Long time behavior of Mean Field Games

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Given a finite horizon $T > 0$, we consider the MFG system

\[
\begin{cases}
-\partial_t u^T - \Delta u^T + H(x, Du^T) = f(x, m^T(t)) & \text{in } (0, T) \times \mathbb{T}^d \\
\partial_t m^T - \Delta m^T - \text{div}(m^T D_p H(x, Du^T)) = 0 & \text{in } (0, T) \times \mathbb{T}^d \\
m^T(0, \cdot) = m_0, \ u^T(T, x) = g(x, m^T(T)) & \text{in } \mathbb{T}^d
\end{cases}
\]

where

- $u = u(t, x)$ and $m = m(t, x)$ are the unknown,
- $H = H(x, p) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is a smooth, unif. convex in $p$, Hamiltonian,
- $f, g : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ are “smooth” and monotone : for any $m, m' \in \mathcal{P}(\mathbb{T}^d)$,

\[\int_{\mathbb{T}^d} (f(x, m) - f(x, m'))d(m - m')(x) \geq 0, \int_{\mathbb{T}^d} (g(x, m) - g(x, m'))d(m - m')(x) \geq 0\]

$(\mathcal{P}(\mathbb{T}^d)$ = the set of Borel probability measures on $\mathbb{T}^d$)
- $m_0 \in \mathcal{P}(\mathbb{T}^d)$ is a smooth positive density

1. Introduced by Lasry-Lions and Huang-Caines-Malhamé to study optimal control problems with infinitely many controllers.
The limit problem

Let \((u^T, m^T)\) be the solution to

\[
\begin{align*}
-\partial_t u^T - \Delta u^T + H(x, Du^T) &= f(x, m^T(t)) & \text{in } (0, T) \times \mathbb{T}^d \\
\partial_t m^T - \Delta m^T - \text{div}(m^T D_p H(x, Du^T)) &= 0 & \text{in } (0, T) \times \mathbb{T}^d \\
m^T(0, \cdot) &= m_0, \quad u^T(T, x) = g(x, m^T(T)) & \text{in } \mathbb{T}^d
\end{align*}
\]

Study the limit as \(T \to +\infty\) of the pair \((u^T, m^T)\).

- **Motivation**: one numerically observes that the pair \((u^T, m^T)\) quickly stabilizes as \(T\) is large (when \(f\) and \(g\) are monotone).

- In contrast with similar problem for HJ equation, **initial** and **terminal** conditions for the system.
The limit problem

One expects that \((u^T, m^T)\) “converges” to the solution of the ergodic MFG problem

\[
\begin{cases}
\tilde{\lambda} - \Delta \tilde{u} + H(x, D\tilde{u}) = f(x, \bar{m}) & \text{in } \mathbb{T}^d \\
-\Delta \bar{m} - \text{div}(\bar{m}D_p H(x, D\tilde{u})) = 0 & \text{in } \mathbb{T}^d \\
\bar{m} \geq 0 & \text{in } \mathbb{T}^d, \\
\int_{\mathbb{T}^d} \bar{m} = 1
\end{cases}
\]

where now the unknown are \(\tilde{\lambda}, \tilde{u} = \tilde{u}(x)\) and \(\bar{m} = \bar{m}(x)\).
For the Fokker-Plank equation driven by a vector-field $V$:

$$\partial_t m^T - \Delta m^T - \text{div}(m^T V(x)) = 0 \quad \text{in} \ (0, T) \times \mathbb{T}^d$$

(exponential) convergence of $m^T$ to the ergodic measure is well-known.

For HJ equations:

$$\partial_t u - \Delta u + H(x, Du) = f(x) \quad \text{in} \ (0, +\infty) \times \mathbb{T}^d$$

- Ergodic constant $\lambda$, convergence of $u(T, \cdot)/T$: Lions-Papanicolaou-Varadhan, ...
- Convergence of $u(T, \cdot) - \bar{\lambda} T$: Fathi, Roquejoffre, Fathi-Siconolfi, Barles-Souganidis, ...

For the MFG system, convergence of $u^T/T$ and $m^T$:

- Lions (Cours in Collège de France)
- Gomes-Mohr-Souza (discrete setting)
- C.-Lasry-Lions-Porretta (viscous setting), C. (Hamilton-Jacobi)
- Turnpike property (Samuelson, Porretta-Zuazua, Trélat,...)

Long-time behavior not known so far.
The long time average

Let \((u^T, m^T)\) be the solution to

\[
\begin{aligned}
-\partial_t u^T - \Delta u^T + H(x, Du^T) &= f(x, m^T(t)) \quad \text{in } (0, T) \times \mathbb{T}^d \\
\partial_t m^T - \Delta m^T - \text{div}(m^T D_p H(x, Du^T)) &= 0 \quad \text{in } (0, T) \times \mathbb{T}^d \\
m^T(0, \cdot) = m_0, \quad u^T(T, x) = g(x, m^T(T)) \quad \text{in } \mathbb{T}^d 
\end{aligned}
\]

**Theorem** (C.-Lasry-Lions-Porretta ('13), C.-Porretta ('17))

There exists \(\gamma > 0\) and \(C\), independent of \(T\) and \(m_0\), such that

\[
\|D u^T(t, \cdot) - \bar{D} \bar{u}\|_{C^1} + \|m^T(t, \cdot) - \bar{m}\|_{\infty} \leq C \left( e^{-\gamma t} + e^{-\gamma (T-t)} \right). 
\]

Moreover, the map \((s, x) \to u^T(Ts, x)/T\) unif. converges to \((1 - s)\bar{\lambda}\) on \([0, 1] \times \mathbb{T}^d\).

The result does not say anything about the limit of \(u^T(t, x) - (T - t)\bar{\lambda}\).
The long time behavior

For $T > 0$ and $m_0 \in \mathcal{P}(\mathbb{T}^d)$, let $(u^T, m^T)$ be the solution to the MFG system:

\[
\begin{cases}
-\partial_t u^T - \Delta u^T + H(x, Du^T) = f(x, m^T(t)) & \text{in } (0, T) \times \mathbb{T}^d \\
\partial_t m^T - \Delta m^T - \text{div}(m^T D_p H(x, Du^T)) = 0 & \text{in } (0, T) \times \mathbb{T}^d \\
m^T(0, \cdot) = m_0, \ u^T(T, x) = g(x, m^T(T)) & \text{in } \mathbb{T}^d
\end{cases}
\]

**Theorem (C.-Porretta (’17))**

There exists a map $\chi : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ (with $\chi$ and $D_x \chi$ Lipschitz) and a constant $\bar{c}$, independent of $m_0$, such that, for any $t \geq 0$,

\[
\lim_{T \to \infty} u^T(t, x) - \bar{\lambda}(T - t) = \chi(x, m(t)) + \bar{c},
\]

where the convergence is uniform in $x$ and $m$ solves the McKean-Vlasov eq.

\[
\partial_t m - \Delta m - \text{div}(m H_p(x, D_x \chi(x, m))) = 0, \quad m(0) = m_0.
\]

Moreover, for any $\kappa \in (0, 1)$,

\[
\lim_{T \to \infty} u^T(\kappa T, x) - (1 - \delta)\bar{\lambda}T = \chi(x, \bar{m}) + \bar{c} = \bar{u}(x) + \bar{c}.
\]
Ideas of proof

We want to show that \( \lim_{T \to \infty} u^T(t, x) - \bar{\lambda}(T - t) = \chi(x, m(t)) + \bar{c} \).

- The proof relies on the master equation:
  \[
  \begin{cases}
    -\partial_t U - \Delta_x U + H(x, D_x U) - f(x, m) \\
    - \int_{\mathbb{T}^d} \text{div}_y [D_m U] \ dm(y) + \int_{\mathbb{T}^d} D_m U \cdot D_p H(y, D_x U) \ dm(y) = 0 \\
    U(0, x, m) = g(x, m) \quad \text{in } \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)
  \end{cases}
\]

  where \( U = U(t, x, m) : (-\infty, 0] \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R} \).

- As observed by Lasry-Lions, solutions of the MFG systems are “characteristics” of the master equation.
  \((\text{existence of solutions : Lasry-Lions, Buckdahn-Li-Peng-Rainer, Gangbo-Swiech, Chassagneux-Delarue-Crisan, C.-Lasry-Lions-Delarue})\).

- The main step is that there exists a constant \( \bar{c} \) such that
  \[
  \lim_{t \to -\infty} U(t, x, m) + \bar{\lambda} t = \chi(x, m) + \bar{c}
  \]

  where \( \chi \) is a weak solution to the master cell problem
  \[
  \bar{\lambda} - \Delta_x \chi(x, m) + H(x, D_x \chi(x, m)) - \int_{\mathbb{T}^d} \text{div}(D_m \chi(x, m)) dm \\
  + \int_{\mathbb{T}^d} D_m \chi(x, m) \cdot H_p(x, D_x \chi(x, m)) dm = f(x, m) \quad \text{in } \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)
  \]
Conclusion

We have established the long time behavior of the solution of the MFG system, of the master equation and of the discounted master equation.

Open problems:

- First order setting for the long time behavior.
- Convergence in the non-monotone setting.