Long time behavior of Mean Field Games

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The MFG system

Given a finite horizon $T > 0$, we consider the MFG system [1]

$$(MFG) \qquad \begin{cases} -\partial_t u^T - \Delta u^T + H(x, Du^T) = f(x, m^T(t)) & \text{in } (0,T)\times \mathbb{T}^d \\ \partial_t m^T - \Delta m^T - \mathrm{div}(m^T D_pH(x, Du^T)) = 0 & \text{in } (0,T)\times \mathbb{T}^d \\ m^T(0,\cdot) = m_0, \ u^T(T,x) = g(x, m^T(T)) & \text{in } \mathbb{T}^d \end{cases}$$

where

- $u = u(t,x)$ and $m = m(t,x)$ are the unknown,
- $H = H(x,p) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is a smooth, unif. convex in p, Hamiltonian,
- $f, g : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ are “smooth" and monotone : for any $m, m' \in \mathcal{P}(\mathbb{T}^d)$,

$$\int_{\mathbb{T}^d} (f(x,m) - f(x,m'))d(m-m')(x) \geq 0, \ \int_{\mathbb{T}^d} (g(x,m) - g(x,m'))d(m-m')(x) \geq 0$$

  ($\mathcal{P}(\mathbb{T}^d)$ = the set of Borel probability measures on $\mathbb{T}^d$)
- $m_0 \in \mathcal{P}(\mathbb{T}^d)$ is a smooth positive density

1. Introduced by Lasry-Lions and Huang-Caines-Malhamé to study optimal control problems with infinitely many controllers.

The limit problem

Let (u^T, m^T) be the solution to

$$(MFG) \qquad \left\{ \begin{array}{ll} -\partial_t u^T - \Delta u^T + H(x, Du^T) = f(x, m^T(t)) & \text{in } (0, T) \times \mathbb{T}^d \\ \partial_t m^T - \Delta m^T - \mathrm{div}(m^T D_p H(x, Du^T)) = 0 & \text{in } (0, T) \times \mathbb{T}^d \\ m^T(0, \cdot) = m_0, \ u^T(T, x) = g(x, m^T(T)) & \text{in } \mathbb{T}^d \end{array} \right.$$

Study the limit as $T \to +\infty$ of the pair (u^T, m^T).

- **Motivation :** one numerically observes that the pair (u^T, m^T) quickly stabilizes as T is large (when f and g are monotone).
- In contrast with similar problem for HJ equation, **initial** and **terminal** conditions for the system.

The limit problem

One expects that (u^T, m^T) “converges" to the solution of the ergodic MFG problem

$$(MFG-erg) \qquad \left\{ \begin{array}{l} \bar\lambda - \Delta \bar u + H(x, D\bar u) = f(x,\bar m) \qquad \text{in } \mathbb{T}^d \\ -\Delta \bar m - \mathrm{div}(\bar m D_pH(x,D\bar u)) = 0 \qquad \text{in } \mathbb{T}^d \\ \bar m \geq 0 \quad \text{in } \mathbb{T}^d, \qquad \displaystyle\int_{\mathbb{T}^d} \bar m = 1 \end{array} \right.$$

where now the unknown are $\bar\lambda$, $\bar u = \bar u(x)$ and $\bar m = \bar m(x)$.

- For the Fokker-Plank equation driven by a vector-field V :

$$\partial_t m^T - \Delta m^T - \mathrm{div}(m^T V(x)) = 0 \qquad \text{in } (0,T) \times \mathbb{T}^d$$

(exponential) convergence of m^T to the ergodic measure is well-known.

- For HJ equations :

$$\partial_t u - \Delta u + H(x, Du) = f(x) \qquad \text{in } (0,+\infty) \times \mathbb{T}^d$$

  - Ergodic constant $\bar{\lambda}$, convergence of $u(T,\cdot)/T$: Lions-Papanicolau-Varadhan, ...
  - Convergence of $u(T,\cdot) - \bar{\lambda}T$: Fathi, Roquejoffre, Fathi-Siconolfi, Barles-Souganidis, ...

- For the MFG system, convergence of u^T/T and m^T :
  - Lions (Cours in Collège de France)
  - Gomes-Mohr-Souza (discrete setting)
  - C.-Lasry-Lions-Porretta (viscous setting), C. (Hamilton-Jacobi)
  - Turnpike property (Samuelson, Porretta-Zuazua, Trélat,...)

- Long-time behavior not known so far.

The long time average

Let (u^T, m^T) be the solution to

$$(MFG) \qquad \left\{ \begin{array}{ll} -\partial_t u^T - \Delta u^T + H(x, Du^T) = f(x, m^T(t)) & \text{in } (0,T)\times\mathbb{T}^d \\ \partial_t m^T - \Delta m^T - \mathrm{div}(m^T D_pH(x, Du^T)) = 0 & \text{in } (0,T)\times\mathbb{T}^d \\ m^T(0,\cdot) = m_0, \; u^T(T,x) = g(x, m^T(T)) & \text{in } \mathbb{T}^d \end{array} \right.$$

Theorem (C.-Lasry-Lions-Porretta ('13), C.-Porretta ('17))

There exists $\gamma > 0$ and C, independent of T and m_0, such that

$$\|Du^T(t,\cdot) - D\bar{u}\|_{C^1} + \|m^T(t,\cdot) - \bar{m}\|_\infty \leq C\left(e^{-\gamma t} + e^{-\gamma(T-t)}\right).$$

Moreover, the map $(s,x) \to u^T(Ts,x)/T$ unif. converges to $(1-s)\bar{\lambda}$ on $[0,1]\times\mathbb{T}^d$.

- The result does not say anything about the limit of $u^T(t,x) - (T-t)\bar{\lambda}$.

The long time behavior

For $T > 0$ and $m_0 \in \mathcal{P}(\mathbb{T}^d)$, let (u^T, m^T) be the solution to the MFG system :

$$\left\{ \begin{array}{ll} -\partial_t u^T - \Delta u^T + H(x, Du^T) = f(x, m^T(t)) & \text{in } (0,T)\times\mathbb{T}^d \\ \partial_t m^T - \Delta m^T - \mathrm{div}(m^T D_pH(x, Du^T)) = 0 & \text{in } (0,T)\times\mathbb{T}^d \\ m^T(0,\cdot) = m_0, \; u^T(T,x) = g(x, m^T(T)) & \text{in } \mathbb{T}^d \end{array} \right.$$

Theorem (C.-Porretta ('17))

There exists a map $\chi : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ (with χ and $D_x\chi$ Lipschitz) and a constant $\bar{c}$, independent of m_0, such that, for any $t \geq 0$,

$$\lim_{T\to\infty} u^T(t,x) - \bar{\lambda}(T-t) = \chi(x, m(t)) + \bar{c},$$

where the convergence is uniform in x and m solves the McKean-Vlasov eq.

$$\partial_t m - \Delta m - \mathrm{div}(mH_p(x, D_x\chi(x,m))) = 0, \qquad m(0) = m_0.$$

Moreover, for any $\kappa \in (0,1)$,

$$\lim_{T\to\infty} u^T(\kappa T, x) - (1-\delta)\bar{\lambda}T = \chi(x, \bar{m}) + \bar{c} = \bar{u}(x) + \bar{c}.$$

Ideas of proof

We want to show that $\lim_{T\to\infty} u^T(t,x) - \bar\lambda(T-t) = \chi(x, m(t)) + \bar c$.

- The proof relies on the master equation :

$$\left\{\begin{array}{l} -\partial_t U - \Delta_x U + H(x, D_x U) - f(x,m) \\ \qquad - \int_{\mathbb{T}^d} \mathrm{div}_y\,[D_m U]\ dm(y) + \int_{\mathbb{T}^d} D_m U \cdot D_p H(y, D_x U)\ dm(y) = 0 \\ \qquad\qquad\qquad \text{in } (-\infty, 0) \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \\ U(0,x,m) = g(x,m) \qquad \text{in } \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \end{array}\right.$$

  where $U = U(t,x,m) : (-\infty, 0] \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$.
- As observed by Lasry-Lions, solutions of the MFG systems are "characteristics" of the master equation.
  (existence of solutions : Lasry-Lions, Buckdahn-Li-Peng-Rainer, Gangbo-Swiech, Chassagneux-Delarue-Crisan, C.-Lasry-Lions-Delarue).
- The main step is that there exists a constant $\bar c$ such that

$$\lim_{t\to-\infty} U(t,x,m) + \bar\lambda t = \chi(x,m) + \bar c$$

  where χ is a weak solution to the master cell problem

$$\bar\lambda - \Delta_x \chi(x,m) + H(x, D_x\chi(x,m)) - \int_{\mathbb{T}^d} \mathrm{div}(D_m\chi(x,m))dm$$
$$+ \int_{\mathbb{T}^d} D_m\chi(x,m).H_p(x, D_x\chi(x,m))dm = f(x,m) \qquad \text{in } \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$$

Conclusion

We have established the long time behavior of the solution of the MFG system, of the master equation and of the discounted master equation.

Open problems :

- First order setting for the long time behavior.
- Convergence in the non-monotone setting.