BEYOND STOCHASTIC CONTROL...

(and MFG....!)

I INTRODUCTION

II BAYESIAN VS PARTIAL INFORMATION

III CONDITION. PROCESSES

(some extracts from this year CdF course
more details may be downloaded...)

I INTRODUCTION

- Bayesian learning and stochastic control
  - (ex. ads ...)

  show to formulate and solve the problems?

  - links with partial information problems ...

- Optimal control
  - natural models
  - how to formulate and solve the problems?
  - feedbacks vs general controls

- more in the context of MFG (instead of control pbs ...)

  to be continued...

II BAYESIAN VS PARTIAL INFORMATION

II.1. EX.

$$dX_t = a\,\alpha_t\,dt + dW_t$$

a unknown parameter

α_t control ("adapted to X_t")

W_t B-motion

only observation : X_t

B. rule : μ_0 initial guess on a (proba)

$$\text{"}\quad \mu_{t+h} = \frac{e^{-\frac{|dX - a\alpha h|^2}{2h}}\,\mu_t}{\int e^{-\frac{|dX - b\alpha h|^2}{2h}}\,\mu_t(b)\,db} \quad\text{"}$$

$$P(A|B) = P(B|A)\,\frac{P(A)}{P(B)}$$

(B) $d\mu_t = \left(a - \left(\int b\mu_t\right)\right)\mu_t \alpha_t \, dX_t - \left(\int b\mu_t\right)\left(a - \left(\int b\mu_t\right)\right)\mu_t \alpha_t^2 \, dt$

<u>Prop</u>: $\exists !$ sol. given by $\mu_t = \dfrac{\rho_t}{\int \rho_t \, db}$ where $\rho_0 = \mu_0$ and

ρ_t solves the (Z) eq.

$$d\rho_t = a \rho_t \left(\alpha_t \, dX_t\right)$$

$$\left(\rho_t = \rho_0 \exp\left(a \int_0^t \alpha_s \, dX_s - \frac{1}{2} a^2 \int_0^t \alpha^2 ds\right)\right).$$

<u>Remarks</u>:

i) $\mu_0 = \rho_0$ Gaussian $\Rightarrow \mu_t, \rho_t$ "Gaussians" hence reduced equations for mean and variance

ii) $\mu_0 = \rho_0 = \sum_{i=1}^{M} \Theta^i \delta_{a_i} \Rightarrow \rho_t = \sum_{i=1}^{M} \theta^i(t) \, \delta_{a_i}$

$$d\theta^i = a_i \Theta^i \alpha_t \, dX_t$$

iii) If $\alpha_t \equiv \alpha$, $dX_t = a_0 \alpha \, dt + dW_t$, $\mu_t \not\to \delta_{a_0}$ unless $a_0 \in \mathrm{Supp}(\mu_0 = \rho_0)$. In fact μ_t concentrates on the set of minima of $\quad \inf\left(|a - a_0| \,/\, a \in \mathrm{Supp}\,\mu_0\right)$

II.2 How to combine with stochastic control?

(inf. hor., $\rho > 0$, cost. $L(x, \alpha)$)

"pick randomly a_t according to μ_t hence dX, $d\mu$ and average the cost wrt μ_t" dynamic programming?

better to rephrase everything in terms of partial info.

$dX_t = A\,\alpha_t\,dt + dW_t$ $\qquad$ A indt of W unknown

observation X_t, α_t "adapted to X_t"

$$E\int_0^\infty e^{-\rho t} L(X_t, \alpha_t)\,dt]$$

$$= E[\int_0^\infty e^{-\rho t} L(X_t, \alpha_t)\, e^{\int_0^t A\alpha_s dW_s - \frac{1}{2}\int_0^t A^2\alpha_s^2 ds}]$$

now $dX_t = dW_t$

$$= E[\int_0^\infty e^{-\rho t} L(X_t, \alpha_t)\, E[e^{\int_0^t A\alpha_s dW_s - \frac{1}{2}\int_0^t A^2\alpha_s^2 ds} | \mathcal{F}_t]]$$

where $\mathcal{F}_t = \sigma(X_s / 0 \le s \le t)$).

Zakai: introduce f_t (random measure adapted to $\mathcal{F}_t$)

$$\int \varphi(a) f_t(a)\,da = E[\varphi(A) \exp(\int_0^t A\alpha_s dW_s - \frac{1}{2}\int_0^t A^2\alpha_s^2 ds) | \mathcal{F}_t]$$

(Z) $df_t = a\, f_t\, \alpha_t\, dX_t$

Conclusion: $V(x, f) = \inf_{\alpha_t} E \int_0^\infty e^{-\rho t} L(x_t, \alpha_t) \int f_t)$

$dX_t = dW_t$, $df_t = b\, f_t\, \alpha_t\, dW_t$, $X_0 = x$, $f_0 = f \in M_b^+$.

∞ D HJB (PL^2 - -) wellposed for instance

if $\alpha_t \to A$ bounded (closed) set of $\mathbb{R}^k$.

Finite D reduction (specific to ex.) fix f

$$\int f_t = \int f \left(\exp\left(b \int_0^t \alpha_s\, dW_s - \frac{b^2}{2} \int_0^t \alpha_s^2 ds\right)\right) db.$$

introduce $\varphi(\lambda, \mu) = \int f \exp\left(\lambda b - \mu \frac{b^2}{2}\right)$

$d\lambda_t = \alpha_t\, dW_t$, $d\mu_t = \alpha_t^2\, dt$:

Prop. $V(x, f) = V(x, \lambda, \mu) = \inf_{\alpha_t} E\left[\int_0^\infty e^{-\rho t} L(X_t, \alpha_t)\, \varphi(\lambda_t, \mu_t)\right]$

formally $\rho V - \frac{1}{2} \frac{\partial^2 V}{\partial x^2} + \sup_{\alpha \in A} \left\{ -\frac{1}{2} \alpha^2 \frac{\partial^2 V}{\partial \lambda^2} - \alpha \frac{\partial^2 V}{\partial x \partial \lambda} - \alpha^2 \frac{\partial V}{\partial \mu} - L(x, \alpha)\varphi \right\} = 0$

singular φ unbounded ("exp." in λ)

[observe that $\frac{\partial^2 \varphi}{\partial \lambda^2} = -\frac{1}{2} \frac{\partial \varphi}{\partial \mu}$ and

use U : $V = U\varphi$]

Then the equation for U is

$$\rho U - \frac{1}{2}\frac{\partial^2 U}{\partial x^2} + \sup_{\alpha \in A}\Big\{ -\frac{1}{2}\alpha^2 \frac{\partial^2 U}{\partial \lambda^2} - \alpha \frac{\partial^2 U}{\partial x \partial \lambda} - \alpha^2 \frac{\partial U}{\partial \mu} - L(x,\alpha)$$
$$- \alpha^2 \Big(\frac{\partial}{\partial \lambda}\log \psi\Big)\frac{\partial U}{\partial \lambda} - \alpha\Big(\frac{\partial}{\partial \lambda}\log \psi\Big)\frac{\partial U}{\partial x}\Big\} = 0$$

for all $x, \lambda, \mu \geq 0$ (no BC at $\mu = 0$)

<u>Prop.</u> If $\left|\frac{\partial}{\partial \lambda}\log \psi\right| \leq C(1 + |\lambda|)$ (OK if f_0 decays at least like a G.), then $\exists!$ viscosity solution U.

<u>Remark</u>: Most general situation is

$$dX_t = \sigma(X_t, \alpha_t)\, dW_t + b(X_t, \alpha_t)\, dt. \quad \text{(observed)}$$

$$dY_t = \gamma(X_t, Y_t, \alpha_t)\, dW_t + \delta(X_t, Y_t, \alpha_t)\, dZ_t + \beta(X_t, Y_t, \alpha_t)\, dt$$
(not observed)

W, Z ind^t B.

Then f_t solves a SPDE ...

(optimal control of SPDE, HJB $\Leftrightarrow$ MFG)

III CONDITION. PROCESSES

III.1 Introduction

- motivation : modelling issue!
- static / dynamic
- condition. by no exit. : domain D how ?

(${D} \uparrow \mathbb{R}^d$ recovering classical dyn. prog. ?)

III.2 Two examples

$$dX_t = \alpha_t \, dt + dW_t \qquad X_0 = x \in \bar{D} \quad \left(\begin{array}{c}\text{smooth}\\ \text{domain}\end{array} \text{ in } \mathbb{R}^d\right)$$

$$① \inf_{|\alpha_t| \le L} \mathbb{E}_x [g(X_T) / \tau_x \ge T] \quad \left(= \frac{\mathbb{E}_x (g(X_T) 1_{\tau_x \ge T})}{\mathbb{P}(\tau_x \ge T)} \right)$$

$$② \inf_{\alpha_t} \int_0^T \mathbb{E}_x [f(X_t) + \tfrac{1}{2}|\alpha_t|^2 / \tau_x \ge t] \, dt + \mathbb{E}_x [g(X_T) / \tau_x \ge T]$$

- feedbacks : $\alpha_t = \alpha(t, x)$

or ?

- general possibly stochastic controls

III.3 Feedbacks

$$dX_t = \alpha(t, X_t)\,dt + dW_t, \quad X_0 = x \in \overline{D}$$

$$\text{(I)} \quad \inf_{|\alpha| \leq L} E_x\left[g(X_T) / \tau_x \geq T\right] \qquad (L > 0)$$

$$E_x\left[g(X_T)\,\mathbb{1}_{(\tau_x \geq T)}\right] / P(\tau_x \geq T)$$

$$= \frac{\int p(T,y)\, g(y)\, dy}{\int p(T,y)\, dy}$$

$$\begin{cases} \frac{\partial p}{\partial t} - \frac{1}{2}\Delta p + \operatorname{div}(\alpha p) = 0, & p|_{t=0} = p_0 \ (\text{ex. } \delta_x) \\ p = 0 \text{ on } \partial D \end{cases}$$

$$\min_{|\alpha| \leq L} \left\{ \frac{\int p(T,y)\, g(y)}{\int p(T,y)\, dy} \Big/ \frac{\partial p}{\partial t} \cdots \right\}.$$

<u>Prop.</u> $\exists$ optimal control and for any optimal control α, $\exists u$ solving

$$-\frac{\partial u}{\partial t} - \frac{1}{2}\Delta u + |\nabla u| = 0, \quad u|_{t=T} = \frac{g}{\int p} - \frac{\int g p}{\left(\int p\right)^2}$$

and $\alpha = -\frac{\nabla u}{|\nabla u|}$ if $\nabla u \neq 0$.

(Opt. control of FP eqs and MFG!)

Sketch: $\dfrac{\int \delta\rho\, g}{\int \rho} - \dfrac{\int \delta\rho \int \rho g}{(\int \rho)^2} \geq 0$

$\delta\rho = q - \rho \quad q \Leftrightarrow \beta$ control $\quad \delta\alpha = \beta - \alpha$

$$\frac{\partial \delta\rho}{\partial t} - \frac{1}{2}\Delta \delta\rho + \operatorname{div}(\delta\alpha\, \rho + \alpha\, \delta\rho) = 0, \quad \delta\rho|_{t=0} = 0$$

adjoint state u: $-\dfrac{\partial u}{\partial t} - \dfrac{1}{2}\Delta u = A$, $u|_{t=T} = \dfrac{g}{\int\rho} - \dfrac{\int g\rho}{(\int \rho)^2}$

$$0 \leq \int u\, \delta\rho = \iint A\, \delta\rho + (\delta\alpha\rho + \alpha\,\delta\rho)\cdot\nabla u$$

choose $A = +\alpha\nabla u$ $\quad = +\iint (\beta - \alpha)\rho\cdot\nabla u \quad \forall |\beta| \leq 1.$

hence $\alpha = -\dfrac{\nabla u}{|\nabla u|}$.

(2) $\displaystyle \min_{\alpha} \int_0^T \frac{\int (f + \frac{1}{2}|\alpha|^2)\rho}{\int \rho}\, dt + \frac{\int g\, \rho(T)}{\int \rho}$

Prop.: $\exists$ optimal control and, for any optimal control, $\exists u$

solving: $u|_{t=T} = \dfrac{g}{\int\rho} - \dfrac{\int g\rho}{(\int\rho)^2}$

$$-\frac{\partial u}{\partial t} - \frac{1}{2}\Delta u + \frac{1}{2}\left(\int\rho\right)|\nabla u|^2 = \frac{f}{\int \rho} \underbrace{- \frac{\int f\rho}{(\int\rho)^2} - \frac{1}{2}\int |\nabla u|^2 \rho}_{-C(t)}$$

and $\alpha = -\left(\int\rho\right)(\nabla u)$.

Rk nonuniqueness! $g \equiv 0$!

III.4 Relaxing the conditioning

$D = B_R \quad R \to +\infty$

$\alpha_R \Longleftrightarrow -\nabla \tilde{u}_R$

where $\tilde{u}_R = u_R + C_R$ for any constant C_R

and $u_R = \dfrac{g}{\int p_R} - C_R$ at time T (Ex. 1)

and $\int p_R \longrightarrow 1$ as $R \to \infty$, hence $\tilde{u}_R$ goes to

the solution of HJB (decoupled from the p eq. i.e.

MFG $\to$ HJB + FP decoupled ... !)

III.5 General controls

Case (1) for instance : "cond. law" SPDE

$$dp + \left(-\frac{1}{2}\Delta p + \operatorname{div}(\alpha p)\right) dt = \nabla p \, dW$$

$$p = 0 \text{ on } \partial D, \quad p|_{t=0} = p_0$$

and

$$\min_{|\alpha| \leq 1} \frac{E\left(\int g(y)\, p(T,y,\omega)\, dy\right)}{E\left(p(T,y,\omega)\right)}$$

(FB stochastic system for optimal controls $\Leftrightarrow$ MFG)

or HJB approach $\operatorname{Min} = \Phi(T, p_0)$ and Φ

solves

$$0 = \frac{\partial \Phi}{\partial t} - \frac{1}{2}\frac{\partial^2 \Phi}{\partial p^2}(\nabla p, \nabla p) + \sup_{\|\alpha\|_\infty \leq 1}\left\{\left\langle \frac{\partial \Phi}{\partial p}, -\frac{1}{2}\Delta p + \operatorname{div}(\alpha p)\right\rangle\right\}$$

with $\Phi|_{t=0} = \frac{\int g p}{\int p}$

<u>Remarks</u>: 1) feedbacks $\alpha(t,x)$ lead to the same HJB eq. without the 2^{nd} order term

2) $\alpha = -\frac{\nabla_x \frac{\partial \Phi}{\partial p}}{\left|\nabla_x \frac{\partial \Phi}{\partial p}\right|}$

"THM" If g is not constant, the two infima are not equal in general (i.e $\underset{G.C.}{\mathrm{Min}} < \underset{F.C.}{\mathrm{Min}}$ in general).

$$0 \stackrel{?}{=} \frac{\partial^2 \overline{\Phi}}{\partial \rho^2}(\nabla\rho, \nabla\rho)$$

yes at time T: $\Phi = \frac{\int g\rho}{\int \rho}$, $\frac{\partial^2 \Phi}{\partial \rho^2} = -2\frac{(\int g\nabla\rho)(\int \nabla\rho)}{(\int \rho)^2}$

and $\int \nabla\rho = 0$ $(\rho = 0 \text{ on } \partial D)$.

but $\frac{\partial}{\partial t}\frac{\partial^2 \Phi}{\partial \rho^2}(\nabla\rho, \nabla\rho)\Big|_{t=0} \neq 0$: because of a piece

$$\Phi_1' = \frac{\int g\rho \int -\Delta\rho}{\int \rho^2}$$

and $\frac{\partial^2 \Phi_1'}{\partial \rho^2}(\nabla\rho, \nabla\rho) = 2\frac{\int g\nabla\rho \int -\Delta(\nabla\rho)}{\int \rho^2} \neq 0$ for some ρ

if $\int g\nabla\rho \neq 0$ for some ρ i.e g non constant...

III $T \longrightarrow +\infty$

Formally if α becomes time indt then

$$\rho \sim e^{-\lambda_1 t}\, m \int \rho_0 \varphi$$

where φ, m are the 1st eigenf., λ_1 1st eigenv

$$-\frac{1}{2}\Delta m + \operatorname{div}(\alpha m) = \lambda_1 m \;, m > 0, \int m = 1$$

$$-\frac{1}{2}\Delta\varphi - \alpha\nabla\varphi = \lambda_1\varphi \;, \varphi > 0, \int\varphi = 1$$

and $m = \varphi = 0$ on ∂D.

And we may expect to obtain the following problem as $T \to \infty$ (for example 1)

$$\operatorname*{Min}_{|\alpha| \le L} \left\{ \int g m \;/\; -\frac{1}{2}\Delta m + \operatorname{div} \alpha m = \lambda_1 m ,\; m > 0, \int m = 1, m = 0 \text{ on } \partial D \right\}$$

Justification (see also MFG $T \to \infty$)

<u>General tool</u> : generalized inv. and QI measures for time dependent coeff. !

<u>Rk 1</u> $\alpha(t_n + \cdot, \cdot) \xrightarrow[n]{} \alpha(t, x) \in L^\infty(\mathbb{R}_t \times \mathcal{V})$

<u>Rk 2</u> from my course on ergodic problems $(+\varepsilon)$, valid for general unif'ly elliptic operators / diffusion processes with time dependent coefficients. Ex. 1) periodic base

<u>Th</u> $\exists!\ m > 0$ "bounded" in (x, t) solution of
$$\begin{cases} \dfrac{\partial m}{\partial t} + A(t)^* m = 0 & \text{in } \mathcal{V} \times \mathbb{R}_t \\ \int m(t)\, dx = 1, & \forall t \in \mathbb{R} \end{cases}$$

<u>Cor</u>: if A ind^t of t, m ind^t of ; periodic / a.per. / stationary $\Rightarrow$ m periodic / a.per. / stationary

In the course (16-17), extension to $Q\,\mathrm{Im}$
$\forall\ \alpha \in L^{\infty}_{x,t}$

<u>TH</u> i) $\exists!\ (m, \lambda)$ "bounded" in $t \in \mathbb{R}$

solution of

$$\begin{cases} \dfrac{\partial m}{\partial t} - \dfrac{1}{2}\Delta m + \mathrm{div}(\alpha m) = \lambda m & \text{for } t \in \mathbb{R},\ x \in D \\ m > 0,\ \displaystyle\int_D m = 1\ \ \forall t \in \mathbb{R},\ m|_{\partial D} = 0 \\ \lambda \text{ bded from below and from above on } \mathbb{R} \end{cases}$$

ii) $\exists!\ (\varphi, \mu)$ "bounded" in $t \in \mathbb{R}$ solution of

$$\frac{\partial \varphi}{\partial t} - \frac{1}{2}\Delta\varphi - \alpha\nabla\varphi = \mu\varphi \quad \text{for } t \in \mathbb{R},\ x \in D$$

$$\varphi > 0,\ \int_D \varphi m = 1,\ \forall t \in \mathbb{R},\ \varphi|_{\partial D} = 0$$

and $\lambda \equiv \mu$.

$\big((\varphi, \mu) \sim (\varphi e^{A(t)}, \mu + \dot{A}) \big)$

$T \to \infty$, limit pb is

$$\left[\begin{array}{l} \text{Min} \left\{ \int g\, m(0)\, dx \;/\; |\alpha| \le L \text{ on } D \times \mathbb{R} \right. \\ \frac{\partial m}{\partial t} - \frac{1}{2}\Delta m + \operatorname{div}(\alpha m) = \lambda m,\ m > 0,\ m|_{\partial D} = 0,\ \left. \int m = 1\ \forall t \right\} \\ \lambda, 1/\lambda \text{ bded on } \mathbb{R} \end{array}\right.$$

or $\int \left(b + \frac{1}{2}|\alpha|^2\right) m(0)\, dx$ (ex. 2)

Rk conditions for unique $\Rightarrow$ α indt of t!

(Opt. cond. "MFG factor with D. BC" ...)

for instance in ex. 2 if $\|b\| < \varepsilon_0$, uniqueness!