Sub-Riemannian Continuous Normalizing Flows and Ensemble Controllability on Grassmannian Spaces

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Abstract

We investigate continuous normalizing flows (CNFs) governed by linear control systems of the form $\dot{x}(t) = \sum_{i=1}^{P} u_i(t) X_i(x(t))$, where u_i are the controls and X_i are given vector fields. Under an infinite-dimensional Lie algebra density hypothesis, we extend the classical ensemble controllability results of Agrachev and Sarychev [1] to Grassmannian spaces. We then apply this geometric framework to the construction of CNFs for singular measures, and empirically show that lifting these measures to the space of varifolds accelerates training convergence. Finally, we discuss the equivalence between the sub-Riemannian Kantorovich optimal transport problem using the squared Carnot-Carathéodory distance as a cost and the Benamou-Brenier reformulation for rectifiable varifolds.

Keywords. Continuous Normalizing Flow, Geometric Control, Optimal Transport, Varifold

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