A Parcimonious Long Term Mean Field Model for Mining Industries

Yves Achdou

Laboratoire J-L. Lions, Université Paris Diderot

(joint work with P-N. Giraud, J-M. Lasry and P-L. Lions)

Outline

Mining industries (e.g. copper, nickel, oil . . .)

Since the resource is not renewable, mining industries rely on a continuous flux of investments on

- prospection for new deposits
- construction of new extraction facilities

Mining industries (e.g. copper, nickel, oil . . .)

Since the resource is not renewable, mining industries rely on a continuous flux of investments on

- prospection for new deposits
- construction of new extraction facilities

Goal: Find a parcimonious long term model for a mining industry.
The model will be reminiscent of Lucas-Prescott (*Investments under uncertainty*, 1971), but

- we will use mean field games instead of general equilibrium theory
- we will address the special features of mining industries

Mining industries (e.g. copper, nickel, oil ...)

Since the resource is not renewable, mining industries rely on a continuous flux of investments on

- prospection for new deposits
- construction of new extraction facilities

Goal: Find a parcimonious long term model for a mining industry.
The model will be reminiscent of Lucas-Prescott (*Investments under uncertainty*, 1971), but

- we will use mean field games instead of general equilibrium theory
- we will address the special features of mining industries

There is a large number of competing production units but only a small number n of different technologies $\Rightarrow$ finite number of admissible states.
Depending on their technology, mining producers have different

- prospection costs and costs for constructing facilities
- annual production capacity
- operational cost of extraction

Mean field games

- The production units will be subject to a common noise linked to the state of the economy
- When there is an infinity of admissible states, MFG models with common noise lead to infinite dimensional PDEs, named *the master equations*, whose solutions are functions depending on the state variables and on the distribution of agents (a measure)
- In the present case, there is n admissible states (n technologies) $\Rightarrow$ the distribution of agents is a linear combination of Dirac masses, whose intensities are the global reserve R_i of the production units of type i $\Rightarrow$ finite dimensional PDE

Main assumptions

1. Long term model: stationarity

Main assumptions

1. Long term model: stationarity
2. Scale invariance:
   - the production capacity of a given production unit is proportional to its reserve, with a factor k depending on the technology
   - There is an exogeneous function ϕ depending on the technology, such that, given a producer whose reserve is ρ, an annual investment of α into prospection increases its reserve by

$$d\rho = \rho\phi(\alpha/\rho)$$

   ϕ is nondecreasing and concave.

The dynamics of the industry does not depend on the sizes of the production units

We may split the production units so that each unit corresponds to a unitary reserve of ore

Main assumptions

1. Long term model: stationarity
2. Scale invariance:
   - the production capacity of a given production unit is proportional to its reserve, with a factor k depending on the technology
   - There is an exogeneous function ϕ depending on the technology, such that, given a producer whose reserve is ρ, an annual investment of α into prospection increases its reserve by

     $$d\rho = \rho\phi(\alpha/\rho)$$

     ϕ is nondecreasing and concave.

   The dynamics of the industry does not depend on the sizes of the production units

   We may split the production units so that each unit corresponds to a unitary reserve of ore
3. A single unit has no impact on the equilibrium.

Outline

A single technology : a model of a mean field equilibrium

- Two global variables: the global reserve R and the state of the economy X:
  - R is the total number of production units
  - The state of the economy affects the global demand:

    $$\text{Demand } = XD(p), \text{ where } p \text{ is the unit price of ore}$$

    The state of the economy brings a noise common to all production units:

    $$dX_t/X_t = bdt + \sigma dW_t$$

A single technology : a model of a mean field equilibrium

- Two global variables: the global reserve R and the state of the economy X:
  - R is the total number of production units
  - The state of the economy affects the global demand:

$$\text{Demand } = XD(p), \text{ where } p \text{ is the unit price of ore}$$

  The state of the economy brings a noise common to all production units:

$$dX_t/X_t = bdt + \sigma dW_t$$

- Two controls for each production unit:
  1. the flux α invested into prospecting or into building infrastructures: Only the already existing production units can invest. An investment αdt increases the reserve by a factor $\phi(\alpha)dt$ with ϕ nondecreasing and concave
  2. the production rate β: $0 \leq \beta \leq k \leq 1$

General orientation

- Let $u(R,X)$ be the expected discounted value of a unit of extracted ore, or equivalently the value of a production unit.
- $u(R,X)$ is the value function of an optimal control problem in α and β solved by each producer, knowing the dynamics of the reserve: R_t
- The evolution of R_t depends on u: mean field equilibrium
- The equilibrium yields the master equation: a PDE solved by $u(R,X)$

Price and Global Production knowing $u(R, X)$

- Recall: $u(R, X)$ is the expected discounted value of a unit of extracted ore
- The unitary price p of ore should satisfy $p \geq c + u(R, X)$, where c is the unitary cost of extraction
- Global production: $q \leq kR$
- p and q are deduced from $u(R, X)$:
  - If $p > c + u(R, X)$, it is optimal to produce at full capacity: $q = kR$. Then, matching demand and supply: $D(p) = kR/X$
  - Else, $p = c + u(R, X)$ and $q = XD(c + u(R, X))$
- Summarizing,

$$
\begin{aligned}
P(R, X, u) &= \max\left(D^{-1}\left(\frac{kR}{X}\right), u + c\right)\\
Q(R, X, u) &= XD\left(P(R, X, u)\right)
\end{aligned}
$$

Mean field equilibrium

- Dynamics of the reserve, knowing the function u and the optimal controls:

$$\frac{dR_t}{dt} = R_t\phi(\alpha^*) - Q(R_t, X_t, u)$$

- Optimal control problem for a single production unit (infinite horizon). Dynamic programming, knowing the (stochastic) dynamics of X_t and of R_t:

$$\frac{u(R,X)}{1-rdt} = \max_{\alpha>0, 0\le\beta\le k} \mathbb{E}\left(\begin{array}{cc} & (\beta(P(\cdot, u(R,X)) - c) - \alpha)\, dt \\ + & \\ & (1 - \beta dt + \phi(\alpha) dt)\, u(R+dR, X+dX) \end{array} \right)$$

- Ito calculus yields a two-dimensional PDE:

$$-ru + k(P(\cdot,u) - c - u) + \max_{\alpha}\left(\phi(\alpha)u - \alpha\right) + \frac{dR}{dt}\,\frac{\partial u}{\partial R} + bX\frac{\partial u}{\partial X} + \frac{\sigma^2 X^2}{2}\frac{\partial^2 u}{\partial X^2} = 0$$

and the optimal β is given by

$$\begin{aligned} \beta^* &= k && \text{if } P(R,X,u) > c+u \\ \beta^* &= XD(u+c)/R && \text{if } P(R,X,u) = u+c \end{aligned}$$

Partial Differential Equations

$$ru - k\left(P^* - c - u\right) + Q^* \partial_R u - \frac{\partial}{\partial R}\left(R \max_\alpha (u\phi(\alpha) - \alpha)\right) - bX\frac{\partial u}{\partial X} - \frac{\sigma^2 X^2}{2}\frac{\partial^2 u}{\partial X^2} = 0,$$

with $P^* = \max\left(D^{-1}\left(kR/X\right), u + c\right)$ and $Q^* = XD\left(P^*\right)$.

A viscous conservation law in a reduced variable

Since Q^*/X and P^* only depend on $x \equiv R/X$ and u, we set $w(x) \equiv u(R, X)$ and get:

$$(b - r)w + \frac{d}{dx}\left(\frac{\sigma^2 x^2}{2}\frac{dw}{dx} - H(x, w) - bxw\right) = 0$$

with

$$\begin{aligned} H(x,w) = &\quad H_1(x,w) + H_2(x,w), \\ H_1(x,w) = &\quad -\mathbb{1}_{\{D(w+c)<kx\}} \int_{w+c}^{M} D(z)dz \\ &\quad +\mathbb{1}_{\{D(w+c)\geq kx\}} \left(kx\left(w + c - D^{-1}(kx)\right) - \int_{D^{-1}(kx)}^{M} D(z)dz\right) \end{aligned}$$

$$H_2(x,w) = -x \max_{\alpha \geq 0}(w\phi(\alpha) - \alpha)$$

The Hamilton-Jacobi equation

- Setting $w = dv/dx$, we look for a nondecreasing solution of the HJ equation:

$$(r-b)v + bx\frac{dv}{dx} + H\left(x, \frac{dv}{dx}\right) - \frac{\sigma^2 x^2}{2}\frac{d^2 v}{dx^2} = 0, \qquad \text{for } x \geq 0$$

- Example: $D(p) = p^{-s}$ and $\phi(\alpha) = C\sqrt{\alpha}$:

$$\left.\begin{array}{l} (r-b)v + bxv' - \dfrac{C^2}{4}x(v')^2 - \dfrac{\sigma^2 x^2}{2}v'' \\ +k\mathbb{1}_{\{v'+c\leq(kx)^{-1/s}\}}\left(x(v'+c) - \dfrac{s}{s-1}k^{-1/s}x^{1-1/s}\right) \\ +\dfrac{1}{1-s}\mathbb{1}_{\{v'+c>(kx)^{-1/s}\}}(c+v')^{1-s} \end{array}\right\} = 0$$

- Unusual HJ equation:
    - degenerate at $x = 0$
    - the Hamiltonian near $x = 0$ is of the form

$$\tilde{H}(x,p) = -c_1 x|p|^2 + c_2 x^{1-1/s}, \qquad 0 < s < 1$$

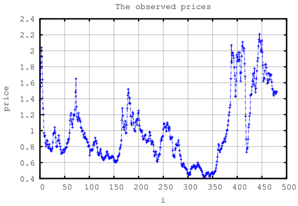
    - 2 regimes $\Rightarrow$ the Hamiltonian is continuous but not smooth
- Mathematical analysis : existence and uniqueness results (Lions-Collège de France 2016/AMO 2016)

Calibration of the model

We find a partial differential equation for the function $u(R, X)$ depending on 7 parameters:

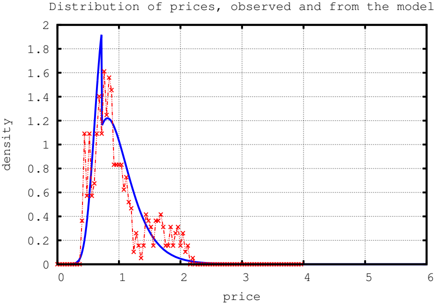
- b : growth rate — σ : volatility
- r : interest rate
- c : production cost — k : maximal extraction rate
- d : characterizes the efficiency of the investments: $\phi(\alpha) = 2d\sqrt{\alpha}$
- s : exponent in the demand function $\widetilde{D} = X/p^s$

We calibrate these parameters in order to fit the historical data.



Data : monthly price of copper from 1974 to 2014

Result of the calibration (maximization of likelyhood)

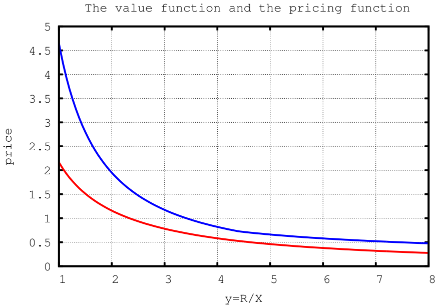


Copper: the distribution of the prices (observed and computed from the model)

Interpretation

- The narrow pike corresponds to periods when the demand is low:
$$p = c + u(R, X) \text{ and } Q(R, X) = X/(c + u(R, X))^s$$
- The wide bump corresponds to periods when the demand is high:
$$p > c + u(R, X) \text{ and } Q(R, X) = kR$$

The curve of prices



Copper: in red : $w = dv/dx$; in blue: the price P^* predicted by the model as a function of $x = R/X$. For large values of $x = R/X$, $P = c + w$.

Outline

The model with two technologies

- Two types of production units (different technologies), with different prospection and production costs $\Rightarrow$ 2 admissible states
- The total reserve of type i will be noted $R_i(t)$. $R_i(t)$ can be viewed as the quantity of production units of type i
- Let $c_i > 0$ be the unitary production cost of units of type i, with $c_1 < c_2$
- An investment rate of α_i increases the reserves of type j at a rate of $\phi_{i,j}(\alpha_i)$. For simplicity, assume that $\phi_{i,j} = 0$ if $i \neq j$

This model leads to a Hamilton-Jacobi equation in $\mathbb{R}_+^2$: with $x = \left(\frac{R_1}{X}, \frac{R_2}{X}\right)$,

$$(b-r)v - H(x, Dv) - b\, x \cdot Dv + \frac{\sigma^2}{2} \text{trace}\left(x \otimes x \cdot D^2 v\right) = 0$$

H is obtained as in the single technology case, but there are now eight different regimes. Moreover, H gets degenerate at $x_i = 0$.

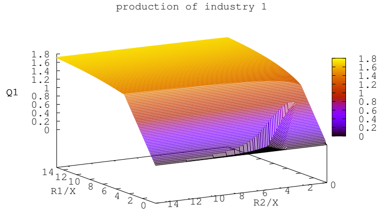
An example

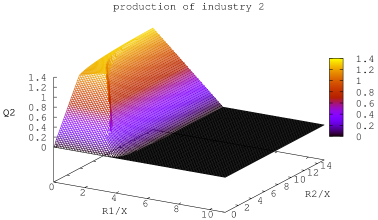
Here, the cost of production is smaller for technology 1: $c_1 < c_2$, whereas the investments into prospection are more efficient for technology 2.

$$r = 0.18;\ c_1 = 0.35;\ c_2 = 0.6;$$
$$k = 0.3;\ s = 0.6;\ \sigma = 0.15;\ b = 0.04;$$

and

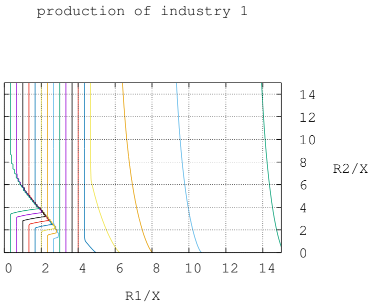
$$\phi_1(\alpha) = 0.895\sqrt{\alpha}, \quad \phi_2(\alpha) = 1.183\sqrt{\alpha},$$

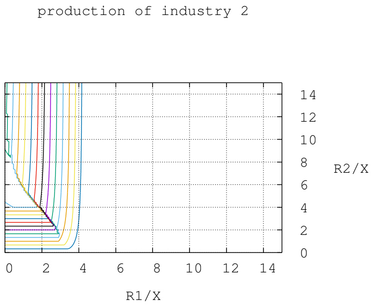




The productions $\tilde{Q}_1 = \frac{Q_1}{X}$ and $\tilde{Q}_2 = \frac{Q_2}{X}$

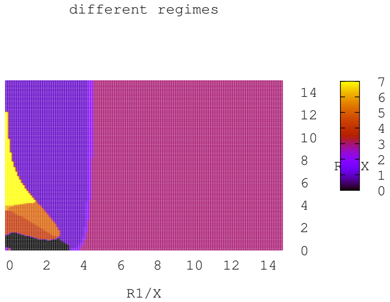
An example (II)





The contours of the productions $\tilde{Q}_1 = \frac{Q_1}{X}$ and $\tilde{Q}_2 = \frac{Q_2}{X}$

An example (III)



There are eight different regimes