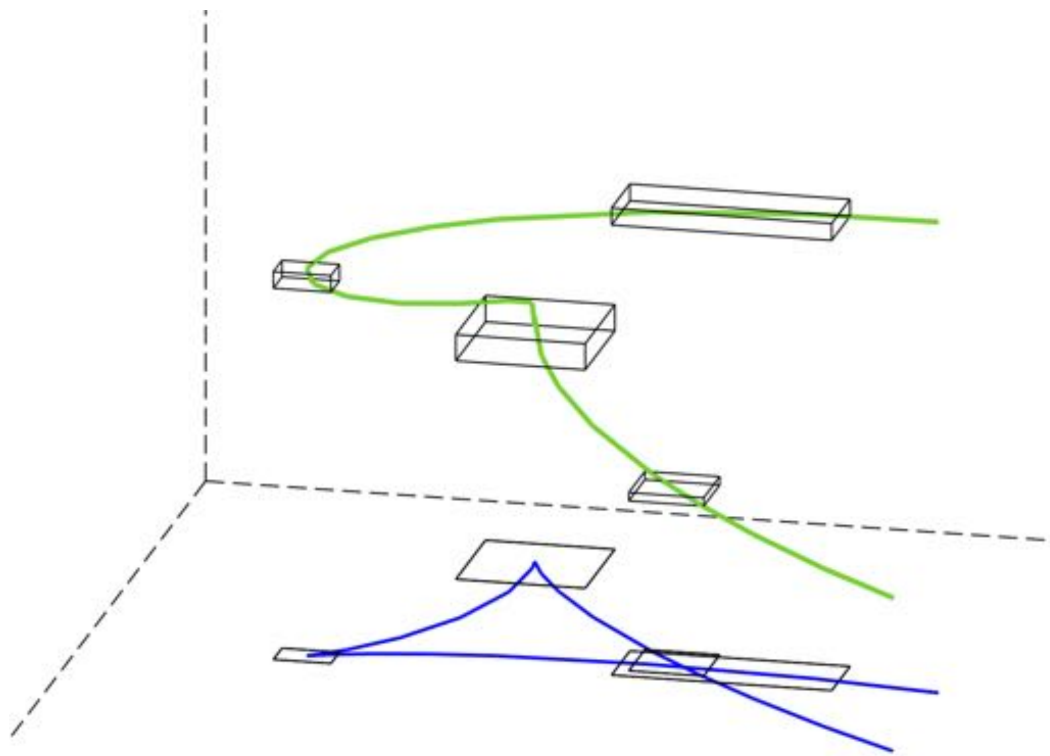
- Enclose $\pi_{(x,y)}(\mathcal{C}) = \mathcal{B} \subset \cup \pi_{(x,y)}(C_k)$
- Restrict the 4D solving domain of $\mathcal{S}_4$
  - Cusp in $\mathcal{B}_k \longleftrightarrow$ sol. in $(x_k, y_k, z_k, [0(\frac{w(z_k)}{2}])^2])$
  - Node in $\mathcal{B}_{ij} = \mathcal{B}_i \cap \mathcal{B}_j \neq \emptyset \longleftrightarrow$ sol. in $(x_{ij}, y_{ij}, \frac{z_i+z_j}{2}, [0, (\frac{z_i-z_j}{2})^2])$



Experiments: isolation of singularities

Degree of $p(x,y,z)$	RSCube $\mathbb{R}^2$	$\mathcal{S}_2$ (Sub-resultant) in $[-1,1]^2$	$\mathcal{S}_4$ in $[-1,1]^2 \times \mathbb{R} \times \mathbb{R}^+$	$\mathcal{S}_4$ with curve tracking
5	3.1	0.05	24.8	1.25
6	32	0.50	8.40	2.36
7	254	4.44	43.8	4.13
8	1898	37.9	70.2	5.91
9	9346	23.1	45.6	5.30

Average running times in seconds for 5 random dense polynomials of degree d, bitsize 8

- Symbolic method becomes intractable (RSCube via triangular decomposition by F. Rouillier)
- Subdivision: working in 4D is more expensive than in 2D
- Subdivision: tracking the curve is efficient

template

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