Two-dimensional pseudo-gravity model: particles motion in a non-potential singular force field

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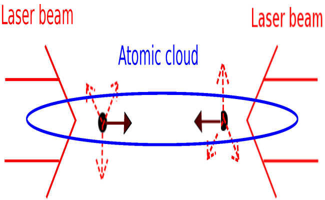
- Model for an atomic cloud in a magneto-optical trap
- Existence and Uniqueness Results
- Particle Approximation
- Conclusion and Further Work

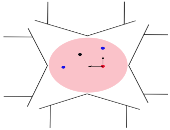
This is joint work with J. Barré (University of Orléans and Institut Universitaire de France), T. Goudon (Laboratoire J.A. Dieudonné, University de Nice-Sophia Antipolis)

Julien Barré, D. C., Thierry Goudon, Two-dimensional pseudo-gravity model https://arxiv.org/abs/1607.08909

Model for atomic clouds dominated by long-range attraction:
In most experimental situations, the repulsive force is stronger.

Goal: make the attractive force dominant.
$\rightarrow$ a self-trapped cloud $\sim$ ”a galaxy in the lab”
$\rightarrow$ strong non potential forces.
$\rightarrow$ In a very flat cloud, attraction is expected to dominate.

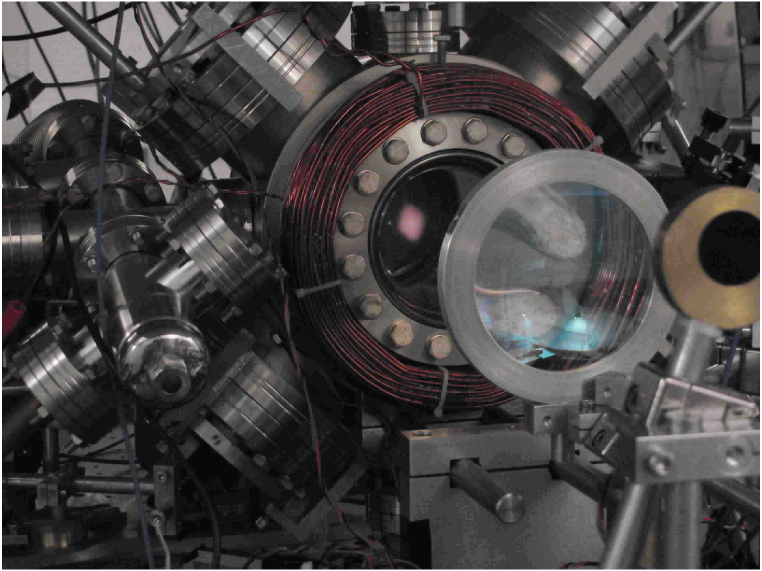




• A (very) simplified description

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \Big(\mathbf{F}[\rho]\rho + D\nabla\rho \Big) \ , \ \mathbf{F}[\rho] = \int \mathbf{K}(\mathbf{x}-\mathbf{x}')\rho(\mathbf{x}')d\mathbf{x}'$$

$$\mathbf{K}(\mathbf{x}-\mathbf{x}') = \begin{pmatrix} -\mathrm{sgn}(x_1 - x_1')\,\delta(x_2 - x_2') \\ -\delta(x_1 - x_1')\,\mathrm{sgn}(x_2 - x_2') \end{pmatrix}$$



$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \Big(\mathbf{F}[\rho]\rho + D\nabla\rho\Big)\ ,\ \mathbf{F}[\rho] = \int \mathbf{K}(\mathbf{x}-\mathbf{x}')\rho(\mathbf{x}')d\mathbf{x}'$$

- $\nabla \cdot \mathbf{F}[\rho] = -\rho \rightarrow$ similar to Keller-Segel equation:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \Big(\mathbf{F}[\rho]\rho + D\nabla\rho\Big)\ ;\ \mathbf{F}[\rho] = -\nabla\phi\ ,\ \Delta\phi = \rho.$$

- Keller-Segel: existence of a global solution if $D > D_c$; finite time singularity if $D < D_c$. Should we expect something similar in this case?
- Not a gradient flow $\rightarrow$ standard methods break down
- Singular interaction; possibility of strong concentrations $\rightarrow$ may be difficult to simulate.
- Three approaches used:
  - Finite difference scheme, implicit for the diffusion (adapted from Saito, Suzuki 2005)
  - Finite volume method, designed for gradient flow structure (Carrillo, Chertock, Huang 2015); no global gradient flow structure, but a partial one along x_1 and x_2, $\rightarrow$ splitting.
  - Particles approximation.
- When the concentration is not too strong, same results for all methods.

Strategy: Mollify the singular interaction.

$$\text{Kernel } \mathbf{K} \;\rightarrow\; \text{kernel } \mathbf{K}^{\varepsilon} = M^{\varepsilon} \star \mathbf{K}\,,\;\; M^{\varepsilon} = \text{mollifier},\; \text{scale } \varepsilon$$

$\rightarrow$ defines the "mollified solution" ρ^{ε}.

Theorem

Let $D > D_2$ (diffusion sufficiently strong); $D_2 =$ known constant. Let $\left(\rho^{(\varepsilon)}\right)_{\varepsilon>0}$ the solutions of the mollified PDE, and any time $T > 0$. Then, up to a subsequence

$$\rho^{(\varepsilon)} \underset{\varepsilon\to 0}{\longrightarrow} \rho\,,\;\text{in } L^p((0,T)\times\mathbb{R}^2),$$

where ρ is a solution of the singular PDE.

1. The "mollified solution" ρ^{ε} exists for all time;
2. Uniform in ε a priori estimates for ρ^{ε}: L^p norms + moments;
3. Convergence of ρ^{ε} by compactness towards ρ (no rate of convergence);
4. Show ρ is indeed the solution of the target equation.

Assume symmetry with respect to x_1 and x_2 axes:

$$\rho(x_1, -x_2) = \rho(-x_1, x_2) = \rho(-x_1, -x_2) = \rho(x_1, x_2).$$

Symmetric cloud $\rightarrow$ force towards the center, $\mathbf{F} \cdot \mathbf{x} \leq 0$

Theorem

Let $D > D_2$ (same $D_2 =$ as before). Assume initial condition ρ_0 symmetric and with "gaussian-like" decay. Let $\left(\rho^{(\varepsilon)}\right)_{\varepsilon>0}$ the (symmetric) solutions of the mollified PDE with initial data ρ_0, and any time $T > 0$. Then there exist $C(\rho_0, T)$ and $\nu(\rho_0, T) < 1$ such that

$$\sup_{t\in[0,T]} ||(\rho^{(\varepsilon)} - \rho)(t)||_{L^1} \leq C(\rho_0, T)\varepsilon^{\frac{1}{2}\nu(\rho_0,T)}$$

where ρ is the unique symmetric solution of the singular PDE.

Remarks:

- Uniqueness for the symmetric solution.
- There is a rate of convergence, which depends on ρ_0 and T.
- Exponential moments $\int e^{\lambda|\mathbf{x}|}\rho(\mathbf{x}, t)d\mathbf{x} < C_1 e^{C_2\lambda^2 t}$.
- Exponential moments $\rightarrow$ **a Cauchy property for** $\rho^{(\varepsilon)}$**.**

Theorem

Same hypotheses (symmetry, $D > D_c$). Let

$$dZ_i^{\varepsilon,N} = \frac{1}{N}\sum_{j\neq i} \mathbf{K}^{\varepsilon}\left(Z_i^{\varepsilon,N} - Z_j^{\varepsilon,N}\right)dt + \sqrt{2D}dB_i \qquad \widehat{\rho}_t^{(\varepsilon,N)} \equiv \frac{1}{N}\sum_i \delta_{Z_i^{\varepsilon,N}}$$

and ρ be the solution of the PDE. Then

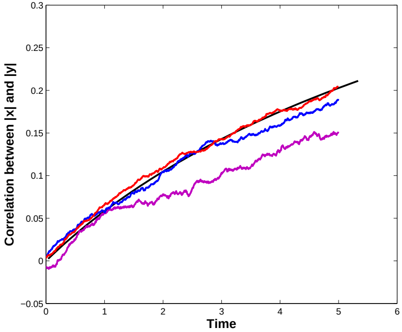
$$\sup_{t\in[0,T]} \mathbb{E}[W_1(\widehat{\rho}_t^{(\varepsilon,N)}, \rho_t)] \leq \tilde{C}\left(\log(N)\right)^{-\frac{1}{4}\nu}$$

$W_1 =$ Wasserstein distance, $\nu = \nu(\rho_0, T)$ from previous theorem.

Remarks:

- $W_1(\mu,\nu) = \sup\left\{\left|\int \varphi(z)\mu(\mathrm{d}z) - \int \varphi(z)\nu(\mathrm{d}z)\right|, \|\varphi\|_{\mathrm{Lip}} \leq 1\right\}$,
- Good part: theorem with a rate of convergence, for a very singular interaction;
- Bad part: (sub-)logarithmic rate of convergence in N.

Plots of the quantity $\left(\int |x_1 x_2|\rho\right) - \left(\int |x_1|\rho\right)\left(\int |x_2|\rho\right)$.
Parameters: $D = 0.15$; mollifier for the particles $\varepsilon = 0.1$.
Black=PDE simulation; purple: $N = 2000$, blue: $N = 4000$, red: $N = 8000$ (one realization).



$\rightarrow$ reasonable agreement for $N \ll \exp(1/\varepsilon^2)$

1. The ”shadow effect” or ”Dalibard force” poses nice physical, mathematical and numerical questions.
2. Non potential, very singular interaction; yet: global existence for D large enough,
3. Rate of convergence, through a *direct proof of a Cauchy property*.
4. Natural Particle Approximation
5. Open question: is there a critical D for a finite time singularity?
6. Experiments in progress in Singapore (David Wilkowski, Vincent Mançois) . See a ”blow up” is out of question; hope = observe some characteristic features of this strange interaction...