Describing the thermodynamic limit of networks of interacting neurons

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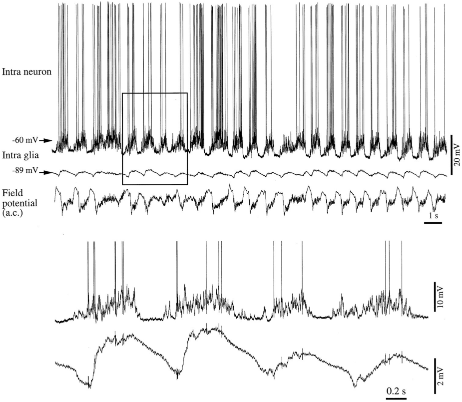
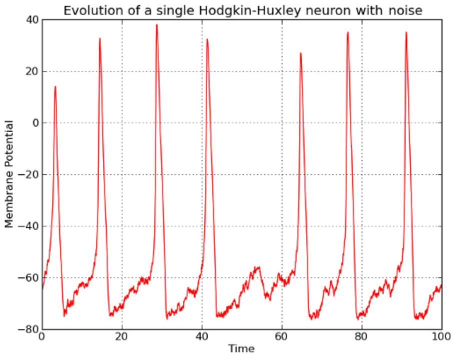
Types of neuronal models

Recording of a real neuron



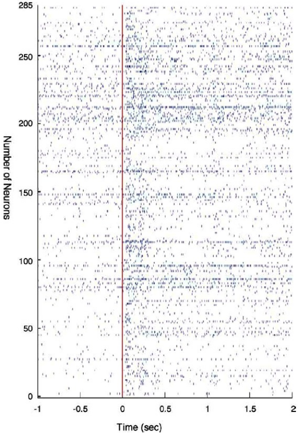
Types of neuronal models

Hodgkin-Huxley model



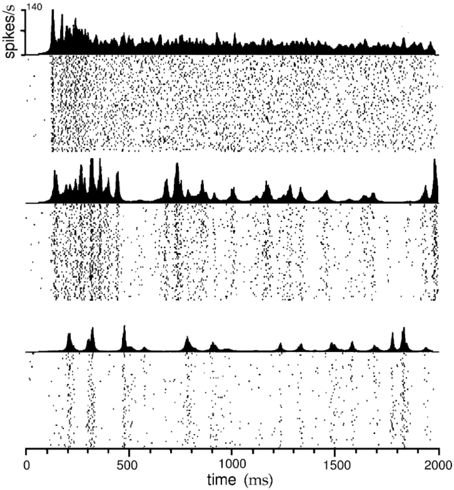
Types of neuronal models

Focusing on the spikes

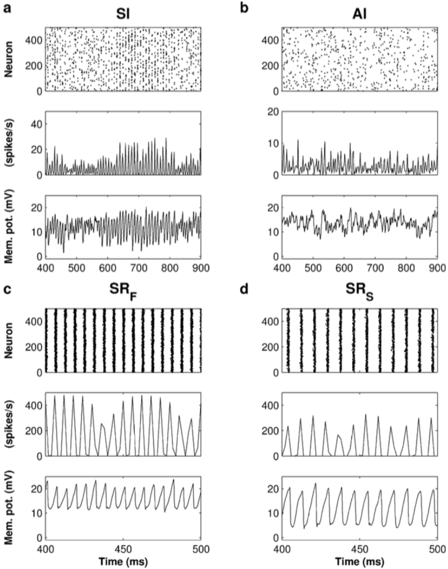


Types of neuronal models

Focusing on the firing rate



The question



- Find concise mathematical descriptions of large networks of neurons

This talk

- Fully connected networks of rate neurons
- Random synaptic weights
- Annealed results

The mathematical model

- Intrinsic dynamics:

$$\mathcal{S} := \begin{cases} dV_t & = & -\alpha V_t dt + \sigma dB_t, \; 0 \leq t \leq T \\ \text{Law of } V_0 & = & \mu_0, \end{cases}$$

- There is a unique strong solution to $\mathcal{S}$ (Ornstein-Uhlenbeck process):

$$V_t = \exp(-\alpha t) V_0 + \sigma \int_0^t \exp(\alpha(s-t)) dB_s$$

- Note P its law on the set $\mathcal{T} := \mathcal{C}([0, T]; \mathbb{R})$ of trajectories

The mathematical model

- N neurons, $N = 2n + 1$; completely connected network
- Coupled dynamics

$$\mathcal{S}(J^N) :=$$

$$\begin{cases} dV_t^i & = & (-\alpha V_t^i + \sum_{j=1}^N J_{ij}^N f(V_t^j))dt + \sigma dB_t^i \\ \text{Law of} & & \\ V_N(0) & = & (V_0^1, \cdots, V_0^N) = \mu_0^{\otimes N} \end{cases}$$

$i \in I_n := [-n, \cdots, n]$.

- f is bounded, Lipschitz continuous (usually a sigmoid), defining the firing rate
- B^i: independent Brownians: intrinsic noise on the membrane potentials

The mathematical model

- There is a unique strong solution to $\mathcal{S}(J^N)$
- Note $P(J^N)$ its law on the set $\mathcal{T}^N$ of N-tuples of trajectories.

Modeling the synaptic weights

- J_{ij}^N: stationary Gaussian field: random synaptic weights

$$\mathbb{E}[J_{ij}^N] = \frac{\overline{J}}{N}$$

$$cov(J_{ij}^N J_{kl}^N) = \frac{\Lambda(k-i, l-j)}{N}$$

- $\Lambda(k, l)$ is a covariance function.
- Analogy with random media

Consequences

- $P(J^N)$ is a random law on $\mathcal{T}^N$
- Consider the law $P^{\otimes N}$ of N independent uncoupled neurons
- Girsanov theorem allows us to compare the law of the solution to the coupled system, $P(J^N)$, with the law of the uncoupled system, $P^{\otimes N}$:

$$\frac{dP(J^N)}{dP^{\otimes N}} = \exp\left\{ \sum_{i \in I_n} \frac{1}{\sigma} \int_0^T \left(\sum_{j \in I_n} J_{ij}^N f(V_t^j) \right) dB_t^i - \frac{1}{2\sigma^2} \int_0^T \left(\sum_{j \in I_n} J_{ij}^N f(V_t^j) \right)^2 dt \right\}$$

Uncorrelated case

- Consider the empirical measure:

$$\hat{\mu}_u^N(V_N) = \frac{1}{N} \sum_{i \in I_n} \delta_{V^i},$$

$V_N = (V^{-n}, \cdots, V^n)$

- It defines the mapping

$$\hat{\mu}_u^N : \mathcal{T}^N \to \mathcal{P}(\mathcal{T})$$

Correlated case

- Consider the empirical measure

$$\hat{\mu}_c^N(V_N) = \frac{1}{N} \sum_{i \in I_n} \delta_{S^i(V_{N,p})},$$

  a probability measure on $\mathcal{T}^{\mathbb{Z}}$.
- $V_{N,p}$ is the periodic extension of the finite sequence of trajectories $V_N = (V^{-n}, \cdots, V^n)$.
- S is the shift operator acting on elements of $\mathcal{T}^{\mathbb{Z}}$.
- It defines the mapping

$$\hat{\mu}_c^N(V_N) : \mathcal{T}^N \to \mathcal{P}_S(\mathcal{T}^{\mathbb{Z}})$$

- We are interested in the laws of $\hat{\mu}_u^N$ and $\hat{\mu}_c^N$ under $P(J^N)$
- Define

$$Q^N = \int_\Omega P(J^N(\omega))\, d\omega,$$

  the average of $P(J^N)$ w.r.t. to the "random medium", i.e. the synaptic weights.
- We study the law of $\hat{\mu}_u^N$ and $\hat{\mu}_c^N$ under Q^N: annealed results.

The strategy

- Consider the law Π_u^N of $\hat{\mu}_u^N$ under Q^N: it is a probability measure on $\mathcal{P}(\mathcal{T})$:

$$\Pi_u^N(B) = \left(Q^N \circ (\hat{\mu}_u^N)^{-1}\right)(B) = Q^N(\hat{\mu}_u^N \in B),$$

  B measurable set of $\mathcal{P}(\mathcal{T})$

- Consider the law Π_c^N of $\hat{\mu}_c^N$ under Q^N: it is a probability measure on $\mathcal{P}(\mathcal{T}^{\mathbb{Z}})$:

$$\Pi_c^N(B) = Q^N(\hat{\mu}_c^N \in B),$$

  B measurable set of $\mathcal{P}(\mathcal{T}^{\mathbb{Z}})$

The strategy

- Establish a Large Deviation Principle for the sequences of probability measures $(\Pi_u^N)_{N\geq 1}$ and $(\Pi_c^N)_{N\geq 1}$, i.e.
- Design a rate function (non-negative lower semi-continuous) H_u (resp. H_c) on $\mathcal{P}(\mathcal{T})$ (resp. $\mathcal{P}(\mathcal{T}^{\mathbb{Z}})$)
- The intuitive meaning of H is the following

$$Q^N(\hat{\mu}^N \simeq Q) \simeq e^{-NH(Q)}$$

- The measures $\hat{\mu}^N$ concentrate on the measures Q such that $H(Q) = 0$.
- If H reaches 0 at a single measure Q then Π^N converges in law toward the Dirac mass δ_Q

Minimum of H_u

By adapting the results of Ben Arous and Guionnet [BAG95] and of Moynot and Samuelides [MS02] one obtains:

Theorem

$$H_u(\mu) = I^{(2)}(\mu; P) - \Gamma_u(\mu),$$

where $I^{(2)}(\mu; P)$ *is the relative entropy of* μ *w.r.t.* P *and* Γ_u *is defined by*

$$\frac{dQ^N}{dP^{\otimes N}} = e^{N\Gamma_u(\hat{\mu}^N)}$$

H_u *achieves its minimum at a unique point* μ_u *of* $\mathcal{P}(\mathcal{T})$.

Minimum of H_u

and

Theorem

μ_u is the law of the solution to a linear non-Markovian stochastic system.

Annealed results

Two main results:

Theorem (1)

The law of the empirical measure $\hat{\mu}_u^N$ under Q^N converges weakly to δ_{μ_u}

This means that

$$\forall F \in C_b(\mathcal{P}(\mathcal{T}))$$

$$\lim_{N\to\infty} \int_\Omega \left(\int_{\mathcal{T}^N} F\left(\frac{1}{N} \sum_1^N \delta_{v^i} \right) P(J^N(\omega))(dv_N) \right) d\gamma(\omega) = F(\mu_u)$$

Annealed results

Theorem (2)

Q^N is μ_u-chaotic.

i.e. for all $m \geq 2$ and f_i, $i = 1, \ldots, m$ in $C_b(\mathcal{T})$

$$\lim_{N \to \infty} \int_{\mathcal{T}^N} f_1(v^1) \cdots f_m(v^m) \, dQ^N(v^1, \cdots, v^N) = \prod_{i=1}^{m} \int_{\mathcal{T}} f_i(v) \, d\mu_u(v)$$

"In the thermodynamic limit ($N \to \infty$) and on average, the neurons in any finite-size group become independent. One observes the propagation of chaos. The neurons become asynchronous."

Joint work with James Maclaurin and Etienne Tanré

1. Note that the sequence $\Pi_0^N = P^{\otimes N} \circ (\hat{\mu}_c^N)^{-1}$ satisfies the LDP with good rate function

$$I^{(3)}(\mu; P^{\mathbb{Z}}) = \lim_{N\to\infty} \frac{1}{N} I^{(2)}(\mu_N; P^{\otimes N})$$

2. Show that there exists a sequence Ψ_m of continuous functions $\mathcal{P}_S(\mathcal{T}^{\mathbb{Z}}) \to \mathcal{P}_S(\mathcal{T}^{\mathbb{Z}})$ and a measurable map $\Psi : \mathcal{P}_S(\mathcal{T}^{\mathbb{Z}}) \to \mathcal{P}_S(\mathcal{T}^{\mathbb{Z}})$ such that for every $\alpha < \infty$

$$\limsup_{m\to\infty} \sup_{\mu: I^{(3)}(\mu)\leq\alpha} D_T(\Psi_m(\mu), \Psi(\mu)) = 0$$

Joint work with James Maclaurin and Etienne Tanré

1. Show that the family $\Pi_0^N \circ \Psi_m^{-1}$ is an exponentially good approximation of the family Π_c^N,
2. and conclude that Π_c^N satisfies the LDP with good rate function

$$H_c(\mu) = \inf \left\{ I^{(3)}(\nu) : \mu = \Psi(\nu) \right\}$$

Definition of Ψ_m

- Note that

$$\left.\frac{dQ^N}{dP^{\otimes N}}\right|_{\mathcal{F}_t} = \exp\left(\sum_{j\in I_n}\int_0^t \theta_s^j dB_s^j - \frac{1}{2}\sum_{j\in I_n}\int_0^t \left(\theta_s^j\right)^2 ds\right)$$

where

$$\theta_t^j = \frac{1}{\sigma}\mathfrak{c}_{\hat{\mu}_c^N(V_N)}(t) + \frac{1}{\sigma^2}\mathbb{E}^{\tilde{\gamma}_t^{\hat{\mu}_c^N(V_N)}}\left[\sum_{k\in I_n} G_t^j \int_0^t G_s^k dB_s^k\right]$$

Definition of Ψ_m

- Prove that the SDE

$$Z_t^j = B_t^j + \int_0^t c_{\hat{\mu}_c^N(Z)}(s)ds + \sigma^{-2} \sum_{k \in I_n} \int_0^t \mathbb{E}^{\tilde{\gamma}_t^{\hat{\mu}_c^N(Z)}} \left[G_s^j \int_0^s G_u^k dZ_u^k \right] ds,$$

  $j \in I_n$, is well-posed in $\mathcal{T}^N$ and that the law of $\hat{\mu}_c^N(Z)$ is Π_c^N.
- Construct the continuous function $\varphi_m : \mathcal{T}^{\mathbb{Z}} \times \mathcal{P}_S(\mathcal{T}^{\mathbb{Z}}) \to \mathcal{T}^{\mathbb{Z}}$ by time-discretizing this equation.
- Construct the continuous function $\Psi_m : \mathcal{P}_S(\mathcal{T}^{\mathbb{Z}}) \to \mathcal{P}_S(\mathcal{T}^{\mathbb{Z}})$ by a fixed-point argument as

$$\Psi_m(\mu) = \nu \text{ such that } \nu = \mu \circ (\varphi_m(\cdot, \nu))^{-1}$$

Minimum of H_c

Theorem (O.F., J. Maclaurin, E. Tanré)

H_c achieves its minimum at a unique point μ_c of $\mathcal{P}_S(\mathcal{T}^{\mathbb{Z}})$.

and

Theorem (O.F., J. Maclaurin, E. Tanré)

μ_c is the law of the solution to an infinite dimensional linear non-Markovian stochastic system, hence it is a Gaussian measure (in $\mathcal{P}_S(\mathcal{T}^{\mathbb{Z}})$) if the initial condition is Gaussian.

Summary

- We have started the analyzis of the thermodynamic limit of completely connected networks of rate neurons in the case of uncorrelated and correlated synaptic weights.
- In the uncorrelated case the network becomes asynchronous (propagation of chaos) on average but in general not almost surely.
- In both cases (uncorrelated and correlated synaptic weights) the thermodynamic limit is described by a Gaussian process if the initial conditions are Gaussian.

Perspectives

- Analyze the limit equations
- Understand the fluctuations

References I

G. Ben-Arous and A. Guionnet, *Large deviations for langevin spin glass dynamics*, Probability Theory and Related Fields **102** (1995), no. 4, 455–509.

Mireille Bossy, Olivier Faugeras, and Denis Talay, *Clarification and complement to "mean-field description and propagation of chaos in networks of Hodgkin–Huxley and FitzHugh–Nagumo neurons"*, The Journal of Mathematical Neuroscience (JMN) **5** (2015), no. 19.

R.S. Ellis, *Entropy, large deviations and statistical mechanics*, Springer, 1985.

References II

- Nicolas Fournier and Eva Löcherbach, *On a toy model of interacting neurons*, Ann. Inst. H. Poincaré Probab. Statist. **52** (2016), no. 4, 1844–1876.
- Eric Luçon and Wilhelm Stannat, *Mean field limit for disordered diffusions with singular interactions*, Ann. Appl. Probab. **24** (2014), no. 5, 1946–1993.
- O. Moynot and M. Samuelides, *Large deviations and mean-field theory for asymmetric random recurrent neural networks*, Probability Theory and Related Fields **123** (2002), no. 1, 41–75.

Metric on $\mathcal{T}^{\mathbb{Z}}$

$$d_T^{\mathbb{Z}}(u, v) = \sum_{i \in \mathbb{Z}} 2^{-|i|}(\|u^i - v^i\|_T \wedge 1)$$

Metric on $\mathcal{P}(\mathcal{T}^{\mathbb{Z}})$

Induced by the Wasserstein-1 distance:

$$D_T(\mu,\nu) = \inf_{\xi \in C(\mu,\nu)} \int d_T^{\mathbb{Z}}(u,v)\, d\xi(u,v)$$

Large deviation principle: I

For all open sets $\mathcal{O}$ of $\mathcal{P}(\mathcal{T})$

$$-\inf_{\mu \in \mathcal{O}} H(\mu) \leq \liminf_{N \to \infty} \frac{1}{N} \log \Pi^N(\mathcal{O})$$

Large deviation principle: II

The sequence Π^N is exponentially tight.

Large deviation principle: III

For every compact set F of $\mathcal{P}(\mathcal{T})$

$$\limsup_{N\to\infty} \frac{1}{N} \log \Pi^N(F) \leq - \inf_{\mu\in F} H(\mu)$$

Exponential approximation

for all $\delta > 0$

$$\lim_{m\to\infty} \overline{\lim_{n\to\infty}} \frac{1}{N} \log P^{\otimes N}\left(D_T\left(\Psi_m\left(\hat{\mu}_c^N(B)\right), \hat{\mu}_c^N(Z)\right) > \delta \right) = -\infty$$