Weak solutions of mean-field stochastic differential equations

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1. Objective of the talk

2. Case 1: The drift coefficient is bounded and measurable.

3. Case 2: The coefficients are bounded, continuous.

Objective of the talk

Let T be a fixed time horizon, b, σ measurable mappings defined over appropriate spaces. We are interested in a weak solution of

Mean-Field (McKean-Vlasov) SDE :

For $t \in [0,T]$, $\xi \in L^2(\Omega, \mathcal{F}_0, P; \mathbb{R}^d)$,

$$X_t = \xi + \int_0^t b(s, X_{\cdot\wedge s}, Q_{X_{\cdot\wedge s}})ds + \int_0^t \sigma(s, X_{\cdot\wedge s}, Q_{X_{\cdot\wedge s}})dB_s, \qquad (1.1)$$

where Q is a probability measure with respect to which B is a B.M.

Remark: $Q_{X_{\cdot\wedge s}}$ is the law of $X_{\cdot\wedge s}$ w.r.t. Q.

Brief state of art

1) Such Mean-Field SDEs have been intensively studied:

• For a longer time as limit equ. for systems with a large number of particles (propagation of chaos)(Bossy, Méléard, Sznitman, Talay,...);

• Mean-Field Games, since 2006-2007 (Lasry, Lions,...);

2) Mean-Field SDEs/FBSDEs and associated nonlocal PDEs:

• Preliminary works in 2009 (AP, SPA);

• Classical solution of non-linear PDE related with the mean-field SDE:

Buckdahn, Peng, Li, Rainer (2014); Chassagneux, Crisan, Delarue (2014);

• For the case with jumps: Li, Hao (2016); Li (2016);

• Weak solution: Oelschläger(1984), Funaki (1984), Gärtner (1988), Lacker (2015), Carmona, Lacker (2015), Li, Hui (2016, 2017)......

Objective of the talk

Our objectives: To prove the existence and the uniqueness in law of the weak solution of mean-field SDE (1.1):

$*$ when the coefficient b is bounded, measurable and with a modulus of continuity w.r.t the measure, while σ is independent of the measure and Lipschitz.

$*$ when the coefficients (b, σ) are bounded and continuous.

We consider

+ $(\Omega, \mathcal{F}, P)$ - complete probability space;

+ W B.M. over $(\Omega, \mathcal{F}, P)$ (for simplicity: all processes 1-dimensional);

+ $\mathbb{F}$-filtration generated by W, and augmented by $\mathcal{F}_0$.

<u>p-Wasserstein metric</u> on

$$\mathcal{P}_p(\mathbb{R}) := \{\mu \mid \mu \text{ probab. on } (\mathbb{R}, \mathcal{B}(\mathbb{R})) \text{ with } \int_{\mathbb{R}} |x|^p \mu(x) < +\infty\};$$

$$W_p(\mu, \nu) := \inf\Big\{\Big(\int_{\mathbb{R}\times\mathbb{R}} |x|^p \rho(dxdy)\Big)^{\frac{1}{p}}, \rho(\cdot \times \mathbb{R}) = \mu, \rho(\mathbb{R} \times \cdot) = \nu\Big\}. \tag{1.2}$$

Preliminaries

Generalization of the def. of a weak sol. of a classical SDE (see, e.g., Karatzas and Shreve, 1988) to (1.1):

Definition 1.1

A six-tuple $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}, Q, B, X)$ is a weak solution of SDE (1.1), if
(i) $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, Q)$ is a complete probability space, and $\widetilde{\mathbb{F}} = \{\widetilde{\mathcal{F}}_t\}_{0 \le t \le T}$ is a filtration on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, Q)$ satisfying the usual conditions.
(ii) $X = \{X_t\}_{0 \le t \le T}$ is a continuous, $\widetilde{\mathbb{F}}$-adapted $\mathbb{R}$-valued process; $B = \{B_t\}_{0 \le t \le T}$ is an $(\widetilde{\mathbb{F}}, Q)$-BM.
(iii) $Q\{\int_0^T (|b(s, X_{\cdot \wedge s}, Q_{X_{\cdot \wedge s}})| + |\sigma(s, X_{\cdot \wedge s}, Q_{X_{\cdot \wedge s}})|^2) ds < +\infty\} = 1$, and equation (1.1) is satisfied, Q-a.s.

Definition 1.2

We say that uniqueness in law holds for the mean-field SDE (1.1), if for any two weak solutions $(\Omega^i, \mathcal{F}^i, \mathbb{F}^i, Q^i, B^i, X^i)$, $i = 1, 2$, we have $Q^1_{X^1} = Q^2_{X^2}$, i.e., the two processes X^1 and X^2 have the same law.

1. Objective of the talk

2. Case 1: The drift coefficient is bounded and measurable.

3. Case 2: The coefficients are bounded, continuous.

Case 1: Existence of a weak solution

Let $b, \ \sigma$ satisfy the following assumption **(H1)**:

(i) $b : [0,T] \times C([0,T];\mathbb{R}) \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ is bounded and measurable;

(ii) $\sigma : [0,T] \times C([0,T];\mathbb{R}) \to \mathbb{R}$ is bounded, measurable, and s.t., for all $(t,\varphi) \in [0,T] \times C([0,T];\mathbb{R})$, $1/\sigma(t,\varphi)$ is bounded in (t,φ);

(iii) (Modulus of continuity) $\exists \rho : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous, with $\rho(0+) = 0$ s.t., for all $t \in [0,T]$, $\varphi \in C([0,T];\mathbb{R})$, $\mu, \ \nu \in \mathcal{P}_1(\mathbb{R})$,

$$|b(t,\varphi_{\cdot\wedge t},\mu) - b(t,\varphi_{\cdot\wedge t},\nu)| \leq \rho(W_1(\mu,\nu));$$

(iv) $\exists L \geq 0$ s.t., for all $t \in [0,T]$, $\varphi, \ \psi \in C([0,T];\mathbb{R})$,

$$|\sigma(t,\varphi_{\cdot\wedge t}) - \sigma(t,\psi_{\cdot\wedge t})| \leq L \sup_{0\leq s\leq t} |\varphi_s - \psi_s|.$$

Case 1: Existence of a weak solution

We want to study weak solutions of the following mean-field SDE:

$$X_t = \xi + \int_0^t \sigma(s, X_{\cdot\wedge s})dB_s + \int_0^t b(s, X_{\cdot\wedge s}, Q_{X_s})ds,\ t \in [0,T], \qquad (2.1)$$

where $(B_t)_{t\in[0,T]}$ is a BM under the probability measure Q.

Now we can give the main statement of this section.

Theorem 2.1

Under assumption (H1) mean-field SDE (2.1) has a weak solution $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}, Q, B, X)$.

Proof: Girsanov's Theorem. Schauder's Fixed Point Theorem.

Case 1: Existence of a weak solution

Let us give two examples.

Example 1. Take diffusion coefficient $\sigma \equiv I_d$ and drift coefficient $\widehat{b}(s, \varphi_{\cdot\wedge s}, \mu_s) := b(s, \varphi_{\cdot\wedge s}, \int \psi d\mu_s)$, $\varphi \in C([0,T])$, $\mu \in \mathcal{P}_1(\mathbb{R})$, $s \in [0,T]$; the function $\psi \in C([0,T];\mathbb{R})$ is arbitrarily given but fixed, and Lipschitz. Then our mean-field SDE (2.1) can be written as follows:

$$X_t = B_t + \int_0^t b(s, X_{\cdot\wedge s}, E_Q[\psi(X_s)])ds, \quad t \in [0,T]. \tag{2.2}$$

Here $b : [0,T] \times C([0,T]) \times \mathbb{R} \to \mathbb{R}$ is bounded, meas., Lips. in y. Then, the coefficients $\widehat{b}$ and σ satisfy (H1), and from Theorem 2.1, we obtain that the mean-field SDE (2.2) has a weak solution $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}, Q, B, X)$.

Example 2. Take diffusion coefficient $\sigma \equiv I_d$ and drift coefficient $\widehat{b}(s, \varphi_{\cdot\wedge s}, \mu_s) := \int b(s, \varphi_{\cdot\wedge s}, y)\mu_s(dy)$, $\varphi \in C([0,T])$, $\mu_s \in \mathcal{P}_1(\mathbb{R})$, $s \in [0,T]$, i.e., we consider the following mean-field SDE:

$$X_t = B_t + \int_0^t \int_{\mathbb{R}} b(s, X_{\cdot\wedge s}, y) Q_{X_s}(dy)ds, \quad t \in [0,T]. \qquad (2.3)$$

Here the coefficient $b : [0,T] \times C([0,T]) \times \mathbb{R} \to \mathbb{R}$ is bounded, meas. and Lips. in y. Then, the coefficients $\widehat{b}$ and σ satisfy (H1), and from Theorem 2.1 the mean-field SDE (2.3) has a weak solution $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}, Q, B, X)$.

Case 1: Uniqueness in law of weak solutions

Let the functions b and σ satisfy the following assumption **(H2)**:

(i) $b : [0,T] \times C([0,T];\mathbb{R}) \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ is bounded and measurable;

(ii) $\sigma : [0,T] \times C([0,T];\mathbb{R}) \to \mathbb{R}$ is bounded and measurable, and $|1/\sigma(t,\varphi)| \le C$, $(t,\varphi) \in [0,T] \times C([0,T];\mathbb{R})$, for some $C \in \mathbb{R}_+$;

(iii) (Modulus of continuity) There exists a continuous and increasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ with

$$\rho(r) > 0, \text{ for all } r > 0, \text{ and } \int_{0+} \frac{du}{\rho(u)} = +\infty,$$

such that, for all $t \in [0,T]$, $\varphi \in C([0,T];\mathbb{R})$, $\mu,\ \nu \in \mathcal{P}_1(\mathbb{R})$,

$$|b(t,\varphi_{\cdot\wedge t},\mu) - b(t,\varphi_{\cdot\wedge t},\nu)|^2 \le \rho(W_1(\mu,\nu)^2);$$

(iv) $\exists L \ge 0$ such that, for all $t \in [0,T]$, $\varphi,\ \psi \in C([0,T];\mathbb{R})$,

$$|\sigma(t,\varphi_{\cdot\wedge t}) - \sigma(t,\psi_{\cdot\wedge t})| \le L \sup_{0 \le s \le t} |\varphi_s - \psi_s|.$$

Case 1: Uniqueness in law of weak solutions

Obviously, under assumption (H2) the coefficients b and σ also satisfy (H1). Thus, due to Theorem 2.1, the following mean-field SDE

$$X_t = \xi + \int_0^t b(s, X_{\cdot\wedge s}, Q_{X_s})ds + \int_0^t \sigma(s, X_{\cdot\wedge s})dB_s, \quad t \in [0,T], \tag{2.1}$$

has a weak solution.

Theorem 2.2

Suppose that assumption (H2) holds, and let $(\Omega^i, \mathcal{F}^i, \mathbb{F}^i, Q^i, B^i, X^i)$, $i = 1, 2$, be two weak solutions of mean-field SDE (2.1). Then (B^1, X^1) and (B^2, X^2) have the same law under their respective probability measures, i.e., $Q^1_{(B^1,X^1)} = Q^2_{(B^2,X^2)}$.

Case 1: Uniqueness in law of weak solutions

Sketch of the proof: For $\varphi \in C([0,T];\mathbb{R})$, $\mu \in \mathcal{P}_1(\mathbb{R})$, we define $\widetilde{b}(s, \varphi_{\cdot\wedge s}, \mu) = \sigma^{-1}(s, \varphi_{\cdot\wedge s}) b(s, \varphi_{\cdot\wedge s}, \mu)$, and we introduce

$$\begin{cases} W_t^i = B_t^i + \int_0^t \widetilde{b}(s, X_{\cdot\wedge s}^i, Q_{X_s^i}^i) ds, \quad t \in [0,T], \\ L_T^i = \exp\{-\int_0^T \widetilde{b}(s, X_{\cdot\wedge s}^i, Q_{X_s^i}^i) dB_s^i - \frac{1}{2}\int_0^T |\widetilde{b}(s, X_{\cdot\wedge s}^i, Q_{X_s^i}^i)|^2 ds\}, \end{cases} \tag{2.4}$$

$i = 1, 2$. Then from the Girsanov Theorem we know that $(W_t^i)_{t\in[0,T]}$ is an $\mathbb{F}^i$-B.M. under the probability measure $\widetilde{Q}^i = L_T^i Q^i$, $i = 1, 2$, respectively. From (H2), for each i, we have a unique strong solution X^i of the SDE

$$X_t^i = X_0^i + \int_0^t \sigma(s, X_{\cdot\wedge s}^i) dW_s^i, \quad t \in [0,T]. \tag{2.5}$$

Case 1: Uniqueness in law of weak solutions

It is by now standard that $\exists$ a meas. and non-anticipating function $\Phi : [0,T] \times \mathbb{R} \times C([0,T];\mathbb{R}) \to \mathbb{R}$ not depending on $i = 1,2$, s.t.

$$X^i_t = \Phi_t(X^i_0, W^i), \quad t \in [0,T], \ \widetilde{Q}^i\text{-a.s. (and, } Q^i\text{-a.s.)}, \ i = 1,2. \qquad (2.6)$$

Then from (2.4) that $W^i_t = B^i_t + \int_0^t \widetilde{b}(s, \Phi_{\cdot\wedge s}(X^i_0, W^i), Q^i_{X^i_s})ds$, $i = 1,2$.
Hence, putting $f(s, \varphi_{\cdot\wedge s}) = \widetilde{b}(s, \varphi_{\cdot\wedge s}, Q^1_{X_s})$, $(s,\varphi) \in [0,T] \times C([0,T];\mathbb{R})$, from (2.4) and (2.6) we have

$$\begin{cases} W^1_t &= B^1_t + \displaystyle\int_0^t f(s, \Phi_{\cdot\wedge s}(X^1_0, W^1))ds, \quad t \in [0,T], \\ W^2_t &= \widetilde{B}^2_t + \displaystyle\int_0^t f(s, \Phi_{\cdot\wedge s}(X^2_0, W^2))ds, \quad t \in [0,T], \end{cases}$$

where, $t \in [0,T]$,

$$\widetilde{B}^2_t = B^2_t + \int_0^t \Big(\widetilde{b}(s, \Phi_{\cdot\wedge s}(X^2_0, W^2), Q^2_{X^2_s}) - \widetilde{b}(s, \Phi_{\cdot\wedge s}(X^2_0, W^2), Q^1_{X^1_s})\Big)ds.$$

Case 1: Uniqueness in law of weak solutions

Hence, $\exists \bar{\Phi} : [0,T] \times \mathbb{R} \times C([0,T];\mathbb{R}) \to \mathbb{R}$ meas. s.t., for both B^1, $\widetilde{B}^2$,

$$B_t^1 = \bar{\Phi}_t(X_0^1, W^1) \text{ and } \widetilde{B}_t^2 = \bar{\Phi}_t(X_0^2, W^2),\ t \in [0,T]. \tag{2.8}$$

Now we define

$$\begin{cases} d\widehat{L}_t^2 & = -(\widetilde{b}(s, \Phi_{\cdot\wedge s}(X_0^2, W^2), Q^2_{X_s^2}) - \widetilde{b}(s, \Phi_{\cdot\wedge s}(X_0^2, W^2), Q^1_{X_s^1}))\widehat{L}_t^2 dB_t^2, \quad t \\ \widehat{L}_0^2 & = 1. \end{cases} \tag{2.9}$$

From the Girsanov Theorem we know that $\widetilde{B}^2$ is an Brownian motion under the probability measure $\widehat{Q}^2 = \widehat{L}_T^2 Q^2$. Moreover, putting

Case 1: Uniqueness in law of weak solutions

$$\begin{cases} \widetilde{L}_T^2 & = \exp\{-\int_0^T f(s, \Phi_{\cdot\wedge s}(X_0^2, W^2))dW_s^2 + \frac{1}{2}\int_0^T |f(s, \Phi_{\cdot\wedge s}(X_0^2, W^2))|^2 ds\}, \\ \bar{Q}^2 & = \widetilde{L}_T^2 \widehat{Q}^2, \end{cases} \tag{2.10}$$

we have that $(W_t^2)_{t\in[0,T]}$ is a B.M. under both $\widetilde{Q}^2$ and $\bar{Q}^2$, while $(W_t^1)_{t\in[0,T]}$ is a B.M. under $\widetilde{Q}^1$.

On the other hand, since f is bounded and meas., we can prove that $\exists$ a meas. function $\widetilde{\Phi} : \mathbb{R} \times C([0,T];\mathbb{R}) \to \mathbb{R}$, s.t.

$$\widetilde{\Phi}(X_0^i, W^i) = \int_0^T f(s, \Phi_{\cdot\wedge s}(X_0^i, W^i))dW_s^i, \quad Q^i\text{-a.s.}, \; i = 1, 2.$$

Therefore, recalling the definition of L_T^1 and (2.10), we have

$$\begin{cases} L_T^1 = \exp\{-\int_0^T f(s,\Phi_{\cdot\wedge s}(X_0^1,W^1))dW_s^1 + \frac{1}{2}\int_0^T |f(s,\Phi_{\cdot\wedge s}(X_0^1,W^1))|^2 ds\}, \\ \widetilde{L}_T^2 = \exp\{-\int_0^T f(s,\Phi_{\cdot\wedge s}(X_0^2,W^2))dW_s^2 + \frac{1}{2}\int_0^T |f(s,\Phi_{\cdot\wedge s}(X_0^2,W^2))|^2 ds\}, \end{cases} \tag{2.11}$$

and we see that $\exists$ a meas. function $\widehat{\Phi} : \mathbb{R} \times C([0,T];\mathbb{R}) \to \mathbb{R}$, s.t.

$$L_T^1 = \widehat{\Phi}(X_0^1, W^1),\ Q^1\text{-a.s., and } \widetilde{L}_T^2 = \widehat{\Phi}(X_0^2, W^2),\ Q^2\text{-a.s. (and, } \bar{Q}^2\text{-a.s.).} \tag{2.12}$$

Case 1: Uniqueness in law of weak solutions

Consequently, as X_0^i is $\mathcal{F}_0^i$-measurable, $i = 1, 2$ and $Q^1_{X_0^1} = Q^2_{X_0^2}$, from (2.8), (2.10), (2.11) and (2.12) we have that, for all bounded measurable function $F : C([0,T];\mathbb{R}^d)^2 \to \mathbb{R}$,

$$E_{Q^1}[F(B^1, W^1)] = E_{\widetilde{Q}^1}[\frac{1}{\widehat{\Phi}(X_0^1, W^1)} F(\bar{\Phi}(X_0^1, W^1), W^1)]$$

$$= E_{\bar{Q}^2}[\frac{1}{\widehat{\Phi}(X_0^2, W^2)} F(\bar{\Phi}(X_0^2, W^2), W^2)] = E_{\widehat{Q}^2}[F(\widetilde{B}^2, W^2)].$$

That is,

$$Q^1_{(B^1, W^1)} = \widehat{Q}^2_{(\widetilde{B}^2, W^2)}. \tag{2.13}$$

Taking into account (2.6), we have

$$Q^1_{(B^1, W^1, X^1)} = \widehat{Q}^2_{(\widetilde{B}^2, W^2, X^2)}, \tag{2.14}$$

and, in particular, $Q^1_{X^1} = \widehat{Q}^2_{X^2}$.

Case 1: Uniqueness in law of weak solutions

On the other hand, we can prove

$\bullet W_1(Q^1_{X^1_s}, Q^2_{X^2_s})^2 = W_1(\widehat{Q}^2_{X^2_s}, Q^2_{X^2_s})^2 \leq C\int_0^s \rho(W_1(Q^1_{X^1_r}, Q^2_{X^2_r})^2)dr;$

• The continuity of $s \to W_1(Q^1_{X^1_s}, Q^2_{X^2_s})$.

Putting $u(s) := W_1(Q^1_{X^1_s}, Q^2_{X^2_s})$, $s \in [0, T]$, then we have from above,

$$u(s)^2 \leq C\int_0^s \rho(u(r)^2)dr,\ 0 \leq s \leq t \leq T.$$

From (H2)-(iii), $\int_{0+} \frac{du}{\rho(u)} = +\infty$, it follows from Bihari's inequality that $u(s) = 0$, for any $s \in [0, T]$, that is, $Q^1_{X^1_s} = Q^2_{X^2_s}$, $s \in [0, T]$. Thus, from (2.7) and (2.9) it follows that $\widetilde{B}^2 = B^2$, $\widehat{L}^2_T = 1$, and, consequently, $\widehat{Q}^2 = Q^2$. Then, $\widehat{Q}^2_{(\widetilde{B}^2, W^2, X^2)} = Q^2_{(B^2, W^2, X^2)}$, and from (2.14)

$$Q^1_{(B^1, W^1, X^1)} = Q^2_{(B^2, W^2, X^2)}. \tag{2.15}$$

This implies, in particular, $Q^1_{(B^1, X^1)} = Q^2_{(B^2, X^2)}$. □

1. Objective of the talk

2. Case 1: The drift coefficient is bounded and measurable.

3. Case 2: The coefficients are bounded, continuous.

Case 2: Preliminaries

Definition 3.1 (see, e.g., Karatzas, Shreve, 1988)

A probability $\widehat{P}$ on $(C([0,T];\mathbb{R}), \mathcal{B}(C([0,T];\mathbb{R})))$ is a solution to the local martingale problem associated with $\mathcal{A}'$, if for every $f \in C^{1,2}([0,T]\times\mathbb{R};\mathbb{R})$,

$$M_t^f := f(t,y(t)) - f(0,y(0)) - \int_0^t (\partial_s + \mathcal{A}')f(s,y(s))ds, \ t \in [0,T], \quad (3.1)$$

is a continuous local martingale w.r.t $(\mathbb{F}^y, \widehat{P})$, where $y = (y(t))_{t\in[0,T]}$ is the coordinate process on $C([0,T];\mathbb{R})$, the considered filtration $\mathbb{F}^y = (\mathcal{F}_t^y)_{t\in[0,T]}$ is that generated by $y = (y(t))_{t\in[0,T]}$ and augmented by all $\widehat{P}$-null sets, and $\mathcal{A}'$ is defined by, $y \in C([0,T];\mathbb{R})$,

$$\mathcal{A}'f(s,y) = b(s,y)\partial_x f(s,y(s)) + \frac{1}{2}\sigma^2(s,y)\partial_x^2 f(s,y(s)). \quad (3.2)$$

Case 2: Preliminaries

Let us first recall a well-known result concerning the equivalence between the weak solution of a functional SDE and the solution to the corresponding local martingale problem (see, e.g., Karatzas, Shreve, 1988).

Lemma 3.1

The existence of a weak solution $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}, \widetilde{P}, \widetilde{W}, X)$ to the following functional SDE with given initial distribution μ on $\mathcal{B}(\mathbb{R})$:

$$X_t = \xi + \int_0^t b(s, X_{\cdot\wedge s})ds + \int_0^t \sigma(s, X_{\cdot\wedge s})d\widetilde{W}_s, \ t \in [0, T],$$

is equivalent to the existence of a solution $\widehat{P}$ to the local martingale problem (3.1) associated with $\mathcal{A}'$ defined by (3.2), with $\widehat{P}_{y(0)} = \mu$. The both solutions are related by $\widehat{P} = \widetilde{P} \circ X^{-1}$, i.e., the probability measure $\widehat{P}$ is the law of the weak solution X on $(C([0, T]; \mathbb{R}), \mathcal{B}(C([0, T]; \mathbb{R})))$.

Case 2: Preliminaries

Recall the definition of the derivative of $f : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ w.r.t probability measure $\mu \in \mathcal{P}_2(\mathbb{R})$ (in the sense of P.L.Lions)(P.L.Lions' lectures at Collège de France, also see the notes of Cardaliaguet).

Definition 3.2

(i) $\widetilde{f} : L^2(\Omega, \mathcal{F}, P; \mathbb{R}) \to \mathbb{R}$ is Fréchet differentiable at $\xi \in L^2(\Omega, \mathcal{F}, P)$, if $\exists$ a linear continuous mapping $D\widetilde{f}(\xi)(\cdot) \in L(L^2(\Omega, \mathcal{F}, P; \mathbb{R}); \mathbb{R})$, s.t. $\widetilde{f}(\xi + \eta) - \widetilde{f}(\xi) = D\widetilde{f}(\xi)(\eta) + o(|\eta|_{L^2})$, with $|\eta|_{L^2} \to 0$ for $\eta \in L^2(\Omega, \mathcal{F}, P)$.

(ii) $f : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ is differentiable at $\mu \in \mathcal{P}_2(\mathbb{R})$, if for $\widetilde{f}(\xi) := f(P_\xi)$, $\xi \in L^2(\Omega, \mathcal{F}, P; \mathbb{R})$, there is some $\zeta \in L^2(\Omega, \mathcal{F}, P; \mathbb{R})$ with $P_\zeta = \mu$ such that $\widetilde{f} : L^2(\Omega, \mathcal{F}, P; \mathbb{R}) \to \mathbb{R}$ is Fréchet differentiable in ζ.

Case 2: Preliminaries

From Riesz' Representation Theorem there exists a P-a.s. unique variable $\vartheta \in L^2(\Omega, \mathcal{F}, P; \mathbb{R})$ such that $D\widetilde{f}(\zeta)(\eta) = (\vartheta, \eta)_{L^2} = E[\vartheta\eta]$, for all $\eta \in L^2(\Omega, \mathcal{F}, P; \mathbb{R})$. P.L. Lions proved that there is a Borel function $h : \mathbb{R} \to \mathbb{R}$ such that $\vartheta = h(\zeta)$, P-a.e., and function h depends on ζ only through its law P_ζ. Therefore,
$f(P_\xi) - f(P_\zeta) = E[h(\zeta) \cdot (\xi - \zeta)] + o(|\xi - \zeta|_{L^2})$, $\xi \in L^2(\Omega, \mathcal{F}, P; \mathbb{R})$.

Definition 3.3

We call $\partial_\mu f(P_\zeta, y) := h(y)$, $y \in \mathbb{R}$, the derivative of function $f : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ at P_ζ, $\zeta \in L^2(\Omega, \mathcal{F}, P; \mathbb{R})$.

Remark: $\partial_\mu f(P_\zeta, y)$ is only $P_\zeta(dy)$-a.e. uniquely determined.

Case 2: Preliminaries

Definition 3.4

We say that $f \in C^1(\mathcal{P}_2(\mathbb{R}))$, if for all $\xi \in L^2(\Omega, \mathcal{F}, P; \mathbb{R})$ there exists a P_ξ-modification of $\partial_\mu f(P_\xi, .)$, also denoted by $\partial_\mu f(P_\xi, .)$, such that $\partial_\mu f : \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ is continuous w.r.t the product topology generated by the 2-Wasserstein metric over $\mathcal{P}_2(\mathbb{R})$ and the Euclidean norm over $\mathbb{R}$, and we identify this modified function $\partial_\mu f$ as the derivative of f.

The function f is said to belong to $C_b^{1,1}(\mathcal{P}_2(\mathbb{R}))$, if $f \in C^1(\mathcal{P}_2(\mathbb{R}))$ is s.t. $\partial_\mu f : \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ is bounded and Lipschitz continuous, i.e., there exists some constant $C \geq 0$ such that

(i) $|\partial_\mu f(\mu, x)| \leq C,\ \mu \in \mathcal{P}_2(\mathbb{R}),\ x \in \mathbb{R}$;

(ii) $|\partial_\mu f(\mu, x) - \partial_\mu f(\mu', x')| \leq C(W_2(\mu, \mu') + |x - x'|),\ \mu, \mu' \in \mathcal{P}_2(\mathbb{R}),\ x, x' \in \mathbb{R}$.

Case 2: Preliminaries

Definition 3.5

We say that $f \in C^2(\mathcal{P}_2(\mathbb{R}))$, if $f \in C^1(\mathcal{P}_2(\mathbb{R}))$ and $\partial_\mu f(\mu, .) : \mathbb{R} \to \mathbb{R}$ is differentiable, and its derivative $\partial_y \partial_\mu f : \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \to \mathbb{R} \otimes \mathbb{R}$ is continuous, for every $\mu \in \mathcal{P}_2(\mathbb{R})$.

Moreover, $f \in C_b^{2,1}(\mathcal{P}_2(\mathbb{R}))$, if $f \in C^2(\mathcal{P}_2(\mathbb{R})) \bigcap C_b^{1,1}(\mathcal{P}_2(\mathbb{R}))$ and its derivative $\partial_y \partial_\mu f : \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \to \mathbb{R} \otimes \mathbb{R}$ is bounded and Lipschitz-continuous.

Remark: $C_b^{2,1}(\mathbb{R} \times \mathcal{P}_2(\mathbb{R})), C_b^{1,2,1}([0,T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}); \mathbb{R})$ are similarly defined.

Case 2: Preliminaries

Now we can give our Itô's formula.

Theorem 3.1

Let $\sigma = (\sigma_s)$, $\gamma = (\gamma_s)$, $b = (b_s)$, $\beta = (\beta_s)$ $\mathbb{R}$-valued adapted stochastic processes, such that

(i) There exists a constant $q > 6$ s.t. $E[(\int_0^T (|\sigma_s|^q + |b_s|^q)ds)^{\frac{3}{q}}] < +\infty$;

(ii) $\int_0^T (|\gamma_s|^2 + |\beta_s|)ds < +\infty$, P-a.s.

Let $F \in C_b^{1,2,1}([0,T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}))$. Then, for the Itô processes

$$X_t = X_0 + \int_0^t \sigma_s dW_s + \int_0^t b_s ds,\ t \in [0,T],\ X_0 \in L^2(\Omega, \mathcal{F}_0, P),$$

$$Y_t = Y_0 + \int_0^t \gamma_s dW_s + \int_0^t \beta_s ds,\ t \in [0,T],\ Y_0 \in L^2(\Omega, \mathcal{F}_0, P),$$

Case 2: Preliminaries

Theorem 3.1 (continued)

we have

$$
\begin{aligned}
& F(t, Y_t, P_{X_t}) - F(0, Y_0, P_{X_0}) \\
= & \int_0^t \Big(\partial_r F(r, Y_r, P_{X_r}) + \partial_y F(r, Y_r, P_{X_r})\beta_r + \frac{1}{2}\partial_y^2 F(r, Y_r, P_{X_r})\gamma_r^2 \\
+ & \bar{E}[(\partial_\mu F)(r, Y_r, P_{X_r}, \bar{X}_r)\bar{b}_r + \frac{1}{2}\partial_z(\partial_\mu F)(r, Y_r, P_{X_r}, \bar{X}_r)\bar{\sigma}_r^2] \Big) dr \\
+ & \int_0^t \partial_y F(r, Y_r, P_{X_r})\gamma_r dW_r, \ t \in [0, T].
\end{aligned}
$$

Here $(\bar{X}, \bar{b}, \bar{\sigma})$ denotes an independent copy of (X, b, σ), defined on a P.S. $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$. The expectation $\bar{E}[\cdot]$ on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ concerns only r.v. endowed with the superscript $^-$.

Case 2: Preliminaries

(H3) The coefficients $(\sigma, b) \in C_b^{1,2,1}([0,T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}); \mathbb{R} \times \mathbb{R})$.

Theorem 3.2 (Buckdahn, Li, Peng and Rainer, 2014)

Let $\Phi \in C_b^{2,1}(\mathbb{R} \times \mathcal{P}_2(\mathbb{R}))$, then under assumption (H3) the following PDE:

$$\begin{cases} 0 = \partial_t V(t,x,\mu) + \partial_x V(t,x,\mu) b(x,\mu) + \dfrac{1}{2}\partial_x^2 V(t,x,\mu)\sigma^2(x,\mu) \\ \qquad + \displaystyle\int_{\mathbb{R}} (\partial_\mu V)(t,x,\mu,y) b(y,\mu)\mu(dy) \\ \qquad + \dfrac{1}{2}\displaystyle\int_{\mathbb{R}} \partial_y(\partial_\mu V)(t,x,\mu,y)\sigma^2(y,\mu)\mu(dy), \\ \qquad\qquad\qquad (t,x,\mu) \in [0,T) \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}); \\ V(T,x,\mu) = \Phi(x,\mu), \ (x,\mu) \in \mathbb{R} \times \mathcal{P}_2(\mathbb{R}). \end{cases}$$

has a unique classical solution $V(t,x,\mu) \in C_b^{1,2,1}([0,T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}); \mathbb{R})$.

Case 2: Existence of a weak solution

Let b and σ satisfy the following assumption:

(H4) $b, \sigma : [0,T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ are continuous and bounded.

We want to study weak solution of the following mean-field SDE:

$$X_t = \xi + \int_0^t b(s, X_s, Q_{X_s})ds \int_0^t \sigma(s, X_s, Q_{X_s})dB_s, \ t \in [0,T], \qquad (3.3)$$

where $\xi \in L^2(\Omega, \mathcal{F}_0, P; \mathbb{R})$ obeys a given distribution law $Q_\xi = \nu \in \mathcal{P}_2(\mathbb{R})$ and $(B_t)_{t\in[0,T]}$ is a B.M. under the probability measure Q.

Extension of the corresponding local martingale problem:

Definition 3.6

A probability measure $\widehat{P}$ on $(C([0,T];\mathbb{R}),\mathcal{B}(C([0,T];\mathbb{R})))$ is a solution to the local martingale problem (resp., martingale problem) associated with $\widetilde{\mathcal{A}}$, if for every $f \in C^{1,2}([0,T]\times\mathbb{R};\mathbb{R})$ (resp., $f \in C_b^{1,2}([0,T]\times\mathbb{R};\mathbb{R})$), the process

$$C^f(t,y,\mu) := f(t,y(t)) - f(0,y(0)) - \int_0^t \big((\partial_s + \widetilde{\mathcal{A}})f\big)(s,y(s),\mu(s))ds, \tag{3.4}$$

is a continuous local $(\mathbb{F}^y,\widehat{P})$-martingale (resp., continuous $(\mathbb{F}^y,\widehat{P})$-martingale),

Case 2: Existence of a weak solution

Definition 3.6 (continued)

where $\mu(t) = \widehat{P}_{y(t)}$ is the law of the coordinate process $y = (y(t))_{t\in[0,T]}$ on $C([0,T];\mathbb{R})$ at time t, the filtration $\mathbb{F}^y$ is that generated by y and completed, and $\widetilde{\mathcal{A}}$ is defined by

$$(\widetilde{\mathcal{A}}f)(s,y,\nu) := \partial_y f(s,y)b(s,y,\nu) + \frac{1}{2}\partial_y^2 f(s,y)\sigma^2(s,y,\nu), \qquad (3.5)$$

$(s,y,\nu) \in [0,T]\times\mathbb{R}\times\mathcal{P}_2(\mathbb{R})$. Here $((\partial_s + \widetilde{A})f)(s,y(s),\mu(s))$ abbreviates

$$((\partial_s + \widetilde{A})f)(s,y(s),\mu(s)) := (\partial_s f)(s,y(s)) + (\widetilde{A}f)(s,y(s),\mu(s)).$$

Case 2: Existence of a weak solution

Proposition 3.1

The existence of a weak solution $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}, Q, B, X)$ to equation (3.3) with initial distribution ν on $\mathcal{B}(\mathbb{R})$ is equivalent to the existence of a solution $\widehat{P}$ to the local martingale problem (3.4) associated with $\widetilde{\mathcal{A}}$ defined by (3.5), with $\widehat{P}_{y(0)} = \nu$.

Case 2: Existence of a weak solution

Lemma 3.2

Let the probability measure $\widehat{P}$ on $(C([0,T];\mathbb{R}), \mathcal{B}(C([0,T];\mathbb{R})))$ be a solution to the local martingale problem associated with $\widetilde{\mathcal{A}}$. Then, for the second order differential operator

$$\begin{aligned}(\mathcal{A}f)(s,y,\nu) \quad := \quad & (\widetilde{\mathcal{A}}f)(s,y,\nu) + \int_{\mathbb{R}} (\partial_\mu f)(s,y,\nu,z) b(s,z,\nu)\nu(dz) \\ & + \frac{1}{2}\int_{\mathbb{R}} \partial_z(\partial_\mu f)(s,y,\nu,z)\sigma^2(s,z,\nu)\nu(dz),\end{aligned} \tag{3.6}$$

Case 2: Existence of a weak solution

Lemma 3.2 (continued)

applying to functions $f \in C^{1,2}([0,T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}); \mathbb{R})$ we have that, for every such $f \in C^{1,2}([0,T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}); \mathbb{R})$, the process

$$\begin{aligned} C^f(t, y, \mu) := & f(t, y(t), \mu(t)) - f(0, y(0), \mu(0)) \\ & - \int_0^t (\partial_s + \mathcal{A}) f(s, y(s), \mu(s)) ds, \ t \in [0,T], \end{aligned} \tag{3.7}$$

is a continuous local $(\mathbb{F}^y, \widehat{P})$-martingale, where $\mu(t) = \widehat{P}_{y(t)}$ is the law of the coordinate process $y = (y(t))_{t \in [0,T]}$ on $C([0,T]; \mathbb{R})$ at time t, the filtration $\mathbb{F}^y$ is that generated by y and completed. Moreover, if $f \in C_b^{1,2,1}([0,T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}); \mathbb{R})$, this process C^f is an $(\mathbb{F}^y, \widehat{P})$-martingale.

Case 2: Existence of a weak solution

Now we can give the main statement of this section.

Theorem 3.3

Under assumption (H4) mean-field SDE (3.3) has a weak solution $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}, \widetilde{Q}, B, X)$.

Remark 2. If $b, \sigma : [0,T] \times C([0,T];\mathbb{R}) \times \mathcal{P}_2(C([0,T];\mathbb{R})) \to \mathbb{R}$ are bounded and continuous, then the following mean-field SDE

$$X_t = \xi + \int_0^t b(s, X_{\cdot\wedge s}, Q_{X_{\cdot\wedge s}})ds + \int_0^t \sigma(s, X_{\cdot\wedge s}, Q_{X_{\cdot\wedge s}})dB_s,\ \ t \in [0,T], \tag{1.1}$$

where $\xi \in L^2(\Omega, \mathcal{F}_0, P)$ obeys a given distribution law $Q_\xi = \nu$, has a weak solution $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}, \widetilde{Q}, X, B)$.

Case 2: Uniqueness in law of weak solutions

Now we want to study the uniqueness in law for the weak solution of the mean-field SDE (3.3).

Definition 3.7

We call $\mathcal{C} \subset b\mathcal{B}(\mathbb{R}) = \{\phi \mid \phi : \mathbb{R} \to \mathbb{R} \text{ bounded Borel-measurable function}\}$ a determining class on $\mathbb{R}$, if for any two finite measures ν_1 and ν_2 on $\mathcal{B}(\mathbb{R})$, $\int_{\mathbb{R}^d} \phi(x)\nu_1(dx) = \int_{\mathbb{R}^d} \phi(x)\nu_2(dx)$ for all $\phi \in \mathcal{C}$ implies $\nu_1 = \nu_2$.

Remark: The class $C_0^\infty(\mathbb{R})$ is a determining class on $\mathbb{R}$.

Case 2: Uniqueness in law of weak solutions

Theorem 3.4

For given $f \in C_0^\infty(\mathbb{R})$, we consider the Cauchy problem

$$\begin{aligned} \frac{\partial}{\partial t} v(t,x,\nu) &= \mathcal{A} v(t,x,\nu),\ (t,x,\nu) \in [0,T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}), \\ v(0,x,\nu) &= f(x),\ x \in \mathbb{R}, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} \mathcal{A} v(t,x,\nu) = \ & (\widetilde{\mathcal{A}} v)(t,x,\nu) + \int_{\mathbb{R}} (\partial_\mu v)(t,x,\nu,u) b(t,u,\nu) \nu(du) \\ & + \frac{1}{2} \int_{\mathbb{R}} \partial_z (\partial_\mu v)(t,x,\nu,u) \sigma^2(t,u,\nu) \nu(du), \\ (\widetilde{\mathcal{A}} v)(t,x,\nu) = \ & \partial_y v(t,x,\nu) b(t,x,\nu) + \frac{1}{2} \partial_y^2 v(t,x,\nu) \sigma^2(t,x,\nu), \end{aligned}$$

$(t,x,\nu) \in [0,\infty) \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R})$.

Case 2: Uniqueness in law of weak solutions

Theorem 3.4 (continued)

We suppose that, for all $f \in C_0^\infty(\mathbb{R})$, (3.8) has a solution $v_f \in C_b([0,\infty) \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R})) \bigcap C_b^{1,2,1}((0,\infty) \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}))$. Then, the local martingale problem associated with $\widetilde{\mathcal{A}}$ (Recall Definition 3.6) and with the initial condition δ_x has at most one solution.

Remark: Theorem 3.4 generalizes a well-known classical uniqueness for weak solutions to the case of mean-field SDE.

Corollary 3.1

Under the assumption of Theorem 3.4, we have for the mean-field SDE (3.3) the uniqueness in law, that is, for any weak solutions, $i = 1, 2$ $(\Omega^i, \mathcal{F}^i, \mathbb{F}^i, Q^i, B^i, X^i)$, of SDE (3.3), we have $Q^1_{X^1} = Q^2_{X^2}$.

Uniqueness in law of weak solutions

Sketch of proof of Theorem 3.4: Let $T > 0$, denote by $y{=}(y(t))_{t\in[0,T]}$ the coordinate process on $C([0,T];\mathbb{R})$. Let P^1 and P^2 be two arbitrary solutions of the local martingale problem associated with $\widetilde{\mathcal{A}}$ and initial condition $x \in \mathbb{R}$: $P^l_{y(0)} = \delta_x$, $l = 1, 2$.

Consequently, due to Lemma 3.2, for any $g \in C_b^{1,2,1}([0,T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}))$,

$$C^g(t, y, P^l_y) := g(t, y(t), P^l_{y(t)}) - g(0, x, \delta_x) - \int_0^t (\partial_s + \mathcal{A}) g(s, y(s), P^l_{y(s)}) ds, \tag{3.9}$$

is a P^l-martingale, $l = 1, 2$, $t \in [0,T]$. For given $f \in C_0^\infty(\mathbb{R})$, let $v_f \in C_b([0,T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R})) \bigcap C_b^{1,2,1}((0,T) \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}))$ be a solution of the Cauchy problem (3.8).

Uniqueness in law of weak solutions

Then putting $g(t,z,\nu) := v_f(T-t,z,\nu)$, $t \in [0,T]$, $z \in \mathbb{R}$, $\nu \in \mathcal{P}_2(\mathbb{R})$, defines a function g of class $C_b([0,T]\times\mathbb{R}\times\mathcal{P}_2(\mathbb{R}))\bigcap C_b^{1,2,1}((0,T)\times\mathbb{R}\times\mathcal{P}_2(\mathbb{R}))$ which satisfies

$$\partial_s g(s,z,\nu)+\mathcal{A}g(s,z,\nu)=0,\ \ g(T,z,\nu)=f(z),\ (s,z,\nu)\in[0,T]\times\mathbb{R}\times\mathcal{P}_2(\mathbb{R}).$$

From (3.9) we see that $\{C^g(s,y,P_y^l), s\in[0,T]\}$ is an $(\mathbb{F}^y,P^l)$-martingale. Hence, for $E^l[\cdot]=\int_{\Omega^l}(\cdot)dP^l$,

$$E^l[f(y(T))]=E^l[g(T,y(T),P_{y(T)}^l)]=g(0,x,\delta_x),\ \ x\in\mathbb{R},\ \ l=1,2,$$

that is $E^1[f(y(T))]=E^2[f(y(T))]$, for all $f\in C_0^\infty(\mathbb{R})$. Combining this with the arbitrariness of $T\geq 0$, we have that $P_{y(t)}^1=P_{y(t)}^2$, for every $t\geq 0$.

Uniqueness in law of weak solutions

Consequently, P^1, P^2 are solutions of the same classical martingale problem, associated with $\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}^l$, $l = 1, 2$,

$$\widetilde{\mathcal{A}}^l \phi(t,z) = \partial_y \phi(t,z) \widetilde{b}^l(t,z) + \partial_y^2 \phi(t,z) (\widetilde{\sigma}^l(t,z))^2, \ \phi \in C^{1,2}([0,T] \times \mathbb{R}; \mathbb{R}),$$

with the coefficients $\widetilde{\sigma}^1 = \widetilde{\sigma}^2$, $\widetilde{b}^1 = \widetilde{b}^2$ (without mean field term),

$$\widetilde{\sigma}^l(t,z) = \sigma(t,z,P^l_{y(t)}), \ \widetilde{b}^l(t,z) = b(t,z,P^l_{y(t)}), \ (t,z) \in [0,T] \times \mathbb{R},$$

and we have seen that $P^1_{y(t)} = P^2_{y(t)}$, $t \in [0,T]$.

....

$\rightsquigarrow P^1 = P^2$, i.e., the local martingale problem has at most one solution. $\square$

Thank you very much!

谢谢!