Dipartimento di Scienze Matematiche G. L. Lagrange

Reduced order methods for parametric optimal control problems: an overview and diverse applications

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Outline

1. Problem Formulation
2. Space-Time POD
3. Space-Time Greedy
4. UQ Applications

Collaborators

- Prof. Gianluigi Rozza and Dr. Francesco Ballarin
- Giuseppe Carere, Prof. Rob Stevenson and Fabio Zoccolan

Tools

- *multiphenics (https://mathlab.sissa.it/multiphenics)*
- *RBniCS (https://www.rbnicsproject.org/)*

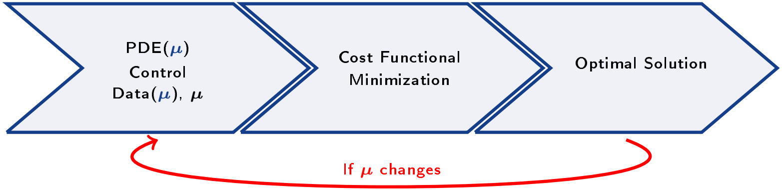
Motivations

Parameteric Optimal Control Problems (OCP($\boldsymbol{\mu}$)s) are

- based on *data*

  (noisy, scattered, difficult to interpret...)
- related several simulations for different values of physical and/or geometrical parameter $\boldsymbol{\mu}$ (**uncertainty quantification**, **parameter estimation problems**...)

ROM: **fast** and **reliable** tool to solve several parametric instances;
OCP($\boldsymbol{\mu}$): classical mathematical tool to **add data information** in the model.



Problem Formulation

Continuous Problem Formulation

Given $\boldsymbol{\mu} \in \mathcal{P} \subset \mathbb{R}^d$, find $(y(\boldsymbol{\mu}), u(\boldsymbol{\mu})) \in \mathcal{Y} \times \mathcal{U}$ which solves

$$\min_{(y,u)\in\mathcal{Y}\times\mathcal{U}} J(y,u;\boldsymbol{\mu}) := \min_{(y,u)\in\mathcal{Y}\times\mathcal{U}} \frac{1}{2}\int_0^T \|y(\boldsymbol{\mu}) - y_d(\boldsymbol{\mu})\|^2_{Y(\Omega_{\mathbf{OBS}})} dt + \int_0^T \frac{\alpha}{2}\|u(\boldsymbol{\mu})\|^2_{U(\Omega_u)} dt$$

$$\text{such that} \qquad \int_0^T \mathcal{B}((y, y_t, u), w; \boldsymbol{\mu}) = \int_0^T \langle f(\boldsymbol{\mu}), w\rangle \; \forall w \in L^2(0,T;Y),$$

where:

- $\Omega \times [0,T]$ is our *space-time domain*,
- $\mathcal{Y}, \mathcal{U}$ are Hilbert Spaces $(H^1(0,T;Y), L^2(0,T;U))$,
- $\Omega_{\text{OBS}} \subseteq \Omega$ is the *observation domain*,
- $\Omega_u \subseteq \bar{\Omega}$ is the *control domain*,
- $y_d(\boldsymbol{\mu}) \in L^2(0,T;Y(\Omega_{\text{OBS}}))$ is our given data in *observation space*,
- $\alpha > 0$ is a penalization parameter.

Continuous Problem Formulation

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Lagrangian Approach [p adjoint variable]

1) define $\mathscr{L}(y,u,p;\boldsymbol{\mu}) = J(y,u;\boldsymbol{\mu}) + \int_0^T \mathcal{B}((y,y_t,u),p;\boldsymbol{\mu}) - \int_0^T \langle f(\boldsymbol{\mu}), p\rangle$

2) given $\boldsymbol{\mu} \in \mathcal{P} \subset \mathbb{R}^p$, find $(y,u,p) \in \mathcal{Y} \times \mathcal{U} \times \mathcal{Y}$

$$\text{s.t.} \quad \begin{cases} \mathcal{D}_y\mathscr{L}(y,u,p;\boldsymbol{\mu})[z] = 0 & \forall z \in \mathcal{Y}, \\ \mathcal{D}_u\mathscr{L}(y,u,p;\boldsymbol{\mu})[v] = 0 & \forall v \in \mathcal{U}, \\ \mathcal{D}_p\mathscr{L}(y,u,p;\boldsymbol{\mu})[\kappa] = 0 & \forall \kappa \in \mathcal{Y}. \end{cases}$$

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Strong Formulation

The optimality system for $\boldsymbol{\mu} \in \mathcal{P}$ find $(y, u, p) \in \mathcal{Y} \times \mathcal{U} \times \mathcal{Y}$ such that

$$
\begin{cases}
y\chi_{\Omega_{\mathbf{obs}}} - p_t + D_a(\boldsymbol{\mu})^* p = y_d & \text{in } \Omega \times [0, T], \\
\alpha u - p = 0 & \text{in } \Omega_u \times (0, T), \\
y_t + D_a(\boldsymbol{\mu}) y - u = f(\boldsymbol{\mu}) & \text{in } \Omega \times (0, T), \\
y(0) = y_0, p(T) = 0 & \text{in } \Omega, \\
\text{boundary conditions} & \text{on } \partial\Omega \times (0, T).
\end{cases}
$$

Existence and uniqueness are proved through Brezzi Theorem (same regularity for y and p)

From steady to time dependent problems

[Negri, Rozza, Manzoni and Quarteroni, Reduced Basis Method for Parametrized Elliptic Optimal Ccontrol Problems, SIAM Journal on Scientific Computing, 2013.]

[Gerner and Veroy. Certified reduced basis methods for parametrized saddle point problems. SIAM Journal on Scientific Computing, 2012.]

Discretized Problem

Truth Problem: $\underbrace{\text{Spatial Discretization (FE) + Time Discretization (Euler)}}_{\mathcal{N}=N_h \cdot N_t}$

One-shot unsteady system

$$\begin{bmatrix} \mathsf{M}_y & 0 & \mathsf{B}^T \\ 0 & \alpha \mathsf{M}_u & -\mathsf{C}^T \\ \mathsf{B} & -\mathsf{C} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{u} \\ \boldsymbol{p} \end{bmatrix} = \begin{bmatrix} \Delta t \mathsf{M}_y \mathsf{y_d} \\ 0 \\ \Delta t \mathsf{f} \end{bmatrix}$$

$$\boldsymbol{y} = [y_1, \dots, y_{N_t}]$$
$$\boldsymbol{u} = [u_1, \dots, u_{N_t}]$$
$$\boldsymbol{p} = [p_1, \dots, p_{N_t}]$$

y, u, p FE discretization (dim $3\mathcal{N}$)

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$$\boldsymbol{p} = [p_1, \dots, p_{N_t}]$$

y, u, p FE discretization (dim $3\mathcal{N}$)

What is the real structure of the matrix?

Discretized Problem

Truth Problem: $\underbrace{\text{Spatial Discretization (FE) + Time Discretization (Euler)}}_{\mathcal{N}=N_h \cdot N_t}$

For example...(state equation)

$$
\mathsf{B} = \begin{bmatrix}
\mathsf{M} + \Delta t \mathsf{D}(\boldsymbol{\mu}) & & & & \\
-\mathsf{M} & \mathsf{M} + \Delta t \mathsf{D}(\boldsymbol{\mu}) & & & \\
& -\mathsf{M} & \mathsf{M} + \Delta t \mathsf{D}(\boldsymbol{\mu}) & & \\
& & \ddots & \ddots & \\
& & & -\mathsf{M} & \mathsf{M} + \Delta t \mathsf{D}(\boldsymbol{\mu})
\end{bmatrix}
$$

$$
\mathsf{C} = \begin{bmatrix}
\mathsf{C}_u(\boldsymbol{\mu}) & & & \\
& \mathsf{C}_u(\boldsymbol{\mu}) & & \\
& & \ddots & \\
& & & \mathsf{C}_u(\boldsymbol{\mu})
\end{bmatrix}
$$

Reduction could be very effective: **space-time** formulation is **unfeasible** for *real-time* applications.

Methodology for OCP(μ)

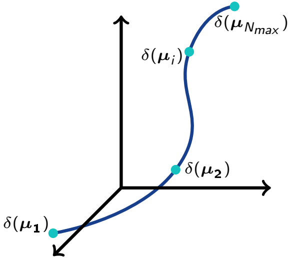
Reduced Order Modelling for OCP(μ)s

Goal: to achieve the **accuracy** of the *truth* solution $\delta^{\mathcal{N}} = y^{\mathcal{N}}, u^{\mathcal{N}}, p^{\mathcal{N}}$ but at greatly **reduced cost** of a **low order model**.

$$\textbf{Strategy}: \ \delta(\boldsymbol{\mu}) \xrightarrow{\text{Space-Time}(\text{dim}=\mathcal{N})} \delta^{\mathcal{N}}(\boldsymbol{\mu}) \xrightarrow[\|\delta(\boldsymbol{\mu})-\delta^{\mathcal{N}}(\boldsymbol{\mu})\|\to 0]{\text{ROM (dim } N)} \delta_N(\boldsymbol{\mu}).$$

- Proper Orthogonal Decomposition (POD):
  - choose $N_{max} \subset \mathcal{D}$ finite,
  - pick $\delta^{\mathcal{N}}(\boldsymbol{\mu}^1), \dots, \delta^{\mathcal{N}}(\boldsymbol{\mu}^{N_{max}})$,
  - solve an eigenvalue problem on $\mathbb{C}_{ij} = (\delta^{\mathcal{N}}(\boldsymbol{\mu}^i)\delta^{\mathcal{N}}(\boldsymbol{\mu}^j))$ for $i, j = 1, \dots, N_{max}$,
  - basis = eigenvectors associated to the largest N eigenvalues (aggragated spaces).

Important: $N \ll \mathcal{N}$



ROMs: Aggregated Spaces

Let us recall the structure of the problem at hand (Saddle Point): $\begin{bmatrix} \mathbb{A}(\boldsymbol{\mu}) & \mathbb{B}^T(\boldsymbol{\mu}) \\ \mathbb{B}(\boldsymbol{\mu}) & 0 \end{bmatrix}$

When is the problem well-posed? For every $\boldsymbol{\mu} \in \mathcal{P}$ it must hold:

1. Continuity of $\mathcal{A}(\cdot, \cdot; \boldsymbol{\mu})$ and $\mathcal{B}(\cdot, \cdot; \boldsymbol{\mu})$ (✓ from space-time),
2. coercivity of $\mathcal{A}(\cdot, \cdot; \boldsymbol{\mu})$ over the kernel of $\mathcal{B}(\cdot, \cdot; \boldsymbol{\mu})$ (✓ from space-time),
3. *inf-sup condition* for $\mathcal{B}(\cdot, \cdot; \boldsymbol{\mu})$, i.e.

$$\beta_N(\boldsymbol{\mu}) = \inf_{p \in Q_N} \sup_{(y,u) \in Y_N \times U_N} \frac{\mathcal{B}((y,u), p; \boldsymbol{\mu})}{\|(y,u)\|_{\mathcal{Y} \times \mathcal{U}} \|p\|_{\mathcal{Q}}} \geq \beta_0 > 0 \qquad (\checkmark \text{ when } Y_N \equiv Q_N).$$

Solution: Aggregated Spaces for State and Adjoint (bigger final dimension):

$$Y_N \equiv Q_N = \text{span}\{y^{\mathcal{N}}(\boldsymbol{\mu}^n), p^{\mathcal{N}}(\boldsymbol{\mu}^n)\}_{n=1}^N \quad \text{and} \quad U_N = \text{span}\{u^{\mathcal{N}}(\boldsymbol{\mu}^n)\}_{n=1}^N$$

POD Application: Coastal Water Height

GOAL: recover parametrized desired height and velocity profiles (bottom, wind...)

Optimization problem - Shallow Waters Equations (SWEs)

Given $\boldsymbol{\mu} \in \mathcal{P} = (10^{-5}, 1.) \times (0.01, 0.5) \times (0.1, 1.)$ find $(\boldsymbol{v}, h, \boldsymbol{u}) \in [H^1(0, T; H^1_0(\Omega))]^2 \times H^1(0, T; L^2(\Omega)) \times L^2(0, T; L^2(\Omega))$ which minimizes

$$\frac{1}{2}\int_0^T \int_{\Omega(\mu_4)} (h - h_d(\mu_3))^2 dxdt + \frac{1}{2}\int_0^T \int_{\Omega(\mu_4)} (\boldsymbol{v} - \boldsymbol{v}_d(\mu_3))^2 dxdt + \frac{\alpha}{2}\int_0^T \int_{\Omega(\mu_4)} \boldsymbol{u}^2 dxdt,$$

constrained to

$$\begin{cases} \boldsymbol{v}_t + \mu_1 \Delta \boldsymbol{v} + \mu_2 (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + g \nabla h - \boldsymbol{u} = 0 & \text{in } \Omega(\mu_4) \times [0, 0.8] \\ h_t + \text{div}(h\boldsymbol{v}) = 0 & \text{in } \Omega(\mu_4) \times [0, 0.8], \\ \boldsymbol{v} = \boldsymbol{v}_0 & \text{on } \Omega(\mu_4) \times \{0\}, \\ h = h_0 & \text{on } \Omega(\mu_4) \times \{0\}, \\ \boldsymbol{v} = 0 & \text{on } \partial\Omega(\mu_4) \times [0, 0.8]. \end{cases}$$

POD Application: Coastal Water Height

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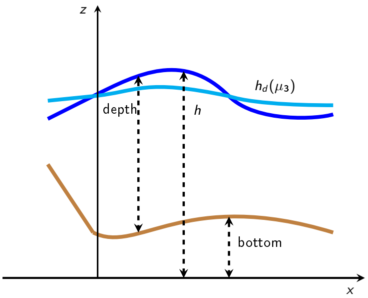


Data

$\mathcal{P} = (10^{-5}, 1.) \times (0.01, 0.5) \times (0.1, 1.) \times (1, 1.5)$, $\mu_1, \mu_2 =$ convection-diffusion, $\mu_3 =$ observation scale factor, $\mu_4 =$ geometry, $\alpha = 10^{-1}$, $\Delta t = 0.1$, $N_{max} = 100$ (Uniform), $N = 30$, $\mathcal{N} = 94016$.

Results

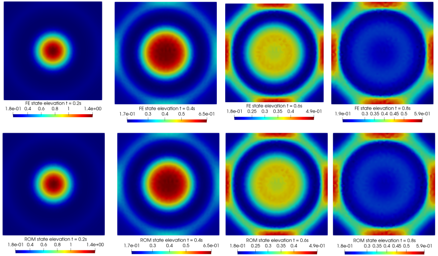


Figure: Space-Time vs ROM t=0.2,0.4,0.6,0.8s for $\mu =$ (0.1, 0.5, 1).

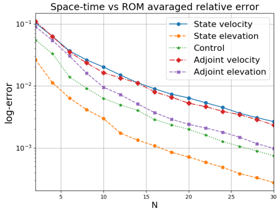


Figure: Space-Time vs ROM relative errors.

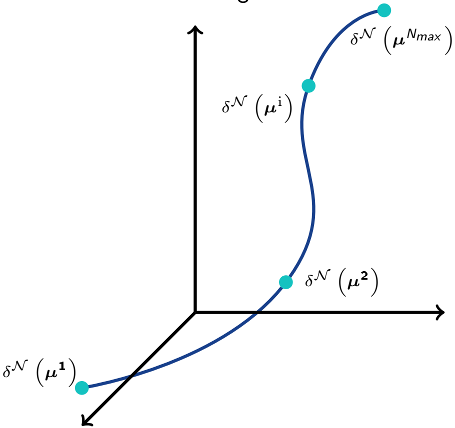
Performances

Errors $\sim 10^{-2}/10^{-3}$, Speedup ~ 30, ROM vs space-time= 270 *vs* 94016 (Aggregated spaces).

[Strazzullo, Ballarin, Rozza, POD-Galerkin Model Order Reduction for Parametrized Nonlinear Time Dependent Optimal Flow Control: an Application to Shallow Water Equations. Journal of Numerical Mathematics, 2022.]
[Ballarin, Rozza, Strazzullo, Space-time POD-Galerkin approach for parametric flow control, Handbook of Numerical Analysis, 2022.]

Space-Time Greedy Strategy

POD–drawback: costly offline phase (N_{max} snapshots).
We need a "smarter" algorithm to be actually useful in the applications.



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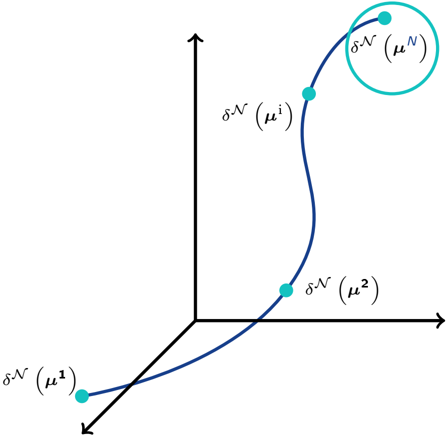

Before starting: the No-Control Framework

For $\boldsymbol{\mu} \in \mathcal{P}$ find $(y, u, p) \in \mathcal{Y} \times \mathcal{U} \times \mathcal{Y}$ such that

$$\text{(POD)} \quad \begin{cases} y\chi_{\Omega_{\mathbf{obs}}} - p_t + D_a(\boldsymbol{\mu})^* p = y_d & \text{in } \Omega \times (0, T), \\ \alpha u - p = 0 & \text{in } \Omega_u \times (0, T), \\ y_t + D_a(\boldsymbol{\mu}) y - u = f(\boldsymbol{\mu}) & \text{in } \Omega \times (0, T), \\ y(0) = y_0, p(T) = 0 & \text{in } \Omega, \\ \text{boundary conditions} & \text{on } \partial\Omega \times (0, T). \end{cases}$$

$\Updownarrow$

$$\text{(RB)} \quad \begin{cases} y\chi_{\Omega_{\mathbf{obs}}} - p_t + D_a(\boldsymbol{\mu})^* p = y_d & \text{in } \Omega \times (0, T), \\ y_t + D_a(\boldsymbol{\mu}) y - \dfrac{1}{\alpha} p\chi_{\Omega_u} = f(\boldsymbol{\mu}) & \text{in } \Omega \times (0, T), \\ y(0) = y_0, p(T) = 0 & \text{in } \Omega, \\ \text{boundary conditions} & \text{on } \partial\Omega \times (0, T). \end{cases}$$

Before starting: the No-Control Framework

For $\boldsymbol{\mu} \in \mathcal{P}$ find $(y, u, p) \in \mathcal{Y} \times \mathcal{U} \times \mathcal{Y}$ such that

$$
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$$

The **control** is recovered in **post-processing**. In weak form, the whole system will be written as

$$
\int_0^T \mathcal{B}_{\mathbf{ocp}}((y, p), (z, q); \boldsymbol{\mu}) dt = \int_0^T \langle \mathcal{F}_{\mathbf{ocp}}(\boldsymbol{\mu}), (z, q) \rangle dt
$$

Existence and uniqueness are proved through Nečas-Babuška (same regularity for y and p)

[Langer, Steinbach, Troltzsch, and Yang. Unstructured space-time finite element methods for optimal control of parabolic equations, submitted, 2020.]

[Urban and Patera, A new error bound for reduced basis approximation of parabolic partial differential equations. Comptes Rendus Mathematique, 2012.]

Reduced Order Modelling for Space-time OCP(μ)s

$$\begin{bmatrix} M_y & B^T \\ B & -\frac{\Delta t}{\alpha} M_u \end{bmatrix} \begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{p} \end{bmatrix} = \begin{bmatrix} \Delta t M_y \mathsf{y_d} \\ \Delta t \mathsf{f} \end{bmatrix}$$

Algorithm(more details later):

- Space-Time Greedy
- Aggregated Spaces (well-posedness)

Important: $N \ll \mathcal{N}$

[Negri, Rozza, Manzoni and Quarteroni, SIAM Journal on Scientific Computing, 2013.]

[Gerner and Veroy. Certified reduced basis methods for parametrized saddle point problems. SIAM Journal on Scientific Computing, 2012.]

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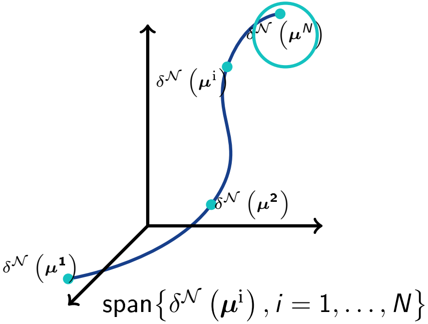


Space-Time Greedy Algorithm

Let us consider the global space-time *truth* solution $\delta^{\mathcal{N}} = (y^{\mathcal{N}}, p^{\mathcal{N}})$

Strategy: build adaptively the spaces thanks to an error estimator $\|\delta^{\mathcal{N}}(\boldsymbol{\mu}) - \delta_N(\boldsymbol{\mu})\|_{\mathcal{Y}\times\mathcal{Y}} \leq \Delta_N(\boldsymbol{\mu})$.

Greedy (given $\Delta_N(\boldsymbol{\mu})$, tolerance τ, initial $\boldsymbol{\mu}_1$, $\mathcal{P}_h \subset \mathcal{P}$):

1. $\mathcal{Y}_N^y = \text{span}\{y^{\mathcal{N}}(\boldsymbol{\mu}_1)\}$, $\mathcal{Y}_N^p = \text{span}\{p^{\mathcal{N}}(\boldsymbol{\mu}_1)\}$,
2. The n–th step of the process we choose the parameter $\boldsymbol{\mu}_n = \arg\max_{\boldsymbol{\mu}\in\mathcal{P}_h} \Delta_N(\boldsymbol{\mu})$,
3. $\mathcal{Y}_N^y = \text{span}\{y^{\mathcal{N}}(\boldsymbol{\mu}_1), \dots, y^{\mathcal{N}}(\boldsymbol{\mu}_n)\}$, $\mathcal{Y}_N^p = \text{span}\{p^{\mathcal{N}}(\boldsymbol{\mu}_1), \dots, p^{\mathcal{N}}(\boldsymbol{\mu}_n)\}$,
4. apply aggregated space $\to Y_N = \mathcal{Y}_N^y \cup \mathcal{Y}_N^p$ to use for state and adjoint variable
5. continue until $\Delta_N(\boldsymbol{\mu}) \leq \tau$.

Important: $N \ll \mathcal{N}$ with N snapshots taken in the building phase

Space-Time Greedy Error Estimator

(From Steady...) Let us define the residual of the whole optimality system

$$\mathcal{R}((z,q);\boldsymbol{\mu}) = \mathcal{B}_{\mathbf{ocp}}((y_N, q_N),(z,q);\boldsymbol{\mu}) - \langle \mathcal{F}_{\mathbf{ocp}}(\boldsymbol{\mu}),(z,q)\rangle$$

and the Babuška inf-sup constant

$$\beta(\boldsymbol{\mu}) := \inf_{(y,p)\in(Y^{N_h}\times Y^{N_h})\setminus\{(0,0)\}} \sup_{(z,q)\in(Y^{N_h}\times Y^{N_h})\setminus\{(0,0)\}} \frac{\mathcal{B}_{\mathbf{ocp}}((y,p),(z,q);\boldsymbol{\mu})}{\sqrt{\|y\|_Y^2+\|p\|_Y^2}\sqrt{\|z\|_Y^2+\|q\|_Y^2}},$$

then we can define an error estimator

$$\|\delta^{\mathcal{N}}(\boldsymbol{\mu}) - \delta_N(\boldsymbol{\mu})\|_{Y\times Y} \leq \frac{\|\mathcal{R}\|}{\beta(\boldsymbol{\mu})} := \Delta_N(\boldsymbol{\mu}) \quad \forall \boldsymbol{\mu}\in\mathcal{D},$$

Problem: $\beta(\boldsymbol{\mu})$ is expensive to compute (even for the steady case).

[Negri, Rozza, Manzoni and Quarteroni, SIAM Journal on Scientific Computing, 2013.]

Space-Time Greedy Error Estimator

(...to Unsteady) Let us define the residual of the whole optimality system

$$\mathcal{R}((z,q);\boldsymbol{\mu}) = \int_0^T \mathcal{B}_{\mathsf{ocp}}((y_N, q_N),(z,q);\boldsymbol{\mu})dt - \int_0^T \langle \mathcal{F}_{\mathsf{ocp}}(\boldsymbol{\mu}),(z,q)\rangle dt$$

and the Babuška inf-sup constant ($\mathcal{Q} = L^2(0,T;Y)$)

$$\beta(\boldsymbol{\mu}) := \inf_{(y,p)\in(\mathcal{Q}^{\mathcal{N}}\times\mathcal{Q}^{\mathcal{N}})\setminus\{(0,0)\}} \sup_{(z,q)\in(\mathcal{Q}^{\mathcal{N}}\times\mathcal{Q}^{\mathcal{N}})\setminus\{(0,0)\}} \frac{\int_0^T \mathcal{B}_{\mathsf{ocp}}((y,p),(z,q);\boldsymbol{\mu})}{\sqrt{\|y\|^2_{\mathcal{Q}}+\|p\|^2_{\mathcal{Q}}}\sqrt{\|z\|^2_{\mathcal{Q}}+\|q\|^2_{\mathcal{Q}}}},$$

Define a **cheap** error estimator $\|\delta^{\mathcal{N}}(\boldsymbol{\mu}) - \delta_N(\boldsymbol{\mu})\|_{\mathcal{Q}\times\mathcal{Q}} \leq \frac{\|\mathcal{R}\|}{\beta(\boldsymbol{\mu})} \leq \frac{\|\mathcal{R}\|}{\beta_{LB}(\boldsymbol{\mu})} := \Delta_N(\boldsymbol{\mu}) \quad \forall \boldsymbol{\mu}\in\mathcal{D}$,

Idea: find a lower bound $\beta_{LB}(\boldsymbol{\mu})$ **cheap to compute**

Space-Time Greedy Error Estimator

Idea: find a lower bound $\beta_{LB}(\boldsymbol{\mu})$ **cheap to compute**

Theorem

The space-time parabolic OCP($\boldsymbol{\mu}$) is well-posed at the continuous and discretized level. Then there exists a lower bound (not dependent on time) for $\beta(\boldsymbol{\mu})$ of the form:

$$\beta_{LB}(\boldsymbol{\mu}) = \begin{cases} \alpha\gamma_a(\boldsymbol{\mu}) & \text{for} \quad \Omega_u = \Omega_{\text{obs}}, \\ \dfrac{\gamma_a(\boldsymbol{\mu})}{\sqrt{2\max\left\{1, \left(\frac{c_c(\boldsymbol{\mu})c_u(\boldsymbol{\mu})}{\alpha\gamma_a(\boldsymbol{\mu})}\right)^2\right\}}} & \text{for} \quad \Omega_u \neq \Omega_{\text{obs}}. \end{cases}$$

[Strazzullo, Ballarin, Rozza, Certified Reduced Basis for Linear Parametrized Parabolic Optimal Control Problems in Space-Time Formulation, submitted, 2022.]

Graetz Flow

GOAL: recover parametrized desired temperature field with geometric boundary control

Optimization problem

given $\boldsymbol{\mu} \in \mathcal{P} = [6.0, 20.0] \times [1.0, 3.0] \times [0.5, 3.0]$ find $(y, u) \in [H^1(0, T; H^1_0(\Omega))]^2 \times L^2(0, T; L^2(\Omega))$ which minimizes

$$\frac{1}{2}\int_0^5 \int_{\Omega_{\mathbf{obs}}(\boldsymbol{\mu})} (y - y_d(\mu_3))^2 dxdt + \frac{\alpha}{2}\int_0^5 \int_{\Gamma_C(\boldsymbol{\mu})} u^2 dxdt,$$

constrained to

$$\begin{cases} y_t - \dfrac{1}{\mu_1}\Delta y - x_2(1 - x_2)\dfrac{\partial y}{\partial x_1} = 0 & \text{in } \Omega(\boldsymbol{\mu}) \times (0, 5), \\ y(0) = y_0 & \text{in } \Omega(\boldsymbol{\mu}), \\ \dfrac{1}{\mu_1}\dfrac{\partial y}{\partial n} = 0 & \text{on } \Gamma_N(\boldsymbol{\mu}) \times (0, 5), \\ \dfrac{1}{\mu_1}\dfrac{\partial y}{\partial n} = u & \text{on } \Gamma_C(\boldsymbol{\mu}) \times (0, 5), \\ y = 1 & \text{on } \Gamma_D \times (0, 5), \end{cases}$$

Graetz Flow

GOAL: recover parametrized desired temperature field with geometric boundary control

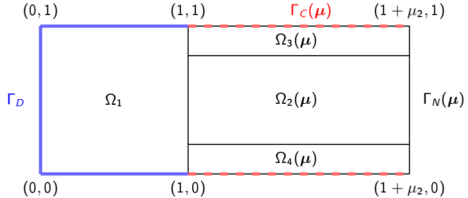
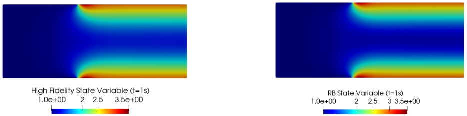


Figure: Domain Ω. *Observation domain:* $\Omega_{\text{obs}}(\boldsymbol{\mu}) = \Omega_3(\boldsymbol{\mu}) \cup \Omega_4(\boldsymbol{\mu})$, *Control domain:* $\Gamma_C(\boldsymbol{\mu})$ (red dashed line). *Blue solid line:* Dirichlet boundary conditions. The reference domain Ω is given by $\mu_2 = 1$.

Data

$\mathcal{P} = [6.0, 20.0] \times [1.0, 3.0] \times [0.5, 3.0]$, μ_1 = convection-advection, μ_2 = geometry, μ_3 = observation, $\tau = 10^{-4}$, $\alpha = 0.07$, $\Delta t = 0.16$, $|\mathcal{P}_h| = 225$ (Uniform), $N = 19$, $\mathcal{N} = 310980$.

Results



(a) (b)

Figure: FOM and ROM state for $t = 1s$ and $\mu_3 = 2.5$

Table: Average relative errors: lower bound $\beta_{LB}(\boldsymbol{\mu})$ vs $\beta(\boldsymbol{\mu})$, with respect to N.

N	$\|\|e\|\|_{\text{rel}}^{\beta_{LB}(\boldsymbol{\mu})}$	$\|\|e\|\|_{\text{rel}}^{\beta(\boldsymbol{\mu})}$
5	8.66e−2	8.01e−2
9	2.46e−2	2.17e−2
13	7.16e−3	5.98e−3
19	1.73e−3	1.72e−3

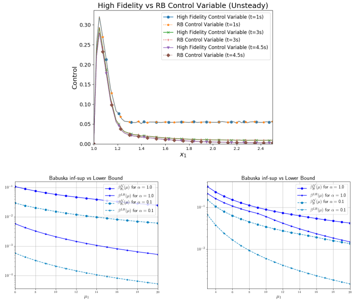
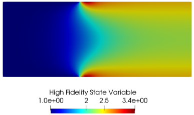


Figure: FOM and ROM control for $t = 1s, 3s, 4.5s$ and $\mu_3 = 2.5$ (top). $\beta_{LB}(\boldsymbol{\mu})$ vs $\beta(\boldsymbol{\mu})$ for geometrical and non-geometrical case (bottom).

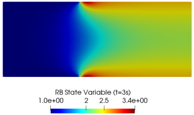
Performances

Effectivity $\sim 10^2/10^3$, Speedup $\sim 10^4$

Results



(a)



(b)

Figure: FOM and ROM state for $t = 3s$ and $\mu_3 = 2.5$

Table: Average relative errors: lower bound $\beta_{LB}(\boldsymbol{\mu})$ vs $\beta(\boldsymbol{\mu})$, with respect to N.

N	$\|\|e\|\|_{\text{rel}}^{\beta_{LB}(\boldsymbol{\mu})}$	$\|\|e\|\|_{\text{rel}}^{\beta(\boldsymbol{\mu})}$
5	8.66e−2	8.01e−2
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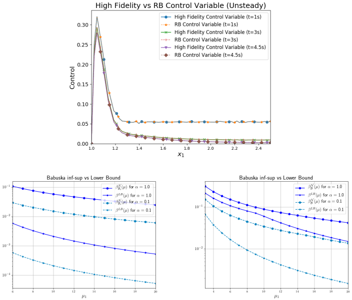


Figure: FOM and ROM control for $t = 1s, 3s, 4.5s$ and $\mu_3 = 2.5$ (top). $\beta_{LB}(\boldsymbol{\mu})$ vs $\beta(\boldsymbol{\mu})$ for geometrical and non-geometrical case (bottom).

Performances

Effectivity $\sim 10^2/10^3$, Speedup $\sim 10^4$

Results

(a) (b)

Figure: FOM and ROM state for $t = 4.5s$ and $\mu_3 = 2.5$

Table: Average relative errors: lower bound $\beta_{LB}(\boldsymbol{\mu})$ vs $\beta(\boldsymbol{\mu})$, with respect to N.

N	$\|\|e\|\|_{\text{rel}}^{\beta_{LB}(\boldsymbol{\mu})}$	$\|\|e\|\|_{\text{rel}}^{\beta(\boldsymbol{\mu})}$
5	8.66e−2	8.01e−2
9	2.46e−2	2.17e−2
13	7.16e−3	5.98e−3
19	1.73e−3	1.72e−3

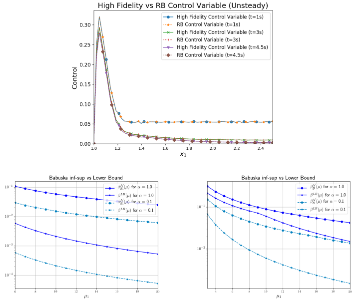


Figure: FOM and ROM control for $t = 1s, 3s, 4.5s$ and $\mu_3 = 2.5$ (top). $\beta_{LB}(\boldsymbol{\mu})$ vs $\beta(\boldsymbol{\mu})$ for geometrical and non-geometrical case (bottom).

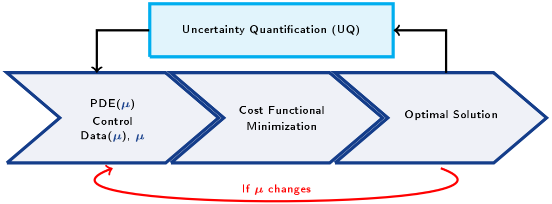
Performances

Effectivity $\sim 10^2/10^3$, Speedup $\sim 10^4$

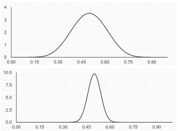
Weighted-POD for Uncertainty Quantification in OFCP(μ)

In collaboration with G. Carere, R. Stevenson, F. Zoccolan

ROMs for Uncertainty Quantification



- parameter is a random variable with given probability distribution ρ
- evaluation of some statistics on the optimal solution (expected solution) [Monte Carlo, a lot of realizations]
- using Weighted-ROM models to accelerate Monte Carlo methods
  - weight solutions during the construction of the ROM,
  - sample the parameter space during the construction of the ROM.



Weighted Reduced Strategies

Standard POD algorithm results in the optimal N-dimensional subspace of the variable space which minimizes

$$\int_{\mathcal{P}} \|\delta(\boldsymbol{\mu}) - \delta_N(\boldsymbol{\mu})\|^2 d\boldsymbol{\mu} \approx \frac{1}{N_{max}} \sum_{i=1}^{N_{max}} \|\delta(\boldsymbol{\mu}^i) - \delta_N(\boldsymbol{\mu}^i)\|^2,$$

Idea: use a more general quadrature rules of the form $\mathcal{U}(f) = \sum_{i=1}^{M} \omega^i f(\boldsymbol{\mu}^i)$ for every integrable function $f : \mathcal{P} \to \mathbb{R}$, where $\boldsymbol{\mu}^1, \dots, \boldsymbol{\mu}^N_{max} \in \mathcal{P}$ are the nodes of the quadrature and $\omega^1, \dots, \omega^N_{max}$ are the respective weights.
This results in the following approximation:

$$\mathbb{E}\left[\|\delta - \delta_N\|^2\right] \approx \frac{1}{N_{max}} \sum_{i=1}^{N_{max}} \underbrace{\omega_i \rho(\boldsymbol{\mu}^i)}_{w(\mu^i)} \|\delta(\boldsymbol{\mu}^i) - \delta_N(\boldsymbol{\mu}^i)\|^2$$

How does it work? $\mathbb{C}_{ij} = (\delta^{\mathcal{N}}(\boldsymbol{\mu}^i), \delta^{\mathcal{N}}(\boldsymbol{\mu}^j)) \to \mathbb{C}^w_{ij} = w(\boldsymbol{\mu}^i)(\delta^{\mathcal{N}}(\boldsymbol{\mu}^i), \delta^{\mathcal{N}}(\boldsymbol{\mu}^j))$

Pollutant Control on the Gulf of Trieste

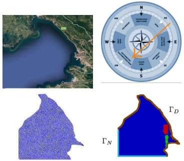
Motivations: *monitor*, *manage* and predict dangerous marine phenomena in a *fast way*

Aim: pollutant loss $y \in H^1_{\Gamma_D}(\Omega)$ under a safeguard threshold y_d

Given $\boldsymbol{\mu} \in [0.5, 1] \times [-1, 1] \times [-1, 1]$, find $(y(\boldsymbol{\mu}), u(\boldsymbol{\mu})) \in Y \times U$ which solves

$$\min_{(y,u)} \frac{1}{2} \int_{\Omega_{OBS}} (y - y_d)^2 \, d\Omega_y + \frac{\alpha}{2} \int_{\Omega_u} u^2 \, d\Omega_u$$

$$\text{s.t.} \begin{cases} \mu_1 \Delta y + [\mu_2, \mu_3] \cdot \nabla y \, d\Omega = u \chi_{\Omega_u} & \text{in } \Omega \\ \frac{\partial y}{\partial n} = 0 & \text{on } \Gamma_N \\ y = 0 & \text{on } \Gamma_D \end{cases}$$



Boundaries:

Γ_D = coasts, Γ_N = Adriatic Sea.

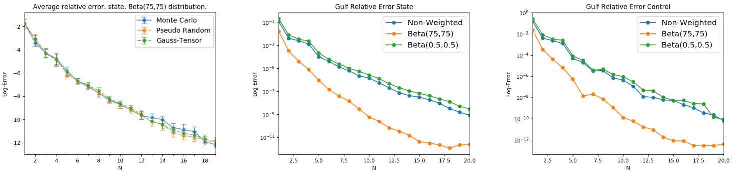
Subdomains:

Ω_{OBS} = Natural area of Miramare;

Ω_u = Source of pollutant (in front of the city of Trieste).

$\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\mu}_3$: wind action, α: 10^{-5}, y_d: 0.2

Numerical Results



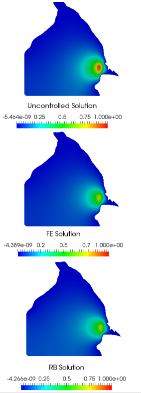


Figure: Relative Error for the state (left), control (middle), using ordinary POD (blue), Beta(0.5, 0.5)-Weighted-POD (green), Beta(75, 75)-Weighted-POD (orange). Three different quadrature rules for Beta(75, 75).

[Carere, Strazzullo, Ballarin, Rozza, Stevenson. Weighted POD-reduction for parametrized PDE-constrained Optimal Control Problems with random inputs and its applications to environmental sciences, Computers & Mathematics with Applications, 2021.]

UQ for stabilized optimal control

We deal with **Advection-Diffusion Problem** (High Péclet→ Stabilization (SUPG)):

Questions

Is OCP(μ) able to avoid instabilities? (Do I still need stabilization? For the adjoint too?)
Is consistent FOM-ROM model convenient?
How can I introduce stochastic knowledge in model order reduction?

[Zoccolan, Strazzullo, Rozza, "A Streamline upwind Petrov-Galerkin Reduced Order Method for Advection-Dominated Partial Differential Equations under Optimal Control", accepted CMAM, 2024.]
[Zoccolan, Strazzullo, Rozza, "Stabilized Reduced Order Method for Advection-Dominated Partial Differential Equations under Optimal Control with random inputs", submitted, 2023.]

The SUPG stabilization for the whole system

[Scott Collis, Heinkenschloss, Analysis of the Streamline Upwind/Petrov Galerkin method applied to the solution of optimal control problems. CAAM TR02-01, 108, 2002.]

Controlled state equation

$$a\left(y^{\mathcal{N}}, q^{\mathcal{N}}; \boldsymbol{\mu}\right) + \sum_{K \in \mathcal{T}_h} \delta_K \left(L y^{\mathcal{N}}, \frac{h_K}{|\mathrm{b}|} L_{SS} q^{\mathcal{N}}\right)_K - \int_\Omega u^{\mathcal{N}} q^{\mathcal{N}} - \sum_{K \in \mathcal{T}_h} \delta_K \left(u^{\mathcal{N}}, \frac{h_K}{|\mathrm{b}|} L_{SS} q^{\mathcal{N}}\right)_K$$
$$= \mathrm{f}(q^{\mathcal{N}}; \boldsymbol{\mu}) + \textstyle\sum_{K \in \mathcal{T}_h} \delta_K \left(f, \frac{h_K}{|\mathrm{b}|} L_{SS} q^{\mathcal{N}}\right)_K \quad \forall q^{\mathcal{N}} \in Y^{\mathcal{N}}.$$

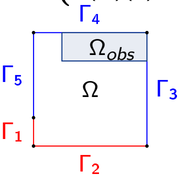
Adjoint equation

$$a^*\left(z^{\mathcal{N}}, p^{\mathcal{N}}\right) + \sum_{K \in \mathcal{T}_h} \delta_K \left(L^* p^{\mathcal{N}}, \frac{h_K}{|\mathrm{b}|} \left(-L_{SS}\right) z^{\mathcal{N}}\right)_K$$
$$+ \int_{\Omega_{obs}} (y^{\mathcal{N}} - y_d) z^{\mathcal{N}} \, \mathrm{dx} + \sum_{K \in \mathcal{T}_{h|_{\Omega_{obs}}}} \delta_K \left(y^{\mathcal{N}} - y_d, \frac{h_K}{|\mathrm{b}|} \left(-L_{SS}\right) z^{\mathcal{N}}\right)_K = 0 \quad \forall z^{\mathcal{N}} \in Y^{\mathcal{N}}$$

Numerical Results

Given $\boldsymbol{\mu} \in \mathcal{P}$ find $\displaystyle \min_{(y,u)\in H(0,T;H^1(\Omega))\times L^2(0,T;L^2(\Omega))} \frac{1}{2}\|y - y_d\|^2_{L^2(0,T;L^2(\Omega_{obs}))} + \frac{\alpha}{2}\|u\|^2_U$ such that

$$
\begin{cases}
y(\boldsymbol{\mu})_t - \dfrac{1}{\mu_1}\Delta y(\boldsymbol{\mu}) + (\cos\mu_2, \sin\mu_2)\cdot\nabla y(\boldsymbol{\mu}) = u, & \text{in } \Omega \times (0,T),\\
y(\boldsymbol{\mu}) = 1 & \text{on } \Gamma_1 \cup \Gamma_2 \times (0,T),\\
y(\boldsymbol{\mu}) = 0 & \text{on } \Gamma_3 \cup \Gamma_4 \cup \Gamma_5 \times (0,T),\\
y(\boldsymbol{\mu})(0) = 0 & \text{in } \Omega.
\end{cases}
$$



Data

$\Omega = (0,1)^2$, $T = 3$, $\Delta t = 0.1$, $N_t = 30$, $\mathcal{N} = 362610$, $y_d = 0.5$, $h = 0.036$, $\delta_k = 1.$, $\mathcal{P} = [10^2, 10^5] \times [0, 1.57]$, $\alpha = 0.01$, $N_{max} = 100$, $N = 30$.

Numerical Results

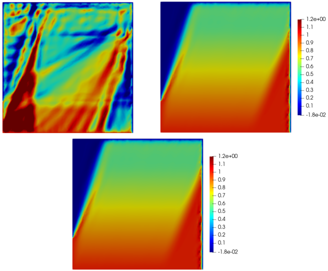


Figure: Only-Offline stabilized, Online-Offline vs FEM (bottom), $\mu = (2 \cdot 10^2, 1.2)$ at the final time.

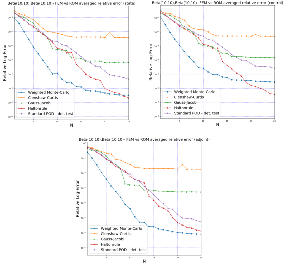


Figure: (Steady case) Relative errors between FEM and reduced solutions w-POD (Beta(10,10), Beta(10,10) and different quadrature points)

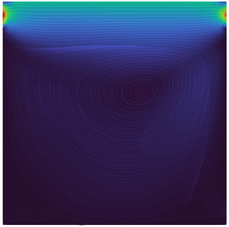
Performances (FOM-ROM Stabilization is preferable)

Relative errors over time and parameters ($P_{train} = 100$) $\sim 10^{-2}, 10^{-3}$. Speedup ~ 5000

Perspectives

Multi-fidelity Statistical POD for OCP

Goal: accelerate the offline phase



- Dirichlet control
- Part of the snapshots collected by optimal control problems
- Part of the snapshots collected by uncontrolled system with random Dirichlet inputs

Results: fast construction + not loosing accuracy
Collaborator: Enrique Delgado (University of Seville)

[Dolgov, Kalise, Saluzzi, "Statistical Proper Orthogonal Decomposition for model reduction in feedback control", arxiv preprint, 2023.]

Dynamical Orthogonal Reduced Basis for Feedback Control

Given a set of initial conditions find the state solution $\mathcal{Y} \in \mathcal{C}^1((0,\infty), \mathbb{R}^{N_h \times p})$ such that

$$\begin{cases} \delta \mathcal{Y}(t) = A\mathcal{Y}(t) + Bu(\mathcal{Y}(t)) & \text{for } t \in (0,\infty) \\ \mathcal{Y}(t_0) = \mathcal{Y}_0(\mu). \end{cases} \quad (1)$$

The main objective is to represent $\mathcal{Y}(t)$ in a **dynamical reduced space**

$$\mathcal{Y}(t) \approx \mathsf{Y}(t) = \mathsf{U}(t)\mathsf{Z}^T(t) = \sum_{i=1}^{n} U_i(t) Z_i(t;\mu), \quad (2)$$

Results: more accurate results and more comprehensive view of the optimality system
Collaborators: Luca Saluzzi (Scuola Normale Superiore di Pisa)

[Sapsis, Lermusiaux, "Dynamically orthogonal field equations for continuous stochastic dynamical systems", Physica D, 2009.]

Conclusions

- Space-time POD and greedy for OCP($\boldsymbol{\mu}$),
- Applications
  - uncertainty quantification

Acknowledgements

European Union Funding for Research and Innovation – Horizon 2020 Program – in the framework of European Research Council Executive Agency: Consolidator Grant H2020 ERC CoG 2015 AROMA-CFD project 681447 "Advanced Reduced Order Methods with Applications in Computational Fluid Dynamics".
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Thank you for your attention

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