Mean-field SDE driven by a fractional BM. A related stochastic control problem

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Introduction

Our objective: Study of the control problem with
the dynamics driven by a fBM B^H with Hurst parameter $H \in (1/2, 1)$

$$X_t^u = x + \int_0^t \sigma(P_{X_s^u}) dB_s^H + \int_0^t b(P_{(X_s^u, u_s)}, X_s^u, u_s) ds,$$

where $x \in R$, $u \in \mathcal{U}(0,T)$ - adapted control process with values in a convex open set $U \subset R^m$, and with
the cost functional:

$$J(u) = E\left[\int_0^T f\left(P_{(X_t^u, u_t)}, X_t^u, u_t\right) dt + g\left(X_T^u, P_{X_T^u}\right)\right];$$

characterisation of an optimal control $u^* \in \mathcal{U}(0,T)$ s.t. J takes its minimum over $\mathcal{U}(0,T)$ at u^*.

Short state-of-art:

• Several recent works on Pontryagin's maximum principle for mean-field control problems driven by a BM: + Carmona, Delarue (2015),..., + Buckdahn, Li, Ma (Pontryagin's principle for a mean-field control problem with partial observation, 2016);

• Several recent works on Peng's maximum principle (using spike control) for mean-field control problems driven by a BM: + Buckdahn, Djehiche, Li (2011), + Buckdahn, Li, Ma (2016);

• The fBM is not a semimartingale which makes the analysis much more subtle; much less works on maximum principle for control problems driven by a fBM: + Biagini, Hu, Oksendal, Sulem (2002), Hu, Zhou (linear control system, Riccati equation is a linear BSDE driven by a BM and a fBM, 2005), Han, Hu, Song

(2013) (Pontryagin's maximum principle with regularity assumptions -as Malliavin differentiability- on the optimal control process; adjoint equation is a linear BSDE driven by a BM and a fBM);
$+$ Buckdahn, Jing: Peng's maximum principle for control problem driven by a BM and a fBM (2014). With Shuai Jing we try to avoid regularity conditions on the optimal control.

Preliminaries

$(\Omega, \mathcal{F}, P)$ complete probability space;
1.**Fractional BM. A minimalist overview**:
- fBM with Hurst parameter $H \in (0,1)$: centered Gaussian process B^H with
$$\text{cov}(B_t^H, B_s^H) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}),\ s, t \geq 0.$$

For $H \in (1/2, 1)$, $\exists$ BM W s.t. $B_t^H = \int_0^t K_H(t,s)dW_s$, $t \in [0,T]$, with kernel

$$K_H(t,s) = c_H s^{1/2-H} \int_s^t (u-s)^{H-3/2} u^{H-1/2} du, \quad t > s,$$

with constant $c_H = [H(2H-1)/\beta(2-2H, H-1/2)]^{1/2}$,
where $\beta(\alpha,\gamma) = \Gamma(\alpha+\gamma)/(\Gamma(\alpha)\Gamma(\gamma))$ (Beta function), $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$ (Gamma function).

• Stochastic integral w.r.t. fBM: Let $f \in L^2([0,T]\times\Omega)$; $f \in$ Dom(δ_H), if $\exists \delta_H(f) \in L^2(\Omega, \mathcal{F}, P)$ s.t., for all $G \in \mathbf{D}_{1,2}^H$,

$$E\left[G\delta_H(f)\right] = \int_0^T E\left[f(t)\mathbf{D}_t^H G\right]dt.$$

If $fI_{[s,t]} \in$ Dom(δ_H), $\int_s^t f(r)dB_r^H := \delta_H(fI_{[s,t]})$.
Fractional calculus:

• <u>Malliavin derivatives</u>. D_t^H - classical Malliavin derivative but now w.r.t. B^H instead of a BM;

$$\mathbf{D}_s^H G = \int_0^T H(2H-1)|s-r|^{2H-2} D_r^H G dr = (K_H K_H^* D_\cdot^H G)(s),\ G \in \mathcal{S},$$

where K_H is an operator on $\mathcal{H} := \left\{ (\int_0^t K_H(t,s)\hat{f}(s)ds)_{t\in[0,T]},\ \hat{f} \in L^2([0,T]) \right\}$ defined by

$$(K_H\psi)(s) = c_H \Gamma(H-1/2) s^{1/2-H} I_{0+}^{H-1/2}(u^{H-1/2}\psi(u))(s)$$

with $I_{0+}^{\alpha}(f)(x) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^x \frac{f(u)}{(x-u)^{1-\alpha}} du$, for almost all $x \in [0,T]$, and K_H^* is its adjoint operator on $\mathcal{H}$. Important: for all $\psi \in \mathcal{H}$,

$$\int_0^T \psi(t) dB^H(t) = \int_0^T (K_H^* \psi)(t) dW(t),\ \int_0^T \psi(t) dW(t) = \int_0^T ({K_H^*}^{-1} \psi)(t) dB^H(t).$$

Therefore, $\mathbf{F} := \mathbf{F}^W = \mathbf{F}^{B^H}$.

- L^2-norm of stochastic integral: For all $f \in \text{Dom}(\delta_H)$,

$$E\big[|\int_0^T f(t)dB_t^H|^2\big] = E\big[\int_0^T |K_H^* f(t)|^2 dt\big] + 2E\big[\int_0^T\int_0^s \mathbf{D}_s^H f(r)\mathbf{D}_r^H f(s)drds\big].$$

Mean-field SDE driven by a fBM

Given $\xi \in L^2(\Omega, \mathcal{F}_0, P)$, $\Theta \in L^2([0,T] \times \Omega; R^m)$, we consider the SDE:

$$(1)\ X_t = \xi + \int_0^t \left(\gamma_s X_s + \sigma(s, P_{(X_s,\Theta_s)}\right) dB_s^H + \int_0^t b(s, P_{(X_s,\Theta_s)}, X_s) ds,$$

where $\sigma : [0,T] \times \mathcal{P}_2(R \times R^m) \to R$ and $b : \Omega \times [0,T] \times \mathcal{P}_2(R \times R^m) \times R \to R$ satisfy the following conditions:

(H1) For any $s \in [0,T]$, $x, x' \in R$, $\eta, \eta' \in L^2(\Omega, \mathcal{F}, P)$ and $\Theta \in L^2(\Omega, \mathcal{F}, P; R^m)$, there exists a constant $C > 0$ such that

$$|\sigma(s, P_{(\eta,\Theta)})| \le C,\ |b(s, P_{(\eta,\Theta)}, x)| \le C(1 + W_2(P_{(\eta,\Theta)}, P_{(0,\Theta)}) + |x|),$$
$$|\sigma(s, P_{(\eta,\Theta)}) - \sigma(s, P_{(\eta',\Theta)})| \le W_2(P_{(\eta,\Theta)}, P_{(\eta',\Theta)}),$$
$$|b(s, P_{(\eta,\Theta)}, x) - b(s, P_{(\eta',\Theta)}, x')| \le C(W_2(P_{(\eta,\Theta)}, P_{(\eta',\Theta)}) + |x - x'|).$$

The SDE will be solved using Girsanov transformation: For $\gamma \in L^\infty([0,T]) \subset \mathcal{H}$,

$$\mathcal{T}_t(\omega) = \omega + \int_0^{t\wedge\cdot} K_H^*(\gamma I_{[0,t]})(s)ds,$$

$$\mathcal{A}_t(\omega) = \omega - \int_0^{t\wedge\cdot} K_H^*(\gamma I_{[0,t]})(s)ds,\ t \in [0,T], \omega \in \Omega,$$

Clearly, $\mathcal{A}_t\mathcal{T}_t(\omega) = \mathcal{T}_t\mathcal{A}_t(\omega) = \omega$. Moreover, for any $F \in \mathcal{S}$, from the Girsanov theorem

$$E[F] = E[F(\mathcal{T}_t)\varepsilon_t^{-1}(\mathcal{T}_t)] = E[F(\mathcal{A}_t)\varepsilon_t],$$

where

$$\begin{aligned}\varepsilon_t &= \exp\left\{\int_0^t \gamma_s dB_s^H - \frac{1}{2}\int_0^t \left(K_H^*(\gamma I_{[0,t]})\right)^2(s)ds\right\}\\ &= \exp\left\{\int_0^t K_H^*(\gamma I_{[0,t]})(s)dW_s - \frac{1}{2}\int_0^t \left(K_H^*(\gamma I_{[0,t]})\right)^2(s)ds\right\},\end{aligned}$$

and

$$E\left[\sup_{t\in[0,T]} \varepsilon_t^p\right] < +\infty \quad \text{and} \quad E\left[\sup_{t\in[0,T]} \varepsilon_t^p(\mathcal{T}_t)\right] < +\infty, \qquad \text{for all} \quad p \in R.$$

Let $L^{2,*}([0,T];R)$ be the Banach space of $\mathbf{F}$-adapted processes $\{\varphi(t), t \in [0,T]\}$ such that

$$\sup_{t\in[0,T]} E\left[|\varphi(t)|^2 \varepsilon_t^{-1}\right] < +\infty.$$

Theorem. The above SDE (1) has a unique solution $X \in L^{2,*}([0,T];R)$.

Remark. Proof is based on the following statement with rather technical proof:
Proposition. 1) $X \in L^{2,*}([0,T];R)$ is a solution of our SDE iff it solves equ. (2)

$$\begin{aligned} & X_t(\mathcal{T}_t)\varepsilon_t^{-1}(\mathcal{T}_t) \\ = \ & \xi + \int_0^t \sigma(s, P_{(X_s,\Theta_s)})\varepsilon_s^{-1}(\mathcal{T}_s)dB_s^H + \int_0^t b(s, \mathcal{T}_s, P_{(X_s,\Theta_s)}, X_s(\mathcal{T}_s))\varepsilon_s^{-1}(\mathcal{T}_s)ds. \end{aligned}$$

2) For all deterministic $\Theta \in L^\infty([0,T])$, $\big(\Theta_s \varepsilon_s^{-1}(\mathcal{T}_s)\big)_{s\in[0,T]} \in \mathrm{Dom}(\delta_H)$.

3) The above SDE (2) has a unique solution $X \in L^{2,*}([0,T]; R)$

Proof. Main tools:

+ The method of Girsanov transformation;

+ The formula for $E\big[|\int_0^T f(t)dB_t^H|^2\big]$ (involving f and its Malliavin derivative $\mathbf{D}f$);

+ Picard's iteration with the distance function $(E[|X_t - X_t'|^2 \varepsilon_t^{-1}])^{1/2}$.

The mean-field control problem

Recall: B^H fBM with Hurst parameter $H \in (1/2, 1)$; control state space $U \subset R^m$ nonempty, convex, bounded; $\mathcal{U}([0,T]) := L^{\infty}_{\mathbf{F}}([0,T]; U)$.

Given $x \in R, u \in \mathcal{U}([0,T])$ we consider the control problem with dynamics

$$(3) \qquad X_t^u = x + \int_0^t \sigma(P_{X_s^u}) dB_s^H + \int_0^t b(P_{(X_s^u, u_s)}, X_s^u, u_s) ds,$$

with the cost functional

$$(4) \qquad J(u) = E\left[\int_0^T f\left(P_{(X_t^u, u_t)}, X_t^u, u_t\right) dt + g\left(X_T^u, P_{X_T^u}\right)\right].$$

Assumptions: (H2) $\sigma : \mathcal{P}_2(R) \to R$, $b : \mathcal{P}_2(R \times U) \times R \times U \to R$, $g : R \times \mathcal{P}_2(R) \to R$ and $f : \mathcal{P}_2(R \times U) \times R \times U \to U$ are bounded and Lipschitz (in all variables);
(H3) σ, b, f and g are C^1 in (x, μ, u) (with $\mu \in \mathcal{P}_2(R)$ and $\mathcal{P}_2(R \times U)$, respectively), and all first order derivatives are bounded and Lipschitz. This means, e.g., that, for some $C \in R$, for any (μ, y), $(\mu', y') \in \mathcal{P}_2(R) \times R$,

$$|\partial_\mu \sigma(\mu, y) - \partial_\mu \sigma(\mu', y')| \leq C(W_2(\mu, \mu') + |y - y'|).$$

Under the above assumptions we have seen the existence and the uniqueness for every $u \in \mathcal{U}([0,T])$.

Assume that there is a $u^* \in \mathcal{U}([0,T])$ minimising $J : \mathcal{U}([0,T]) \to R$; $X^* := X^{u^*}$.
Objective: Characterisation of u^* by convex perturbation: For $u \in \mathcal{U}([0,T])$, $\varepsilon \geq 0$, put $u^\varepsilon := u^* + \varepsilon(u - u^*) \in \mathcal{U}([0,T])$, $X^\varepsilon := X^{u^\varepsilon}$.

Lemma The following SDE obtained by formal differentiation of (3) for $u = u^\varepsilon$ w.r.t. ε at $\varepsilon = 0$, for $t \in [0,T]$,

$$\begin{aligned} Y_t = \quad & \int_0^t \widetilde{E}\left[\partial_\mu \sigma\left(P_{X_s^*}, \widetilde{X}_s^*\right)\widetilde{Y}_s\right] dB_s^H \\ & + \int_0^t \widetilde{E}\left[\left\langle \partial_\mu b\left(P_{(X_s^*,u_s^*)}, X_s^*, u_s^*, \widetilde{X}_s^*, \widetilde{u}_s^*\right), \left(\widetilde{Y}_s, \widetilde{u}_s - \widetilde{u}_s^*\right)\right\rangle\right] ds \\ & + \int_0^t \partial_x b\left(P_{(X_s^*,u_s^*)}, X_s^*, u_s^*\right) Y_s ds + \int_0^t \partial_u b\left(P_{(X_s^*,u_s^*)}, X_s^*, u_s^*\right)(u_s - u_s^*) ds, \end{aligned}$$

has a unique solution $Y \in L^2_{\mathbf{F}}([0,T])$. Moreover,

$$\lim_{\varepsilon \searrow 0} \sup_{t \in [0,T]} E\left[\left|Y_t - \frac{X_t^\varepsilon - X_t^*}{\varepsilon}\right|^2\right] = 0.$$

Above, $(\widetilde{X}^*, \widetilde{Y}, \widetilde{u}, \widetilde{u}^*)$ is an independent copy of (X^*, Y, u, u^*) defined on a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$. The expectation $\widetilde{E}[\cdot]$ under $\widetilde{P}$ only concerns $(\widetilde{X}^*, \widetilde{Y}, \widetilde{u}, \widetilde{u}^*)$ but not (X^*, Y, u, u^*).

As (X^*, u^*) is optimal, $J(u^\varepsilon) \geq J(u^*), \varepsilon \in [0,1]$. Thus,

$$\frac{d}{d\varepsilon} J(u^\varepsilon)\Big|_{\varepsilon=0} := \lim_{0<\varepsilon\searrow 0} \frac{1}{\varepsilon}(J(u^\varepsilon) - J(u^*)) \geq 0, \text{ and}$$

$$\begin{aligned}
0 \leq \tfrac{d}{d\varepsilon} J(u^\varepsilon)\Big|_{\varepsilon=0} &= E\left[\partial_x g(X_T^*, P_{X_T^*}) Y_T\right] + E\left[\widetilde{E}\left[\partial_\mu g(X_T^*, P_{X_T^*}, \widetilde{X}_T^*)\widetilde{Y}_T\right]\right] \\
&+ E\Big[\int_0^T \partial_x f(P_{(X_t^*, u_t^*)}, X_t^*, u_t^*) Y_t dt\Big] \\
&+ E\Big[\int_0^T \widetilde{E}\left[(\partial_\mu f)_1(P_{(X_t^*, u_t^*)}, X_t^*, u_t^*, \widetilde{X}_t^*, \widetilde{u}_t^*)\widetilde{Y}_t\right] dt\Big] \\
&+ E\Big[\int_0^T \widetilde{E}\left[(\partial_\mu f)_2(P_{(X_t^*, u_t^*)}, X_t^*, u_t^*, \widetilde{X}_t^*, \widetilde{u}_t^*)(\widetilde{u}_t - \widetilde{u}_t^*)\right] dt\Big] \\
&+ E\Big[\int_0^T \partial_u f(P_{(X_t^*, u_t^*)}, X_t^*, u_t^*)(u_t - u_t^*) dt\Big]. \qquad (5)
\end{aligned}$$

We introduce the adjoint BSDE with its $\mathbf{F}$-adpated solution (P,β):

$$\begin{aligned} P_t &= P_T - \int_t^T \alpha_s ds - \int_t^T \beta_s dW_s,\ t\in[0,T],\\ P_T &= \partial_x g(X_T^*, P_{X_T^*}) + \widetilde{E}\Big[\partial_\mu g(\widetilde{X}_T^*, P_{X_T^*}, X_T^*)\Big], \end{aligned}$$

where $\alpha \in L^2_{\mathcal{F}}([0,T])$ will be specified later. Applying Itô's formula (with B^H) taking the expectation and rearranging terms, we get

$$\begin{aligned} &E\big[Y_T P_T\big]\\ = &\int_0^T E\big[Y_s\big\{\widetilde{E}\big[\widetilde{P}_s(\partial_\mu b)_1\big(P_{(X_s^*,u_s^*)},\widetilde{X}_s^*,\widetilde{u}_s^*,X_s^*\big)\big] + P_s\partial_x b\big(P_{(X_s^*,u_s^*)},X_s^*,u_s^*\big)\\ &\qquad +\alpha_s + \partial_\mu\sigma\big(P_{X_s^*},X_s^*\big)E\big[\mathbf{D}_s^H P_s\big]\big\}\big]ds\\ &+\int_0^T E\Big[\Big(\widetilde{E}\big[\widetilde{P}_s(\partial_\mu b)_2\big(P_{(X_s^*,u_s^*)},\widetilde{X}_s^*,\widetilde{u}_s^*,u_s^*\big)\big]\\ &\qquad +P_s\partial_u b\big(P_{(X_s^*,u_s^*)},X_s^*,u_s^*\big)\Big)(u_s-u_s^*)\Big]ds. \qquad (6) \end{aligned}$$

Combining (5) with (6) we obtain

$$\begin{aligned}
0 \leq \ & \int_0^T E\Big[Y_s\big\{\widetilde{E}\big[\widetilde{P}_s(\partial_\mu b)_1\big(P_{(X_s^*,u_s^*)},\widetilde{X}_s^*,\widetilde{u}_s^*,X_s^*\big)\big]+P_s\partial_x b\big(P_{(X_s^*,u_s^*)},X_s^*,u_s^*\big)\\
& +\partial_\mu\sigma\big(P_{X_s^*},X_s^*\big)E\big[\mathbf{D}_s^H P_s\big]+\partial_x f(P_{(X_s^*,u_s^*)},X_s^*,u_s^*)+\alpha_s\\
& +\widetilde{E}\big[(\partial_\mu f)_1(P_{(X_s^*,u_s^*)},\widetilde{X}_s^*,\widetilde{u}_s^*,X_s^*,u_s^*)\big]\big\}\Big]ds\\
+ \ & \int_0^T E\Big[(u_s-u_s^*)\big\{\widetilde{E}\big[(\partial_\mu f)_2(P_{(X_s^*,u_s^*)},\widetilde{X}_s^*,\widetilde{u}_s^*,X_s^*,u_s^*)\big] \qquad (7)\\
& +\partial_u f(P_{(X_s^*,u_s^*)},X_s^*,u_s^*)\\
& +\widetilde{E}\big[\widetilde{P}_s(\partial_\mu b)_2\big(P_{(X_s^*,u_s^*)},\widetilde{X}_s^*,\widetilde{u}_s^*,X_s^*,u_s^*\big)\big]+P_s\partial_u b\big(P_{(X_s^*,u_s^*)},X_s^*,u_s^*\big)\big\}\Big]ds.
\end{aligned}$$

Putting the first integral equal to zero, we choose

$$\begin{aligned}
\alpha_s = \ & -\widetilde{E}\big[\widetilde{P}_s(\partial_\mu b)_1\big(P_{(X_s^*,u_s^*)},\widetilde{X}_s^*,\widetilde{u}_s^*,X_s^*\big)\big]-P_s\partial_x b\big(P_{(X_s^*,u_s^*)},X_s^*,u_s^*\big)\\
& -\partial_\mu\sigma\big(P_{X_s^*},X_s^*\big)E\big[\mathbf{D}_s^H P_s\big]-\partial_x f(P_{(X_s^*,u_s^*)},X_s^*,u_s^*)\\
& -\widetilde{E}\big[(\partial_\mu f)_1(P_{(X_s^*,u_s^*)},\widetilde{X}_s^*,\widetilde{u}_s^*,X_s^*,u_s^*)\big].
\end{aligned}$$

This gives the following form of the BSDE for $P_t = P_T - \int_t^T \alpha_s ds - \int_t^T \beta_s dW_s$:

$$
\begin{aligned}
P_t = \quad & P_T + \int_t^T \Big\{ \widetilde{E}\big[\widetilde{P}_s(\partial_\mu b)_1\big(P_{(X_s^*,u_s^*)}, \widetilde{X}_s^*, \widetilde{u}_s^*, X_s^*, u_s^*\big)\big] \\
& \qquad + P_s \partial_x b\big(P_{(X_s^*,u_s^*)}, X_s^*, u_s^*\big) \\
& \qquad + \partial_\mu \sigma\big(P_{X_s^*}, X_s^*\big) E\big[\mathbf{D}_s^H P_s\big] + \partial_x f(P_{(X_s^*,u_s^*)}, X_s^*, u_s^*) \\
& \qquad + \widetilde{E}\big[(\partial_\mu f)_1(P_{(X_s^*,u_s^*)}, \widetilde{X}_s^*, \widetilde{u}_s^*, X_s^*, u_s^*)\big] \Big\} ds - \int_t^T \beta_s dW_s,
\end{aligned}
\tag{8}
$$

which is a mean-field BSDE driven by the standard Brownian motion W.
Let us assume that the above BSDE has an $\mathbf{F}$-adapted solution (P, β) (Discussion on existence and unique for the BSDE later).

Then (7) becomes

$$
\begin{aligned}
0 \leq \quad & \int_0^T E\big[(u_s - u_s^*)\big\{\widetilde{E}\big[(\partial_\mu f)_2(P_{(X_s^*,u_s^*)}, \widetilde{X}_s^*, \widetilde{u}_s^*, X_s^*, u_s^*)\big] \\
& +\partial_u f(P_{(X_s^*,u_s^*)}, X_s^*, u_s^*) \\
& +\widetilde{E}\big[\widetilde{P}_s(\partial_\mu b)_2\big(P_{(X_s^*,u_s^*)}, \widetilde{X}_s^*, \widetilde{u}_s^*, X_s^*, u_s^*\big)\big] + P_s\partial_u b\big(P_{(X_s^*,u_s^*)}, X_s^*, u_s^*\big)\big\}\big]ds.
\end{aligned} \tag{9}
$$

As U is open and $u \in \mathcal{U}([0,T])$ arbitrary, we have

$$
\begin{aligned}
0 = \quad & \widetilde{E}\big[(\partial_\mu f)_2(P_{(X_t^*,u_t^*)}, \widetilde{X}_t^*, \widetilde{u}_t^*, X_t^*, u_t^*)\big] + \partial_u f(P_{(X_t^*,u_t^*)}, X_t^*, u_t^*) \\
& +\widetilde{E}\big[\widetilde{P}_t(\partial_\mu b)_2\big(P_{(X_t^*,u_t^*)}, \widetilde{X}_t^*, \widetilde{u}_t^*, X_t^*, u_t^*\big)\big] + P_t\partial_u b\big(P_{(X_t^*,u_t^*)}, X_t^*, u_t^*\big),
\end{aligned}
$$

dP-a.s., dt-a.e.

Hence, we have following necessary conditions of Pontryagin-type.

Theorem. If (X^*, u^*) is an optimal pair of our control problem, then (X^*, u^*) satisfies the following system:

$$
\begin{aligned}
X_t^* = &\ x + \int_0^t \sigma(P_{X_s^*}) dB_s^H + \int_0^t b(P_{(X_s^*, u_s^*)}, X_s^*, u_s^*) ds, \qquad (10)\\
P_T = &\ \partial_x g(X_T^*, P_{X_T^*}) + \widetilde{E}\big[\partial_\mu g(\widetilde{X_T^*}, P_{X_T^*}, X_T^*)\big],\\
P_t = &\ P_T - \int_t^T \beta_s dW_s + \int_t^T \Big\{ \widetilde{E}\big[\widetilde{P}_s (\partial_\mu b)_1 \big(P_{(X_s^*, u_s^*)}, \widetilde{X}_s^*, \widetilde{u}_s^*, X_s^*, u_s^*\big)\big]\\
&+ P_s \partial_x b\big(P_{(X_s^*, u_s^*)}, X_s^*, u_s^*\big) + \partial_\mu \sigma\big(P_{X_s^*}, X_s^*\big) E\big[\mathbf{D}_s^H P_s\big]\\
&+ \partial_x f(P_{(X_s^*, u_s^*)}, X_s^*, u_s^*) + \widetilde{E}\big[(\partial_\mu f)_1 (P_{(X_s^*, u_s^*)}, \widetilde{X}_s^*, \widetilde{u}_s^*, X_s^*, u_s^*)\big] \Big\} ds,
\end{aligned}
$$

$$
\begin{aligned}
0 = &\ \widetilde{E}\big[\widetilde{P}_t (\partial_\mu b)_2 \big(P_{(X_t^*, u_t^*)}, \widetilde{X}_t^*, \widetilde{u}_t^*, X_t^*, u_t^*\big)\big] + P_t \partial_u b\big(P_{(X_t^*, u_t^*)}, X_t^*, u_t^*\big)\\
&+ \widetilde{E}\big[(\partial_\mu f)_2 (P_{(X_t^*, u_t^*)}, \widetilde{X}_t^*, \widetilde{u}_t^*, X_t^*, u_t^*)\big] + \partial_u f(P_{(X_t^*, u_t^*)}, X_t^*, u_t^*),\\
&\ dP\text{-a.s.}, dt\text{-a.e.}
\end{aligned}
$$

After this necessary condition we can give also a sufficient one:

Theorem. In addition to our standard assumptions, assume that $g = g(x, \mu)$ is convex in (x, μ) and the Hamiltonian

$$H(\mu, x, u, y, z) := f(\mu, x, u) + b(\mu, x, u)y + \sigma(\mu)z$$

be jointly convex in (μ, x, u). Let (u^*, X^*) satisfy the above system. Then (u^*, X^*) is optimal: $J(u^*) = \inf_{u \in \mathcal{U}([0,T])} J(u)$.

Remark. The convexity assumption on the Hamiltonian implies practically the linearity of b and σ.

Another assumption in which b and σ don't need to be linear:

$+$ $g : R \times \mathcal{P}_2(R) \to R$ jointly convex in (x, μ), with $\partial_x g \geq 0$, $\partial_\mu g \geq 0$;

$+$ $b(\mu, x, u) : \mathcal{P}_2(R \times U) \times R \times U \to R$ is jointly convex in (μ, x, u) with $(\partial_\mu b)_1(\mu, x, u, y) \geq 0$, strictly convex in (μ, u);

$+$ $f(\mu, x, u) : \mathcal{P}_2(R \times U) \times R \times U \to R$ is jointly convex in (μ, x, u), and strictly convex in (μ, u), with $(\partial_\mu f)_1(\eta, x, u, y) \geq 0$, $\partial_x f(\eta, x, u, y) \geq 0$;

$+$ $\sigma(\mu) \equiv \sigma \in R$.

Recall (Carmona, Delarue, 2015): g convex (strictly convex), if there exists exists $\lambda \geq 0$ (resp., > 0) such that for all $x, x' \in R$, $\xi, \xi' \in L^2(\Omega, \mathcal{F}, P)$,
$g(x', P_{\xi'}) - g(x, P_\xi) \geq$
$\partial_x g(x, P_\xi)(x' - x) + E\big[\partial_\mu g(x, P_\xi, \xi)(\xi' - \xi)\big] + \lambda\big(|x - x'|^2 + E[|\xi - \xi'|^2]\big)$.

Theorem. If in addition to our standard assumptions the above assumption is satisfied and (u^*, X^*) solves the above system, then (u^*, X^*) is optimal: $J(u^*) = \inf_{u \in \mathcal{U}([0,T])} J(u)$.

Concerning the solvability of the coupled forward-backward system (10) under the conditions of the preceding Theorem, we proceed as follows:
For any given $(P, \xi) \in L^2(\mathcal{F}_t) \times L^2(\mathcal{F}_t)$, we suppose that there is some $\eta \in L^2(\mathcal{F}_t; U)$ such that:

$$\begin{aligned} 0 = \quad & \widetilde{E}\big[(\partial_\mu f)_2\big(P_{(\xi,\eta)},\widetilde{\xi},\widetilde{\eta},\xi,\eta\big)\big] + \partial_u f(P_{(\xi,\eta)},\xi,\eta) \\ & +\widetilde{E}\big[\widetilde{P}(\partial_\mu b)_2\big(P_{(\xi,\eta)},\widetilde{\xi},\widetilde{\eta},\xi,\eta\big)\big] + P(\partial_u b)(P_{(\xi,\eta)},\xi,\eta). \end{aligned}$$

Lemma. The mapping $(P,\xi) \to \eta$ is unique and $\eta = \eta(P,\xi)$ is Lipschitz in (P,ξ) under L^2-norm.

Theorem. Under the assumptions of the preceding theorem (i.e., in particular $\sigma(\mu) = \sigma \in R$) and the preceding assumption, then, if the time horizon $T > 0$ is sufficiently small, there exists a unique solution $(X^*, (P,\beta), u^* = \eta(X^*, P))$ of the coupled FBSDE (10).

THANK YOU VERY MUCH!