Rough Mean Field Equations

PDE and Probability Methods

François Delarue (Nice – J.-A. Dieudonné)

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Joint work with I. Bailleul (Rennes) and R. Catellier (Nice)

Background

- General structure of a McKean Vlasov equation

$$dX_t = b(t, X_t, \mathcal{L}(X_t))dt + \sigma(t, X_t, \mathcal{L}(X_t))dB_t$$

 ◦ use the probabilistic structure of the noise to define the stochastic integration with respect to $(B_t)_t$

 ◦ standard example is Wiener process

- Understood as the asymptotic version of a mean interacting particle system

$$dX_t^i = b(t, X_t^i, \bar{\mu}_t^N) + \sigma(t, X_t^i, \bar{\mu}_t^N)dB_t^i$$

 ◦ i is an integer in $\{1, \cdots, N\}$

 ◦ $(B_t^i)_t$ are independent copies of $(B_t)_t$

 ◦ $\bar{\mu}_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}$ is the empirical distribution

- What about rough signals?

 ◦ $\perp\!\!\!\perp$ copies of a rough signal “à la Lyons” (Cass Lyons, 13)

Motivation

- Theory for general signals like Gaussian (non-Brownian) processes
- Have continuity of the Itô-Lyons map *input* ↦ *output*
  - in the asymptotic regime

$$input = \mathcal{L}((B_t)_t), \quad output = \mathcal{L}((X_t)_t)$$

  - in the particle system

$$input = \frac{1}{N}\sum_{i=1}^{N} \delta_{(B_t^i)_t(\omega)}, \quad output = \frac{1}{N}\sum_{i=1}^{N} \delta_{(X_t^i)_t(\omega)}$$

but for a fixed ω!!

- Ask for a diagram
  - propagation chaos
  - LDP

$$\begin{array}{ccc}
\frac{1}{N}\sum_{i=1}^{N} \delta_{(B_t^i)_t(\omega)} & \xrightarrow{\text{LLN}} & \mathcal{L}((B_t)_t) \\
\big\downarrow \text{Itô-Lyons} & & \text{Itô-Lyons} \big\downarrow \\
\frac{1}{N}\sum_{i=1}^{N} \delta_{(X_t^i)_t(\omega)} & \longrightarrow & \mathcal{L}((X_t)_t)
\end{array}$$

Rough signal

- Rough trajectory $W(\omega)$ with same regularity as a Brownian path

$$|W_t(\omega) - W_s(\omega)| \leq C(\omega)|t-s|^\alpha, \quad \alpha \in (1/3, 1/2]$$

∘ assume that we can define an integral with respect to W and the "iterated integral" of W

$$\mathbb{W}_{s,t}(\omega) = \int_s^t (W_r(\omega) - W_s(\omega)) \otimes dW_r(\omega)$$

∘ if W is $1d$ $\Rightarrow$ natural candidate is

$$\mathbb{W}_{s,t}(\omega) = \tfrac{1}{2}(W_t(\omega) - W_s(\omega))^2 \leq C|t-s|^{2\alpha}$$

∘ if dim greater than 2, "crossed iterated integrals" may not exist $\Rightarrow$ probabilistic structure provides a construction (Stratonovich, Itô...)

- McKV involve infinitely many rough trajectories even if $d = 1$!!

$$\sigma(X_t^i, \bar{\mu}_t^N) dB_t^i = \text{"}\sigma\text{"}(X_t^1, \cdots, X_t^N) dB_t^i$$

∘ X^j involves B^j for $j \neq i$!

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• McKV involve infinitely many rough trajectories even if $d = 1$!!

∘ ⇒ requires a non-trivial rough structure (so far $\sigma = \sigma(x)$, Cass-Lyons, Deuschel et al...)

Rough integral

- Once $\boldsymbol{W}(\omega) = (W(\omega), \mathbb{W}(\omega))$ is given, one may define an integral for curves that behave like $W(\omega)$

  - ∘ controlled trajectory X

$$X_t(\omega) - X_s(\omega) = \delta_x X_s(\omega)(W_t(\omega) - W_s(\omega)) + R_{s,t}(\omega)$$

$$|\delta_x X_t(\omega) - \delta_x X_s(\omega)| \leq C^X(\omega)|t-s|^\alpha, \quad |R_{s,t}(\omega)| \leq C^X(\omega)|t-s|^{2\alpha}$$

  - ∘ rough integral ((t_i) mesh of $[s,t]$)

$$\int_s^t X_r(\omega) d\boldsymbol{W}_r(\omega) \approx \sum_i X_{t_i}(\omega)(W_{t_{i+1}} - W_{t_i})(\omega) + \sum_i \delta_x X_{t_i}(\omega) \mathbb{W}_{t_i, t_{i+1}}(\omega)$$

- Back to our case $\rightsquigarrow$ if $\sigma = \sigma(x)$ smooth, define

$$\int_s^t \sigma(X_r(\omega)) d\boldsymbol{W}_r(\omega)$$

by expanding

$$\begin{aligned}\sigma(X_t(\omega)) &= \sigma(X_s(\omega)) + \sigma'(X_s(\omega))(X_t - X_s)(\omega) + R^X_{s,t}(\omega) \\ &= \sigma(X_s(\omega)) + \sigma'(X_s(\omega))\delta_x X_s(\omega)(W_t - W_s)(\omega) + R^X_{s,t}(\omega)\end{aligned}$$

Rough integral

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  - rough integral ((t_i) mesh of $[s,t]$)

$$\int_s^t X_r(\omega) d\boldsymbol{W}_r(\omega) \approx \sum_i X_{t_i}(\omega)(W_{t_{i+1}} - W_{t_i})(\omega) + \sum_i \delta_x X_{t_i}(\omega)\mathbb{W}_{t_i,t_{i+1}}(\omega)$$

- Back to our case $\rightsquigarrow$ first step is to define

$$\int_s^t \sigma(X_r(\omega), \mathcal{L}(X_r)) d\boldsymbol{W}_r(\omega)$$

  - requires to expand $\sigma(X_r(\omega), \mathcal{L}(X_r))$ including the measure

Wasserstein derivative

- Lions' approach for differentiating on $\mathcal{P}_2(\mathbb{R})$
- Given $\mathcal{U} : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$
- Lift of $\mathcal{U}$

$$\hat{\mathcal{U}} : L^2(\Omega, \mathbb{Q}) \ni X \mapsto \mathcal{U}(\mathcal{L}(X))$$

  - $\mathcal{U}$ differentiable if $\hat{\mathcal{U}}$ Fréchet differentiable
- Derivative of $\mathcal{U}$
  - Fréchet of $\hat{\mathcal{U}}$

$$D\hat{\mathcal{U}}(X) = \partial_\mu \mathcal{U}(\mu)(X), \quad \partial_\mu \mathcal{U}(\mu) : \mathbb{R} \ni x \mapsto \partial_\mu \mathcal{U}(\mu)(x) \quad \mu = \mathcal{L}(X)$$

  - derivative of $\mathcal{U}$ in $\mu \rightsquigarrow \partial_\mu \mathcal{U}(\mu) \in L^2(\mathbb{R}, \mu; \mathbb{R})$
- Finite-dimensional projection

$$\partial_{x_i}\Big[\mathcal{U}\Big(\frac{1}{N}\sum_{j=1}^{N}\delta_{x_j}\Big)\Big] = \frac{1}{N}\partial_\mu \mathcal{U}\Big(\frac{1}{N}\sum_{j=1}^{N}\delta_{x_j}\Big)(x_i), \quad x_1, \ldots, x_N \in \mathbb{R}$$

Wasserstein derivative

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- X and X' two random variables

$$\begin{aligned}\mathcal{U}(\mathcal{L}(X')) - \mathcal{U}(\mathcal{L}(X)) &= \mathbb{E}^{\mathbb{Q}}[\partial_\mu \mathcal{U}(\mathcal{L}(X))(X(\cdot))(X' - X)(\cdot)] + \cdots \\ &= \int \partial_\mu \mathcal{U}(\mathcal{L}(X))(x) \times (x' - x) d\mathcal{L}(X, X')(x, x') + \cdots\end{aligned}$$

Extended rough structure

- Expand $\sigma_t = \sigma(X_t(\omega), \mathcal{L}(X_t))$

$$\begin{aligned}\sigma_t - \sigma_s &= \partial_x\sigma(X_s(\omega), \mathcal{L}(X_s))\partial_x X_s(\omega)(W_t(\omega) - W_s(\omega)) \\ &\quad + \mathbb{E}[\partial_\mu\sigma(X_s(\omega), \mathcal{L}(X_s))(X_s(\cdot))\partial_x X_s(\cdot)(W_s - W_t)(\cdot)] + R^\sigma_{s,t}(\omega) \\ &= [\delta_x\sigma]_s(\omega)(W_t(\omega) - W_s(\omega)) \\ &\quad + \mathbb{E}[[\delta_\mu\sigma]_s(\omega, \cdot)(W_t - W_s)(\cdot)] + R^\sigma_{s,t}(\omega) \\ &= [\delta_x\sigma]_s(\omega)(W_t(\omega) - W_s(\omega)) \\ &\quad + \int_\Omega [\delta_\mu\sigma]_s(\omega, \omega')(W_t - W_s)(\omega')d\mathbb{P}(\omega') + R^\sigma_{s,t}(\omega)\end{aligned}$$

◦ regularity on the derivatives of σ (need second order derivatives, but Fréchet is too demanding)

Extended rough structure

- Expand $\sigma_t = \sigma(X_t(\omega), \mathcal{L}(X_t))$

$$\sigma_t - \sigma_s = [\delta_x \sigma]_s(\omega)(W_t(\omega) - W_s(\omega)) + \int [\delta_\mu \sigma]_s(\omega, \omega')(W_t - W_s)(\omega') d\mathbb{P}(\omega') + R^\sigma_{s,t}(\omega)$$

- By analogy with above, need to define another cross-integral

$$\mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\omega, \omega') = \int_s^t (W'_r - W'_s)(\omega') dW_r(\omega)$$

  - W' independent copy of W on a copy Ω' of Ω
  - pay attention!!! $\mathbb{P}$ refers to the law in the mean-field interaction
  - asymptotic setting $\rightsquigarrow$ equip Ω' with $\mathcal{L}((B_t)_t)$ (works for Gaussian processes of dimension 2)
  - particle system $\rightsquigarrow$ equip Ω' with $\frac{1}{N}\sum_{j=1}^N \delta_{B^j_t(\omega)}$ $\Rightarrow$ need to define

$$\Big(\int_s^t (B^j_r - B^j_s) dB^i_r\Big)(\omega)$$

Extended rough structure

- Expand $\sigma_t = \sigma(X_t(\omega), \mathcal{L}(X_t))$

$$\sigma_t - \sigma_s = [\delta_x \sigma]_s(\omega)(W_t(\omega) - W_s(\omega))$$
$$+ \int [\delta_\mu \sigma]_s(\omega, \omega')(W_t - W_s)(\omega') d\mathbb{P}(\omega') + R^{\sigma}_{s,t}(\omega)$$

- By analogy with above, need to define another cross-integral

$$\mathbb{W}^{\perp\!\!\!\perp}_{s,t}(\omega, \omega') = \int_s^t (W'_r - W'_s)(\omega') dW_r(\omega)$$

  - W' independent copy of W on a copy Ω' of Ω

- Rough integral should be

$$\int_s^t \sigma_r(\omega) d\mathbf{W}_s = \sum_i \sigma_{t_i}(\omega)(W_{t_{i+1}}(\omega) - W_{t_i}(\omega))$$
$$+ \sum_i \partial_x \sigma_{t_i}(\omega) \mathbb{W}_{t_i, t_{i+1}}(\omega) + \sum_i \mathbb{E}[\partial_\mu \sigma_{t_i}(\omega, \cdot) \mathbb{W}^{\perp\!\!\!\perp}_{t_i, t_{i+1}}(\omega, \cdot)]$$

Solving the equation

- Search for a fixed point

$$\Gamma : \boldsymbol{X} = (X_\cdot(\omega), \delta_x X_\cdot(\omega), R^X_\cdot)_\omega \mapsto \left(X_0 + \int_0^\cdot \sigma_r(\omega,\cdot)d\boldsymbol{W}_r, \sigma_\cdot(\omega,\cdot), \cdots\right)_\omega$$

  - suffices to work with T small

- When $\sigma = \sigma(x)$ strategy is to localize on the variation of $W(\omega)$

$$w(0,T,\omega) = \frac{1}{\alpha}\text{-var}_{[0,T]}\big([W(\omega)]\big) + \frac{1}{2\alpha}\text{-var}_{[0,T]}\big(\mathbb{W}(\omega)\big)$$

where

$$\frac{1}{\alpha}\text{-var}_{[0,T]} = \sup_{(t_i)} \sum_i |W_{t_{i+1}} - W_{t_i}|^{\frac{1}{\alpha}}$$

  - here $\rightsquigarrow$ no more possible to do that because of McKV!
  - need a variant of Gronwall (Cass Litterer Lyons)

Solving the equation

- Search for a fixed point

$$\Gamma : \boldsymbol{X} = (X_\cdot(\omega), \delta_x X_\cdot(\omega), R^X_\cdot)_\omega \mapsto \left(X_0 + \int_0^\cdot \sigma_r(\omega,\cdot)d\boldsymbol{W}_r, \sigma_\cdot(\omega,\cdot), \cdots\right)_\omega$$

- Find a norm $\|\cdot\|_\omega$ on $(X_\cdot(\omega), \delta_x X_\cdot(\omega), R^X_\cdot(\omega))$

$$\|\Gamma(\boldsymbol{X})(\omega) - \Gamma(\boldsymbol{X}')(\omega)\|_\omega \leq \rho C^{N(\omega)}\left(\int_\Omega \|\boldsymbol{X}(\omega) - \boldsymbol{X}'(\omega)\|^p_\omega d\mathbb{P}(\omega)\right)^{1/p}$$

  - where $\rho < 1$ and $\int_\Omega C^{pN(\omega)} d\mathbb{P}(\omega) \to 1$ as T tends to 0

$$\|\boldsymbol{X}(\omega)\|_\omega = |(X_0, \delta_x X_0)(\omega)| + \sup_{[s,t]\subset[0,T]}\left(\frac{|\delta_x X_t(\omega) - \partial_x X_s(\omega)|}{w(s,t,\omega)^\alpha} + \frac{|R_{s,t}(\omega)|}{w(s,t,\omega)^{2\alpha}}\right)$$

  - $w(s,t,\omega) \sim \frac{1}{\alpha}\text{-var}_{[s,t]} W(\omega) + \frac{1}{2\alpha}\text{-var}_{[s,t]}(\mathbb{W}(\omega), \mathbb{W}^\perp(\omega,\cdot))$

- $N(\omega)$ s.t. $(t_i)_{1\leq i\leq N(\omega)}$ with $w(t_i, t_{i+1}, \omega) = \epsilon < 1$ and $t_{N(\omega)} \geq T$

Continuity and propagation of chaos

• Kind of statement: If we can control accordingly the tails of the variations of W, $\mathbb{W}$ and $\mathbb{W}^{\perp}$, then existence and uniqueness

• Continuity of the law of the output with respect to the law of $(W, \mathbb{W}, \mathbb{W}^{\perp})$.

◦ example: Gaussian processes with Hölder covariance of Hölder exponent $> 2/3$

• Revisit propagation of chaos

◦ for N particle system, the law of the triplet takes the form

$$\frac{1}{N^2} \sum_{i,j=1}^{N} \delta_{B^i(\omega), \mathbb{B}^{i,i}(\omega), \mathbb{B}^{i,j}(\omega)}$$

where

$$\mathbb{B}^{i,j} = \int B^j dB^i$$

◦ converges to the law of $(B, \mathbb{B}, \mathbb{B}^{\perp})$