Two-dimensional pseudo-gravity model: particles motion in a non-potential singular force field

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PDE & Probability Methods for Interactions
Inria (Sophia Antipolis)
30-31 March 2017
Model for an atomic cloud in a magneto-optical trap
Existence and Uniqueness Results
Particle Approximation
Conclusion and Further Work

This is joint work with J. Barré (University of Orléans and Institut Universitaire de France), T. Goudon (Laboratoire J.A. Dieudonné, University de Nice-Sophia Antipolis)

Julien Barré, D. C., Thierry Goudon, Two-dimensional pseudo-gravity model
https://arxiv.org/abs/1607.08909
Model for atomic clouds dominated by long-range attraction: In most experimental situations, the repulsive force is stronger.

**Goal:** make the attractive force dominant.

→ a self-trapped cloud \(\sim\) "a galaxy in the lab"

→ strong non potential forces.

→ In a very flat cloud, attraction is expected to dominate.

- A (very) simplified description

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot \left( F[\rho] \rho + D \nabla \rho \right), \quad F[\rho] = \int K(x - x') \rho(x') \, dx'
\]

\[
K(x - x') = \begin{pmatrix}
-\text{sgn}(x_1 - x'_1) \delta(x_2 - x'_2) \\
-\delta(x_1 - x'_1) \, \text{sgn}(x_2 - x'_2)
\end{pmatrix}
\]
\[ \frac{\partial \rho}{\partial t} = -\nabla \cdot \left( F[\rho] \rho + D \nabla \rho \right), \quad F[\rho] = \int K(x - x')\rho(x')dx' \]

\[ \nabla \cdot F[\rho] = -\rho \rightarrow \text{similar to Keller-Segel equation:} \]
\[ \frac{\partial \rho}{\partial t} = -\nabla \cdot \left( F[\rho] \rho + D \nabla \rho \right) ; \quad F[\rho] = -\nabla \phi, \quad \Delta \phi = \rho. \]

- Keller-Segel: existence of a global solution if \( D > D_c \); finite time singularity if \( D < D_c \). Should we expect something similar in this case?
- Not a gradient flow \( \rightarrow \) standard methods break down
- Singular interaction; possibility of strong concentrations \( \rightarrow \) may be difficult to simulate.
- Three approaches used:
  - Finite difference scheme, implicit for the diffusion (adapted from Saito, Suzuki 2005)
  - Finite volume method, designed for gradient flow structure (Carrillo, Chertock, Huang 2015); no global gradient flow structure, but a partial one along \( x_1 \) and \( x_2 \), \( \rightarrow \) splitting.
  - Particles approximation.
- When the concentration is not too strong, same results for all methods.
Strategy: Mollify the singular interaction.

Kernel $K \rightarrow$ kernel $K^\varepsilon = M^\varepsilon \ast K$, $M^\varepsilon =$ mollifier, scale $\varepsilon$

$\rightarrow$ defines the ”mollified solution” $\rho^\varepsilon$.

Theorem

Let $D > D_2$ (diffusion sufficiently strong); $D_2 =$ known constant. Let $(\rho^{(\varepsilon)})_{\varepsilon > 0}$ the solutions of the mollified PDE, and any time $T > 0$. Then, up to a subsequence

$\rho^{(\varepsilon)} \xrightarrow{\varepsilon \rightarrow 0} \rho$, in $L^p((0, T) \times \mathbb{R}^2)$,

where $\rho$ is a solution of the singular PDE.

1. The ”mollified solution” $\rho^\varepsilon$ exists for all time;
2. Uniform in $\varepsilon$ a priori estimates for $\rho^\varepsilon$: $L^p$ norms + moments;
3. Convergence of $\rho^\varepsilon$ by compactness towards $\rho$ (no rate of convergence);
4. Show $\rho$ is indeed the solution of the target equation.
Assume symmetry with respect to $x_1$ and $x_2$ axes:

$$
\rho(x_1, -x_2) = \rho(-x_1, x_2) = \rho(-x_1, -x_2) = \rho(x_1, x_2).
$$

Symmetric cloud $\rightarrow$ force towards the center, $\mathbf{F} \cdot \mathbf{x} \leq 0$

**Theorem**

Let $D > D_2$ (same $D_2$ = as before). Assume initial condition $\rho_0$ symmetric and with "gaussian-like" decay. Let $(\rho^{(\epsilon)})_{\epsilon > 0}$ the (symmetric) solutions of the mollified PDE with initial data $\rho_0$, and any time $T > 0$. Then there exist $C(\rho_0, T)$ and $\nu(\rho_0, T) < 1$ such that

$$
\sup_{t \in [0, T]} ||(\rho^{(\epsilon)} - \rho)(t)||_{L^1} \leq C(\rho_0, T)\epsilon^{\frac{1}{2}}\nu(\rho_0, T)
$$

where $\rho$ is the unique symmetric solution of the singular PDE.

**Remarks:**
- Uniqueness for the symmetric solution.
- There is a rate of convergence, which depends on $\rho_0$ and $T$.
- Exponential moments $\int e^{\lambda|x|} \rho(x, t) d\mathbf{x} < C_1 e^{C_2 \lambda^2 t}$.
- Exponential moments $\rightarrow$ a Cauchy property for $\rho^{(\epsilon)}$. 
**Theorem**

*Same hypotheses (symmetry, $D > D_c$). Let*

$$
\frac{1}{N} \sum_{j \neq i} K^\varepsilon (Z_i^{\varepsilon,N} - Z_j^{\varepsilon,N}) dt + \sqrt{2D} dB_i 
$$

*and $\rho$ be the solution of the PDE. Then*

$$
\sup_{t \in [0,T]} \mathbb{E} [W_1(\hat{\rho}_t^{\varepsilon,N}, \rho_t)] \leq \tilde{C} (\log(N))^{-\frac{1}{4}} \nu
$$

$W_1 = \text{Wasserstein distance}$, $\nu = \nu(\rho_0, T)$ *from previous theorem.*

**Remarks:**

- $W_1(\mu, \nu) = \sup \{ \left| \int \varphi(z) \mu(dz) - \int \varphi(z) \nu(dz) \right|, \|\varphi\|_{\text{Lip}} \leq 1 \}$,
- Good part: theorem with a rate of convergence, for a very singular interaction;
- Bad part: (sub-)logarithmic rate of convergence in $N$. 
Numerical Simulations

Plots of the quantity \( \left( \int |x_1 x_2| \rho \right) - \left( \int x_1 \rho \right) \left( \int x_2 \rho \right) \).

Parameters: \( D = 0.15 \); mollifier for the particles \( \varepsilon = 0.1 \).

Black=PDE simulation; purple: \( N = 2000 \), blue: \( N = 4000 \), red: \( N = 8000 \) (one realization).

→ reasonable agreement for \( N \ll \exp(1/\varepsilon^2) \)
The "shadow effect" or "Dalibard force" poses nice physical, mathematical and numerical questions.

Non potential, very singular interaction; yet: global existence for $D$ large enough,

Rate of convergence, through a *direct proof of a Cauchy property*.

Natural Particle Approximation

Open question: is there a critical $D$ for a finite time singularity?

Experiments in progress in Singapore (David Wilkowski, Vincent Manços). See a "blow up" is out of question; hope = observe some characteristic features of this strange interaction...