Weak solutions of mean-field stochastic
differential equations

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Objective of the talk

Case 1: The drift coefficient is bounded and measurable.

Case 2: The coefficients are bounded, continuous.
Objective of the talk

Let $T$ be a fixed time horizon, $b$, $\sigma$ measurable mappings defined over appropriate spaces. We are interested in a weak solution of

**Mean-Field (McKean-Vlasov) SDE**:

For $t \in [0, T]$, $\xi \in L^2(\Omega, \mathcal{F}_0, P; \mathbb{R}^d)$,

$$X_t = \xi + \int_0^t b(s, X_{\wedge s}, Q_{X_{\wedge s}})ds + \int_0^t \sigma(s, X_{\wedge s}, Q_{X_{\wedge s}})dB_s, \quad (1.1)$$

where $Q$ is a probability measure with respect to which $B$ is a B.M.

**Remark**: $Q_{X_{\wedge s}}$ is the law of $X_{\wedge s}$ w.r.t. $Q$. 
1) Such Mean-Field SDEs have been intensively studied:
- For a longer time as limit equ. for systems with a large number of particles (propagation of chaos) (Bossy, Méléard, Sznitman, Talay,…);
- Mean-Field Games, since 2006-2007 (Lasry, Lions,…);

2) Mean-Field SDEs/FBSDEs and associated nonlocal PDEs:
- Preliminary works in 2009 (AP, SPA);
- Classical solution of non-linear PDE related with the mean-field SDE: Buckdahn, Peng, Li, Rainer (2014); Chassagneux, Crisan, Delarue (2014);
- For the case with jumps: Li, Hao (2016); Li (2016);
Our objectives: To prove the existence and the uniqueness in law of the weak solution of mean-field SDE (1.1):

* when the coefficient $b$ is bounded, measurable and with a modulus of continuity w.r.t the measure, while $\sigma$ is independent of the measure and Lipschitz.

* when the coefficients $(b, \sigma)$ are bounded and continuous.
We consider
+ $(\Omega, \mathcal{F}, P)$ - complete probability space;
+ $W$ B.M. over $(\Omega, \mathcal{F}, P)$ (for simplicity: all processes 1-dimensional);
+ $\mathcal{F}$-filtration generated by $W$, and augmented by $\mathcal{F}_0$.

$p$-Wasserstein metric on

$$\mathcal{P}_p(\mathbb{R}) := \{ \mu \mid \mu \text{ probab. on } (\mathbb{R}, \mathcal{B}(\mathbb{R})) \text{ with } \int_{\mathbb{R}} |x|^p \mu(x) < +\infty \}$$

$$W_p(\mu, \nu):= \inf \left\{ \left( \int_{\mathbb{R} \times \mathbb{R}} |x|^p \rho(dx,dy) \right)^{\frac{1}{p}} \mid \rho(\cdot \times \mathbb{R}) = \mu, \rho(\mathbb{R} \times \cdot) = \nu \right\}.$$

(1.2)
Generalization of the def. of a weak sol. of a classical SDE (see, e.g., Karatzas and Shreve, 1988) to (1.1):

**Definition 1.1**

A six-tuple \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, Q, B, X)\) is a weak solution of SDE (1.1), if

(i) \((\tilde{\Omega}, \tilde{\mathcal{F}}, Q)\) is a complete probability space, and \(\tilde{\mathbb{F}} = \{\tilde{\mathcal{F}}_t\}_{0 \leq t \leq T}\) is a filtration on \((\tilde{\Omega}, \tilde{\mathcal{F}}, Q)\) satisfying the usual conditions.

(ii) \(X = \{X_t\}_{0 \leq t \leq T}\) is a continuous, \(\tilde{\mathbb{F}}\)-adapted \(\mathbb{R}\)-valued process;

\(B = \{B_t\}_{0 \leq t \leq T}\) is an \((\tilde{\mathbb{F}}, Q)\)-BM.

(iii) \(Q\{\int_0^T (|b(s, X_{\cdot \wedge s}, Q_{X_{\cdot \wedge s}})| + |\sigma(s, X_{\cdot \wedge s}, Q_{X_{\cdot \wedge s}})|^2)ds < +\infty\} = 1\), and equation (1.1) is satisfied, \(Q\)-a.s.
Definition 1.2

We say that uniqueness in law holds for the mean-field SDE (1.1), if for any two weak solutions \((\Omega^i, \mathcal{F}^i, \mathbb{F}^i, Q^i, B^i, X^i), i = 1, 2\), we have \(Q^{1}_{X^1} = Q^{2}_{X^2}\), i.e., the two processes \(X^1\) and \(X^2\) have the same law.
1 Objective of the talk

2 Case 1: The drift coefficient is bounded and measurable.

3 Case 2: The coefficients are bounded, continuous.
Case 1: Existence of a weak solution

Let $b$, $\sigma$ satisfy the following assumption \textbf{(H1)}:

(i) $b : [0, T] \times C([0, T]; \mathbb{R}) \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ is bounded and measurable;

(ii) $\sigma : [0, T] \times C([0, T]; \mathbb{R}) \to \mathbb{R}$ is bounded, measurable, and s.t., for all $(t, \varphi) \in [0, T] \times C([0, T]; \mathbb{R})$, $1/\sigma(t, \varphi)$ is bounded in $(t, \varphi)$;

(iii) (Modulus of continuity) $\exists \rho : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous, with $\rho(0+) = 0$ s.t., for all $t \in [0, T]$, $\varphi \in C([0, T]; \mathbb{R})$, $\mu$, $\nu \in \mathcal{P}_1(\mathbb{R})$,

$$|b(t, \varphi \wedge t, \mu) - b(t, \varphi \wedge t, \nu)| \leq \rho(W_1(\mu, \nu));$$

(iv) $\exists L \geq 0$ s.t., for all $t \in [0, T]$, $\varphi$, $\psi \in C([0, T]; \mathbb{R})$,

$$|\sigma(t, \varphi \wedge t) - \sigma(t, \psi \wedge t)| \leq L \sup_{0 \leq s \leq t} |\varphi_s - \psi_s|.$$
Case 1: Existence of a weak solution

We want to study weak solutions of the following mean-field SDE:

\[ X_t = \xi + \int_0^t \sigma(s, X_{\cdot \wedge s})dB_s + \int_0^t b(s, X_{\cdot \wedge s}, Q_{X_s})ds, \quad t \in [0, T], \quad (2.1) \]

where \((B_t)_{t \in [0,T]}\) is a BM under the probability measure \(Q\).

Now we can give the main statement of this section.

**Theorem 2.1**

Under assumption (H1) mean-field SDE (2.1) has a weak solution \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, Q, B, X)\).

**Proof:** Girsanov’s Theorem. Schauder’s Fixed Point Theorem.
Case 1: Existence of a weak solution

Let us give two examples.

**Example 1.** Take diffusion coefficient $\sigma \equiv I_d$ and drift coefficient

$\hat{b}(s, \varphi_s, \mu_s) := b(s, \varphi_s, \int \psi d\mu_s)$, $\varphi \in C([0, T])$, $\mu \in \mathcal{P}_1(\mathbb{R})$, $s \in [0, T]$;

the function $\psi \in C([0, T]; \mathbb{R})$ is arbitrarily given but fixed, and Lipschitz.

Then our mean-field SDE (2.1) can be written as follows:

$$X_t = B_t + \int_0^t b(s, X_s, E_Q[\psi(X_s)]) ds, \quad t \in [0, T]. \quad (2.2)$$

Here $b : [0, T] \times C([0, T]) \times \mathbb{R} \to \mathbb{R}$ is bounded, meas., Lips. in $y$. Then, the coefficients $\hat{b}$ and $\sigma$ satisfy (H1), and from Theorem 2.1, we obtain that the mean-field SDE (2.2) has a weak solution $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, Q, B, X)$. 
Case 1: Existence of a weak solution

**Example 2.** Take diffusion coefficient \( \sigma \equiv I_d \) and drift coefficient 
\[
\tilde{b}(s, \varphi \wedge s, \mu_s) := \int b(s, \varphi \wedge s, y) \mu_s(dy), \quad \varphi \in C([0, T]), \quad \mu_s \in \mathcal{P}_1(\mathbb{R}),
\]
\( s \in [0, T], \) i.e., we consider the following mean-field SDE:

\[
X_t = B_t + \int_0^t \int_{\mathbb{R}} b(s, X \wedge s, y) Q_X(s)(dy) ds, \quad t \in [0, T]. \tag{2.3}
\]

Here the coefficient \( b : [0, T] \times C([0, T]) \times \mathbb{R} \to \mathbb{R} \) is bounded, meas. and Lips. in \( y \). Then, the coefficients \( \tilde{b} \) and \( \sigma \) satisfy \((H1)\), and from Theorem 2.1 the mean-field SDE \((2.3)\) has a weak solution \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{F}, Q, B, X)\).
Case 1: Uniqueness in law of weak solutions

Let the functions $b$ and $\sigma$ satisfy the following assumption (H2):

(i) $b : [0, T] \times C([0, T]; \mathbb{R}) \times \mathcal{P}_1(\mathbb{R}) \rightarrow \mathbb{R}$ is bounded and measurable;

(ii) $\sigma : [0, T] \times C([0, T]; \mathbb{R}) \rightarrow \mathbb{R}$ is bounded and measurable, and $|1/\sigma(t, \varphi)| \leq C$, $(t, \varphi) \in [0, T] \times C([0, T]; \mathbb{R})$, for some $C \in \mathbb{R}_+$;

(iii) (Modulus of continuity) There exists a continuous and increasing function $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with

$$\rho(r) > 0, \text{ for all } r > 0, \text{ and } \int_{0+} \frac{du}{\rho(u)} = +\infty,$$

such that, for all $t \in [0, T]$, $\varphi \in C([0, T]; \mathbb{R})$, $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$,

$$|b(t, \varphi \cdot \wedge t, \mu) - b(t, \varphi \cdot \wedge t, \nu)|^2 \leq \rho(W_1(\mu, \nu)^2);$$

(iv) $\exists L \geq 0$ such that, for all $t \in [0, T]$, $\varphi$, $\psi \in C([0, T]; \mathbb{R})$,

$$|\sigma(t, \varphi \cdot \wedge t) - \sigma(t, \psi \cdot \wedge t)| \leq L \sup_{0 \leq s \leq t} |\varphi_s - \psi_s|.$$
Case 1: Uniqueness in law of weak solutions

Obviously, under assumption (H2) the coefficients $b$ and $\sigma$ also satisfy (H1). Thus, due to Theorem 2.1, the following mean-field SDE

$$X_t = \xi + \int_0^t b(s, X_{\cdot \wedge s}, Q_X s) \, ds + \int_0^t \sigma(s, X_{\cdot \wedge s}) \, dB_s, \quad t \in [0, T], \quad (2.1)$$

has a weak solution.

Theorem 2.2

Suppose that assumption (H2) holds, and let $(\Omega^i, \mathcal{F}^i, \mathbb{F}^i, Q^i, B^i, X^i)$, $i = 1, 2$, be two weak solutions of mean-field SDE (2.1). Then $(B^1, X^1)$ and $(B^2, X^2)$ have the same law under their respective probability measures, i.e., $Q^1_{(B^1, X^1)} = Q^2_{(B^2, X^2)}$. 
Case 1: Uniqueness in law of weak solutions

Sketch of the proof: For \( \varphi \in C([0, T]; \mathbb{R}) \), \( \mu \in \mathcal{P}_1(\mathbb{R}) \), we define

\[
\tilde{b}(s, \varphi \cdot \wedge s, \mu) = \sigma^{-1}(s, \varphi \cdot \wedge s)b(s, \varphi \cdot \wedge s, \mu),
\]

and we introduce

\[
\begin{cases}
W^i_t = B^i_t + \int_0^t \tilde{b}(s, X^i_{\cdot \wedge s}, Q^i_{X^i_s})ds, & t \in [0, T], \\
L^i_T = \exp\left\{-\int_0^T \tilde{b}(s, X^i_{\cdot \wedge s}, Q^i_{X^i_s})dB^i_s - \frac{1}{2} \int_0^T |\tilde{b}(s, X^i_{\cdot \wedge s}, Q^i_{X^i_s})|^2 ds\right\},
\end{cases}
\]

(2.4) \( i = 1, 2 \). Then from the Girsanov Theorem we know that \( (W^i_t)_{t \in [0, T]} \) is an \( \mathbb{F}^i \)-B.M. under the probability measure \( \tilde{Q}^i = L^i_T Q^i \), \( i = 1, 2 \), respectively.

From (H2), for each \( i \), we have a unique strong solution \( X^i \) of the SDE

\[
X^i_t = X^i_0 + \int_0^t \sigma(s, X^i_{\cdot \wedge s})dW^i_s, \quad t \in [0, T].
\]

(2.5)
Case 1: Uniqueness in law of weak solutions

It is by now standard that $\exists$ a meas. and non-anticipating function $\Phi : [0, T] \times \mathbb{R} \times C([0, T]; \mathbb{R}) \rightarrow \mathbb{R}$ not depending on $i = 1, 2$, s.t.

$$X_t^i = \Phi_t(X_0^i, W^i), \ t \in [0, T], \ \tilde{Q}^i\text{-a.s. (and, } Q^i\text{-a.s.), } i = 1, 2. \quad (2.6)$$

Then from (2.4) that $W_t^i = B_t^i + \int_0^t \tilde{b}(s, \Phi \cdot \wedge_s (X_0^i, W^i), Q_s^i)ds$, $i = 1, 2$. Hence, putting $f(s, \varphi \cdot \wedge_s) = \tilde{b}(s, \varphi \cdot \wedge_s, Q^1_{X^i_s})$, $(s, \varphi) \in [0, T] \times C([0, T]; \mathbb{R})$, from (2.4) and (2.6) we have

$$\begin{cases}
W_t^1 = B_t^1 + \int_0^t f(s, \Phi \cdot \wedge_s (X_0^1, W^1))ds, \ t \in [0, T], \\
W_t^2 = \tilde{B}_t^2 + \int_0^t f(s, \Phi \cdot \wedge_s (X_0^2, W^2))ds, \ t \in [0, T],
\end{cases} \quad (2.7)$$

where, $t \in [0, T]$,

$$\tilde{B}_t^2 = B_t^2 + \int_0^t \left(\tilde{b}(s, \Phi \cdot \wedge_s (X_0^2, W^2), Q^2_{X^1_s}) - \tilde{b}(s, \Phi \cdot \wedge_s (X_0^2, W^2), Q^1_{X^1_s})\right)ds.$$
Case 1: Uniqueness in law of weak solutions

Hence, \( \exists \Phi : [0, T] \times \mathbb{R} \times C([0, T]; \mathbb{R}) \rightarrow \mathbb{R} \) meas. s.t., for both \( B^1, \tilde{B}^2 \),

\[
B_t^1 = \Phi_t(X_0^1, W^1) \quad \text{and} \quad \tilde{B}_t^2 = \Phi_t(X_0^2, W^2), \quad t \in [0, T]. \tag{2.8}
\]

Now we define

\[
\begin{cases}
\hat{dL}_t^2 = -(\tilde{b}(s, \Phi \cdot \wedge s(X_0^2, W^2), Q^2_{X_s^2}) - \tilde{b}(s, \Phi \cdot \wedge s(X_0^2, W^2), Q^1_{X_1^s})) \hat{L}_t^2 dB_t^2, \quad t \in [0, T] \\
\hat{L}_0^2 = 1.
\end{cases} \tag{2.9}
\]

From the Girsanov Theorem we know that \( \tilde{B}^2 \) is an Brownian motion under the probability measure \( \hat{Q}^2 = \hat{L}_T^2 Q^2 \). Moreover, putting
Case 1: Uniqueness in law of weak solutions

\[
\begin{cases}
\tilde{L}_T^2 = \exp\left\{-\int_0^T f(s, \Phi_{\cdot \wedge s}(X_0^2, W^2))dW_s^2 + \frac{1}{2}\int_0^T |f(s, \Phi_{\cdot \wedge s}(X_0^2, W^2))|^2 ds\right\}, \\
\bar{Q}^2 = \tilde{L}_T^2 \hat{Q}^2.
\end{cases}
\]

we have that \((W_t^2)_{t \in [0, T]}\) is a B.M. under both \(\tilde{Q}^2\) and \(\bar{Q}^2\), while \((W_t^1)_{t \in [0, T]}\) is a B.M. under \(\tilde{Q}^1\).

On the other hand, since \(f\) is bounded and meas., we can prove that \(\exists\) a meas. function \(\tilde{\Phi} : \mathbb{R} \times C([0, T]; \mathbb{R}) \to \mathbb{R}\), s.t.

\[
\tilde{\Phi}(X_0^i, W^i) = \int_0^T f(s, \Phi_{\cdot \wedge s}(X_0^i, W^i))dW_s^i, \quad \hat{Q}^i\text{-a.s.}, \ i = 1, 2.
\]
Case 1: Uniqueness in law of weak solutions

Therefore, recalling the definition of $L^1_T$, and (2.10), we have

\[
\begin{align*}
L^1_T &= \exp\left\{ -\int_0^T f(s, \Phi \cdot \wedge_s (X^1_0, W^1)) \, dW^1_s + \frac{1}{2} \int_0^T |f(s, \Phi \cdot \wedge_s (X^1_0, W^1))|^2 \, ds \right\}, \\
\tilde{L}^2_T &= \exp\left\{ -\int_0^T f(s, \Phi \cdot \wedge_s (X^2_0, W^2)) \, dW^2_s + \frac{1}{2} \int_0^T |f(s, \Phi \cdot \wedge_s (X^2_0, W^2))|^2 \, ds \right\},
\end{align*}
\]

and we see that $\exists$ a meas. function $\hat{\Phi} : \mathbb{R} \times C([0, T]; \mathbb{R}) \to \mathbb{R}$, s.t.

\[
L^1_T = \hat{\Phi}(X^1_0, W^1), \quad Q^1\text{-a.s.}, \quad \text{and} \quad \tilde{L}^2_T = \hat{\Phi}(X^2_0, W^2), \quad Q^2\text{-a.s.} \quad \text{(and, } \bar{Q}^2\text{-a.s.)}.
\]
Consequently, as $X^i_0$ is $\mathcal{F}^i_0$-measurable, $i = 1, 2$ and $Q^{1}_{X_0^1} = Q^{2}_{X_0^2}$, from (2.8), (2.10), (2.11) and (2.12) we have that, for all bounded measurable function $F : C([0, T]; \mathbb{R}^d)^2 \to \mathbb{R}$,

$$E_{Q^1}[F(B^1, W^1)] = E_{\tilde{Q}^1}[\frac{1}{\tilde{\Phi}(X^1_0, W^1)} F(\tilde{\Phi}(X^1_0, W^1), W^1)]$$

$$= E_{\tilde{Q}^2}[\frac{1}{\tilde{\Phi}(X^2_0, W^2)} F(\tilde{\Phi}(X^2_0, W^2), W^2)] = E_{\tilde{Q}^2}[F(\tilde{B}^2, W^2)].$$

That is,

$$Q^1_{(B^1, W^1)} = \tilde{Q}^2_{(\tilde{B}^2, W^2)}. \quad (2.13)$$

Taking into account (2.6), we have

$$Q^1_{(B^1, W^1, X^1)} = \tilde{Q}^2_{(\tilde{B}^2, W^2, X^2)}, \quad (2.14)$$

and, in particular, $Q^1_{X^1} = \tilde{Q}^2_{X^2}$. 
Case 1: Uniqueness in law of weak solutions

On the other hand, we can prove

\[ W_1(Q^1_{X^1_s}, Q^2_{X^2_s})^2 = W_1(\hat{Q}^2_{X^2_s}, Q^2_{X^2_s})^2 \leq C \int_0^s \rho(W_1(Q^1_{X^1_r}, Q^2_{X^2_r})^2)dr; \]

• The continuity of \( s \rightarrow W_1(Q^1_{X^1_s}, Q^2_{X^2_s}) \).

Putting \( u(s) := W_1(Q^1_{X^1_s}, Q^2_{X^2_s}), s \in [0, T] \), then we have from above,

\[ u(s)^2 \leq C \int_0^s \rho(u(r)^2)dr, \quad 0 \leq s \leq t \leq T. \]

From (H2)-(iii), \( \int_0^+ \frac{du}{\rho(u)} = +\infty \), it follows from Bihari’s inequality that \( u(s) = 0 \), for any \( s \in [0, T] \), that is, \( Q^1_{X^1_s} = Q^2_{X^2_s}, s \in [0, T] \). Thus, from (2.7) and (2.9) it follows that \( \hat{B}^2 = B^2, \hat{L}^2_T = 1 \), and, consequently, \( \hat{Q}^2 = Q^2 \). Then, \( \hat{Q}^2_{(\hat{B}^2, W^2, X^2)} = Q^2_{(B^2, W^2, X^2)} \), and from (2.14)

\[ Q^1_{(B^1, W^1, X^1)} = Q^2_{(B^2, W^2, X^2)}. \quad (2.15) \]

This implies, in particular, \( Q^1_{(B^1, X^1)} = Q^2_{(B^2, X^2)}. \)
Objective of the talk

Case 1: The drift coefficient is bounded and measurable.

Case 2: The coefficients are bounded, continuous.
Case 2: Preliminaries

Definition 3.1 (see, e.g., Karatzas, Shreve, 1988)

A probability \( \hat{\mathbb{P}} \) on \( (C([0, T]; \mathbb{R}), \mathcal{B}(C([0, T]; \mathbb{R})) \) is a solution to the local martingale problem associated with \( \mathcal{A}' \), if for every \( f \in C^{1,2}([0, T] \times \mathbb{R}; \mathbb{R}) \),

\[
M^f_t := f(t, y(t)) - f(0, y(0)) - \int_0^t (\partial_s + \mathcal{A}') f(s, y(s)) ds, \quad t \in [0, T],
\]

(3.1)
is a continuous local martingale w.r.t \((\mathbb{F}^y, \hat{\mathbb{P}})\), where \( y = (y(t))_{t \in [0, T]} \) is the coordinate process on \( C([0, T]; \mathbb{R}) \), the considered filtration \( \mathbb{F}^y = (\mathcal{F}^y_t)_{t \in [0, T]} \) is that generated by \( y = (y(t))_{t \in [0, T]} \) and augmented by all \( \hat{\mathbb{P}} \)-null sets, and \( \mathcal{A}' \) is defined by, \( y \in C([0, T]; \mathbb{R}) \),

\[
\mathcal{A}' f(s, y) = b(s, y) \partial_x f(s, y(s)) + \frac{1}{2} \sigma^2(s, y) \partial_x^2 f(s, y(s)).
\]

(3.2)
Case 2: Preliminaries

Let us first recall a well-known result concerning the equivalence between the weak solution of a functional SDE and the solution to the corresponding local martingale problem (see, e.g., Karatzas, Shreve, 1988).

**Lemma 3.1**

The existence of a weak solution \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W}, X)\) to the following functional SDE with given initial distribution \(\mu\) on \(\mathcal{B}(\mathbb{R})\):

\[
X_t = \xi + \int_0^t b(s, X_{\cdot \wedge s}) \, ds + \int_0^t \sigma(s, X_{\cdot \wedge s}) \, d\tilde{W}_s, \quad t \in [0, T],
\]

is equivalent to the existence of a solution \(\hat{\mathbb{P}}\) to the local martingale problem (3.1) associated with \(\mathcal{A}'\) defined by (3.2), with \(\hat{\mathbb{P}}_{y(0)} = \mu\). The both solutions are related by \(\hat{\mathbb{P}} = \tilde{\mathbb{P}} \circ X^{-1}\), i.e., the probability measure \(\hat{\mathbb{P}}\) is the law of the weak solution \(X\) on \((C([0, T]; \mathbb{R}), \mathcal{B}(C([0, T]; \mathbb{R})))\).
Case 2: Preliminaries

Recall the definition of the derivative of $f : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ w.r.t probability measure $\mu \in \mathcal{P}_2(\mathbb{R})$ (in the sense of P.L.Lions)(P.L.Lions’ lectures at Collège de France, also see the notes of Cardaliaguet).

**Definition 3.2**

(i) $\tilde{f} : L^2(\Omega, \mathcal{F}, P; \mathbb{R}) \to \mathbb{R}$ is Fréchet differentiable at $\xi \in L^2(\Omega, \mathcal{F}, P)$, if there exists a linear continuous mapping $D\tilde{f}(\xi)(\cdot) \in L(L^2(\Omega, \mathcal{F}, P; \mathbb{R}); \mathbb{R})$, s.t.

$$\tilde{f}(\xi + \eta) - \tilde{f}(\xi) = D\tilde{f}(\xi)(\eta) + o(|\eta|_{L^2}), \text{ with } |\eta|_{L^2} \to 0 \text{ for } \eta \in L^2(\Omega, \mathcal{F}, P).$$

(ii) $f : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ is differentiable at $\mu \in \mathcal{P}_2(\mathbb{R})$, if for $\tilde{f}(\xi) := f(P\xi)$, $\xi \in L^2(\Omega, \mathcal{F}, P; \mathbb{R})$, there is some $\zeta \in L^2(\Omega, \mathcal{F}, P; \mathbb{R})$ with $P\zeta = \mu$ such that $\tilde{f} : L^2(\Omega, \mathcal{F}, P; \mathbb{R}) \to \mathbb{R}$ is Fréchet differentiable in $\zeta$. 
From Riesz’ Representation Theorem there exists a $P$-a.s. unique variable $\vartheta \in L^2(\Omega, \mathcal{F}, P; \mathbb{R})$ such that $D\tilde{f}(\zeta)(\eta) = (\vartheta, \eta)_{L^2} = E[\vartheta \eta]$, for all $\eta \in L^2(\Omega, \mathcal{F}, P; \mathbb{R})$. P.L. Lions proved that there is a Borel function $h : \mathbb{R} \to \mathbb{R}$ such that $\vartheta = h(\zeta)$, $P$-a.e., and function $h$ depends on $\zeta$ only through its law $P_\zeta$. Therefore,

$$f(P_{\xi}) - f(P_{\zeta}) = E[h(\zeta) \cdot (\xi - \zeta)] + o(|\xi - \zeta|_{L^2}), \quad \xi \in L^2(\Omega, \mathcal{F}, P; \mathbb{R}).$$

**Definition 3.3**

We call $\partial_\mu f(P_\zeta, y) := h(y)$, $y \in \mathbb{R}$, the derivative of function $f : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ at $P_\zeta$, $\zeta \in L^2(\Omega, \mathcal{F}, P; \mathbb{R})$.

**Remark:** $\partial_\mu f(P_\zeta, y)$ is only $P_\zeta(dy)$-a.e. uniquely determined.
Case 2: Preliminaries

Definition 3.4

We say that \( f \in C^1(\mathcal{P}_2(\mathbb{R})) \), if for all \( \xi \in L^2(\Omega, \mathcal{F}, P; \mathbb{R}) \) there exists a \( P_\xi \)-modification of \( \partial_\mu f(P_\xi, .) \), also denoted by \( \partial_\mu f(P_\xi, .) \), such that \( \partial_\mu f : \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \rightarrow \mathbb{R} \) is continuous w.r.t the product topology generated by the 2-Wasserstein metric over \( \mathcal{P}_2(\mathbb{R}) \) and the Euclidean norm over \( \mathbb{R} \), and we identify this modified function \( \partial_\mu f \) as the derivative of \( f \).

The function \( f \) is said to belong to \( C_b^{1,1}(\mathcal{P}_2(\mathbb{R})) \), if \( f \in C^1(\mathcal{P}_2(\mathbb{R})) \) is s.t. \( \partial_\mu f : \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \rightarrow \mathbb{R} \) is bounded and Lipschitz continuous, i.e., there exists some constant \( C \geq 0 \) such that

(i) \[ |\partial_\mu f(\mu, x)| \leq C, \ \mu \in \mathcal{P}_2(\mathbb{R}), \ x \in \mathbb{R}; \]

(ii) \[ |\partial_\mu f(\mu, x) - \partial_\mu f(\mu', x')| \leq C(W_2(\mu, \mu') + |x - x'|), \ \mu, \mu' \in \mathcal{P}_2(\mathbb{R}), \ x, x' \in \mathbb{R}. \]
Case 2: Preliminaries

**Definition 3.5**

We say that \( f \in C^2(\mathcal{P}_2(\mathbb{R})) \), if \( f \in C^1(\mathcal{P}_2(\mathbb{R})) \) and \( \partial_{\mu} f(\mu,.) : \mathbb{R} \rightarrow \mathbb{R} \) is differentiable, and its derivative \( \partial_y \partial_{\mu} f : \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \rightarrow \mathbb{R} \otimes \mathbb{R} \) is continuous, for every \( \mu \in \mathcal{P}_2(\mathbb{R}) \).

Moreover, \( f \in C^{2,1}_b(\mathcal{P}_2(\mathbb{R})) \), if \( f \in C^2(\mathcal{P}_2(\mathbb{R})) \cap C^{1,1}_b(\mathcal{P}_2(\mathbb{R})) \) and its derivative \( \partial_y \partial_{\mu} f : \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \rightarrow \mathbb{R} \otimes \mathbb{R} \) is bounded and Lipschitz-continuous.

**Remark:** \( C^{2,1}_b(\mathbb{R} \times \mathcal{P}_2(\mathbb{R})) \), \( C^{1,2,1}_b([0,T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}); \mathbb{R}) \) are similarly defined.
Case 2: Preliminaries

Now we can give our Itô’s formula.

**Theorem 3.1**

Let $\sigma = (\sigma_s), \gamma = (\gamma_s), b = (b_s), \beta = (\beta_s)$ $\mathbb{R}$-valued adapted stochastic processes, such that

(i) There exists a constant $q > 6$ s.t. $E\left[\left(\int_0^T (|\sigma_s|^q + |b_s|^q)ds\right)^{\frac{3}{q}}\right] < +\infty$;

(ii) $\int_0^T (|\gamma_s|^2 + |\beta_s|)ds < +\infty$, $P$-a.s.

Let $F \in C_b^{1,2,1}([0, T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}))$. Then, for the Itô processes

$$X_t = X_0 + \int_0^t \sigma_s dW_s + \int_0^t b_s ds, \quad t \in [0, T], \; X_0 \in L^2(\Omega, \mathcal{F}_0, P),$$

$$Y_t = Y_0 + \int_0^t \gamma_s dW_s + \int_0^t \beta_s ds, \quad t \in [0, T], \; Y_0 \in L^2(\Omega, \mathcal{F}_0, P),$$
we have

\[ F(t, Y_t, P_{X_t}) - F(0, Y_0, P_{X_0}) \]

\[ = \int_0^t \left( \partial_r F(r, Y_r, P_{X_r}) + \partial_y F(r, Y_r, P_{X_r}) \beta_r + \frac{1}{2} \partial_y^2 F(r, Y_r, P_{X_r}) \gamma_r^2 \right. \]

\[ + \bar{E}\left[ (\partial_{\mu} F)(r, Y_r, P_{X_r}, \bar{X}_r) \bar{b}_r + \frac{1}{2} \partial_z (\partial_{\mu} F)(r, Y_r, P_{X_r}, \bar{X}_r) \bar{\sigma}_r^2 \right] \left) \right. \]

\[ + \int_0^t \partial_y F(r, Y_r, P_{X_r}) \gamma_r dW_r, \ t \in [0, T]. \]

Here \((\bar{X}, \bar{b}, \bar{\sigma})\) denotes an independent copy of \((X, b, \sigma)\), defined on a P.S. \((\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})\). The expectation \(\bar{E}[]\) on \((\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})\) concerns only r.v. endowed with the superscript \(\bar{\cdot}\).
(H3) The coefficients $(\sigma, b) \in C^{1,2,1}_b([0, T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}); \mathbb{R} \times \mathbb{R})$.

**Theorem 3.2 (Buckdahn, Li, Peng and Rainer, 2014)**

Let $\Phi \in C^{2,1}_b(\mathbb{R} \times \mathcal{P}_2(\mathbb{R}))$, then under assumption (H3) the following PDE:

\[
\begin{align*}
0 &= \partial_t V(t, x, \mu) + \partial_x V(t, x, \mu)b(x, \mu) + \frac{1}{2} \partial_x^2 V(t, x, \mu)\sigma^2(x, \mu) \\
&\quad + \int_{\mathbb{R}} (\partial_\mu V)(t, x, \mu, y)b(y, \mu)\mu(dy) \\
&\quad + \frac{1}{2} \int_{\mathbb{R}} \partial_y (\partial_\mu V)(t, x, \mu, y)\sigma^2(y, \mu)\mu(dy),
\end{align*}
\]

\[
(t, x, \mu) \in [0, T') \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R});
\]

\[V(T, x, \mu) = \Phi(x, \mu), \quad (x, \mu) \in \mathbb{R} \times \mathcal{P}_2(\mathbb{R}).\]

has a unique classical solution $V(t, x, \mu) \in C^{1,2,1}_b([0, T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}); \mathbb{R})$. 
Case 2: Existence of a weak solution

Let $b$ and $\sigma$ satisfy the following assumption:

(H4) $b, \sigma : [0, T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ are continuous and bounded.

We want to study weak solution of the following mean-field SDE:

$$X_t = \xi + \int_0^t b(s, X_s, Q_{X_s})ds \int_0^t \sigma(s, X_s, Q_{X_s})dB_s, \ t \in [0, T], \quad (3.3)$$

where $\xi \in L^2(\Omega, \mathcal{F}_0, P; \mathbb{R})$ obeys a given distribution law $Q_\xi = \nu \in \mathcal{P}_2(\mathbb{R})$ and $(B_t)_{t \in [0,T]}$ is a B.M. under the probability measure $Q$. 

Case 2: Existence of a weak solution

Extension of the corresponding local martingale problem:

**Definition 3.6**

A probability measure \( \hat{P} \) on \((C([0, T]; \mathbb{R}), \mathcal{B}(C([0, T]; \mathbb{R})))\) is a solution to the local martingale problem (resp., martingale problem) associated with \( \tilde{A} \), if for every \( f \in C^{1,2}([0, T] \times \mathbb{R}; \mathbb{R}) \) (resp., \( f \in C^{1,2}_b([0, T] \times \mathbb{R}; \mathbb{R}) \)), the process

\[
C^f(t, y, \mu) := f(t, y(t)) - f(0, y(0)) - \int_0^t ((\partial_s + \tilde{A})f)(s, y(s), \mu(s))ds,
\]

is a continuous local \((\mathbb{F}^y, \hat{P})\)-martingale (resp., continuous \((\mathbb{F}^y, \hat{P})\)-martingale),

\(3.4\)
Case 2: Existence of a weak solution

Definition 3.6 (continued)

where \( \mu(t) = \hat{P}_y(t) \) is the law of the coordinate process \( y = (y(t))_{t \in [0,T]} \) on \( C([0, T]; \mathbb{R}) \) at time \( t \), the filtration \( \mathbb{F}^y \) is that generated by \( y \) and completed, and \( \tilde{A} \) is defined by

\[
(\tilde{A}f)(s, y, \nu) := \partial_y f(s, y)b(s, y, \nu) + \frac{1}{2} \partial^2_y f(s, y)\sigma^2(s, y, \nu), \tag{3.5}
\]

\((s, y, \nu) \in [0, T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}). \) Here \( ((\partial_s + \tilde{A})f)(s, y(s), \mu(s)) \) abbreviates

\[
((\partial_s + \tilde{A})f)(s, y(s), \mu(s)) := (\partial_s f)(s, y(s)) + (\tilde{A}f)(s, y(s), \mu(s)).
\]
Case 2: Existence of a weak solution

Proposition 3.1

The existence of a weak solution \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{F}, Q, B, X)\) to equation (3.3) with initial distribution \(\nu\) on \(\mathcal{B}(\mathbb{R})\) is equivalent to the existence of a solution \(\hat{P}\) to the local martingale problem (3.4) associated with \(\mathcal{A}\) defined by (3.5), with \(\hat{P}_y(0) = \nu\).
Case 2: Existence of a weak solution

Lemma 3.2

Let the probability measure $\hat{P}$ on $(C([0, T]; \mathbb{R}), \mathcal{B}(C([0, T]; \mathbb{R})))$ be a solution to the local martingale problem associated with $\tilde{A}$. Then, for the second order differential operator

$$(Af)(s, y, \nu) := (\tilde{A}f)(s, y, \nu) + \int_{\mathbb{R}} (\partial_\mu f)(s, y, \nu, z)b(s, z, \nu)\nu(dz)$$
$$+ \frac{1}{2} \int_{\mathbb{R}} \partial_z (\partial_\mu f)(s, y, \nu, z)\sigma^2(s, z, \nu)\nu(dz),$$

(3.6)
Case 2: Existence of a weak solution

Lemma 3.2 (continued)

applying to functions \( f \in C^{1,2}(\mathbb{R} \times \mathcal{P}_2(\mathbb{R}); \mathbb{R}) \) we have that, for every such \( f \in C^{1,2}(\mathbb{R} \times \mathcal{P}_2(\mathbb{R}); \mathbb{R}) \), the process

\[
C^f(t, y, \mu) := f(t, y(t), \mu(t)) - f(0, y(0), \mu(0)) \tag{3.7}
\]

\[
- \int_0^t (\partial_s + \mathcal{A}) f(s, y(s), \mu(s)) ds, \quad t \in [0, T],
\]

is a continuous local \((\mathbb{F}^y, \hat{\mathcal{P}})\)-martingale, where \( \mu(t) = \hat{P}_{y(t)} \) is the law of the coordinate process \( y = (y(t))_{t \in [0, T]} \) on \( C([0, T]; \mathbb{R}) \) at time \( t \), the filtration \( \mathbb{F}^y \) is that generated by \( y \) and completed. Moreover, if \( f \in C^{1,2,1}(\mathbb{R} \times \mathcal{P}_2(\mathbb{R}); \mathbb{R}) \), this process \( C^f \) is an \((\mathbb{F}^y, \hat{\mathcal{P}})\)-martingale.
Case 2: Existence of a weak solution

Now we can give the main statement of this section.

**Theorem 3.3**

Under assumption (H4) mean-field SDE (3.3) has a weak solution $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{Q}, B, X)$.

**Remark 2.** If $b, \sigma : [0, T] \times C([0, T]; \mathbb{R}) \times \mathcal{P}_2(C([0, T]; \mathbb{R})) \rightarrow \mathbb{R}$ are bounded and continuous, then the following mean-field SDE

$$
X_t = \xi + \int_0^t b(s, X_{\cdot \wedge s}, Q_{X_{\cdot \wedge s}})ds + \int_0^t \sigma(s, X_{\cdot \wedge s}, Q_{X_{\cdot \wedge s}})dB_s, \ t \in [0, T],
$$

(1.1)

where $\xi \in L^2(\Omega, \mathcal{F}_0, P)$ obeys a given distribution law $Q_{\xi} = \nu$, has a weak solution $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{Q}, X, B)$. 
Case 2: Uniqueness in law of weak solutions

Now we want to study the uniqueness in law for the weak solution of the mean-field SDE (3.3).

**Definition 3.7**

We call $\mathcal{C} \subset b\mathcal{B}(\mathbb{R}) = \{\phi \mid \phi : \mathbb{R} \to \mathbb{R} \text{ bounded Borel-measurable function}\}$ a determining class on $\mathbb{R}$, if for any two finite measures $\nu_1$ and $\nu_2$ on $\mathcal{B}(\mathbb{R})$, $\int_{\mathbb{R}} \phi(x) \nu_1(dx) = \int_{\mathbb{R}} \phi(x) \nu_2(dx)$ for all $\phi \in \mathcal{C}$ implies $\nu_1 = \nu_2$.

**Remark:** The class $C_0^\infty(\mathbb{R})$ is a determining class on $\mathbb{R}$. 
Case 2: Uniqueness in law of weak solutions

**Theorem 3.4**

For given $f \in C_0^\infty(\mathbb{R})$, we consider the Cauchy problem

$$
\frac{\partial}{\partial t} v(t, x, \nu) = \mathcal{A}v(t, x, \nu), \quad (t, x, \nu) \in [0, T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}),
$$

where

$$
\mathcal{A}v(t, x, \nu) = (\tilde{\mathcal{A}}v)(t, x, \nu) + \int_{\mathbb{R}} (\partial_\mu v)(t, x, \nu, u) b(t, u, \nu) \nu(du)
$$

$$
+ \frac{1}{2} \int_{\mathbb{R}} \partial_z (\partial_\mu v)(t, x, \nu, u) \sigma^2(t, u, \nu) \nu(du),
$$

$$(\tilde{\mathcal{A}}v)(t, x, \nu) = \partial_y v(t, x, \nu) b(t, x, \nu) + \frac{1}{2} \partial_y^2 v(t, x, \nu) \sigma^2(t, x, \nu),$$

$$(t, x, \nu) \in [0, \infty) \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}).$$
Theorem 3.4 (continued)

We suppose that, for all $f \in C_0^\infty(\mathbb{R})$, (3.8) has a solution $v_f \in C_b([0, \infty) \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R})) \cap C_b^{1,2,1}((0, \infty) \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}))$. Then, the local martingale problem associated with $\tilde{\mathcal{A}}$ (Recall Definition 3.6) and with the initial condition $\delta_x$ has at most one solution.

Remark: Theorem 3.4 generalizes a well-known classical uniqueness for weak solutions to the case of mean-field SDE.

Corollary 3.1

Under the assumption of Theorem 3.4, we have for the mean-field SDE (3.3) the uniqueness in law, that is, for any weak solutions, $i = 1, 2$ $(\Omega^i, \mathcal{F}^i, \mathbb{F}^i, Q^i, B^i, X^i)$, of SDE (3.3), we have $Q^1_{X^1} = Q^2_{X^2}$. 

Uniqueness in law of weak solutions

Sketch of proof of Theorem 3.4: Let $T > 0$, denote by $y=(y(t))_{t\in[0,T]}$ the coordinate process on $C([0, T]; \mathbb{R})$. Let $P^1$ and $P^2$ be two arbitrary solutions of the local martingale problem associated with $\tilde{A}$ and initial condition $x \in \mathbb{R}$: $P^l_y(0) = \delta_x$, $l = 1, 2$.

Consequently, due to Lemma 3.2, for any $g \in C_b^{1,2,1}([0, T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}))$,

$$C^g(t, y, P^l_y) := g(t, y(t), P^l_{y(t)}) - g(0, x, \delta_x) - \int_0^t (\partial_s + A)g(s, y(s), P^l_{y(s)})ds,$$

(3.9)

is a $P^l$-martingale, $l = 1, 2$, $t \in [0, T]$. For given $f \in C_0^\infty(\mathbb{R})$, let $v_f \in C_b([0, T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R})) \cap C_b^{1,2,1}((0, T) \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}))$ be a solution of the Cauchy problem (3.8).
Uniqueness in law of weak solutions

Then putting \( g(t, z, \nu) := v_f(T-t, z, \nu), \ t \in [0, T], \ z \in \mathbb{R}, \ \nu \in \mathcal{P}_2(\mathbb{R}) \), defines a function \( g \) of class
\[
C_b([0, T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R})) \cap C^{1,2,1}_b((0, T) \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}))
\]
which satisfies
\[
\partial_s g(s, z, \nu) + A g(s, z, \nu) = 0, \ g(T, z, \nu) = f(z), \ (s, z, \nu) \in [0, T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}).
\]

From (3.9) we see that \( \{C^g(s, y, P^l_y), s \in [0, T]\} \) is an \((\mathbb{F}_y^l, P^l)\)-martingale. Hence, for \( E^l[\cdot] = \int_{\Omega} \cdot \ dP^l \),
\[
E^l[f(y(T))] = E^l[g(T, y(T), P^l_{y(T)})] = g(0, x, \delta_x), \ x \in \mathbb{R}, \ l = 1, 2,
\]
that is \( E^1[f(y(T))] = E^2[f(y(T))] \), for all \( f \in C_0^\infty(\mathbb{R}) \). Combining this with the arbitrariness of \( T \geq 0 \), we have that \( P^1_{y(t)} = P^2_{y(t)} \), for every \( t \geq 0 \).
Consequently, $P^1, P^2$ are solutions of the same classical martingale problem, associated with $\tilde{\mathcal{A}} = \tilde{\mathcal{A}}^l$, $l = 1, 2$,

$\tilde{\mathcal{A}}^l \phi(t, z) = \partial_y \phi(t, z)\tilde{b}^l(t, z) + \partial^2_y \phi(t, z)(\tilde{\sigma}^l(t, z))^2$, $\phi \in C^{1,2}([0, T] \times \mathbb{R}; \mathbb{R})$,

with the coefficients $\tilde{\sigma}^1 = \tilde{\sigma}^2$, $\tilde{b}^1 = \tilde{b}^2$ (without mean field term),

$\tilde{\sigma}^l(t, z) = \sigma(t, z, P^l_{y(t)}), \quad \tilde{b}^l(t, z) = b(t, z, P^l_{y(t)}), \quad (t, z) \in [0, T] \times \mathbb{R}$,

and we have seen that $P^1_{y(t)} = P^2_{y(t)}$, $t \in [0, T]$.

$\Rightarrow P^1 = P^2$, i.e., the local martingale problem has at most one solution. $\square$
Thank you very much!
谢谢！