Network of interacting neurons.

PDE and Probability Methods for Interactions

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Introduction

$$\begin{cases} V_t = V_0 + \int_0^t b(V_s)ds + \sigma W_t \\ \\ \tau_k = \inf\{t \geq \tau_{k-1},\ V_{t-} \geq V^{th}\} & (\tau_0 = 0) \\ V_{\tau_{k+}} = V_r \end{cases}$$

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Introduction

A singular McKean-Vlasov equation

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V_t = V_0 + \int_0^t b(V_s)ds + \sigma W_t - M_t \left(V^{th} - V^r \right) + \alpha \mathbb{E}(M_t) \\
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Stochastic process

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Associated Fokker-Planck equation $p(v,t) = \mathbb{P}(V_t \in dv)$

$$\begin{cases} \frac{\partial}{\partial t} p(v,t) - \frac{1}{2} \frac{\partial^2}{\partial v^2} p(v,t) + \frac{\partial}{\partial v} \left[(b(v) + \alpha e'(t))p(v,t)\right] = \delta_{V^r}(v) e'(t) \\ p(V^{th}, t) = 0 \\ e'(t) = -\frac{1}{2} \frac{\partial}{\partial v} p(V^{th}, t) \end{cases}$$

$$e'(t) = \frac{d}{dt} \mathbb{E}(M_t)$$

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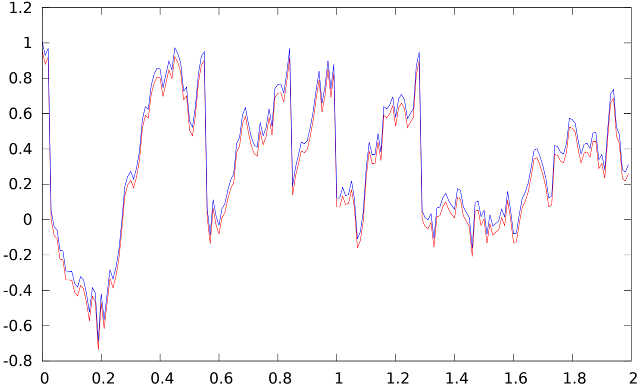
Main difficulties in our setting

- The processes have jumps
- The interactions are not smooth functionals of the trajectories



An existence result with continuous $e(t) = \mathbb{E}(M_t)$

"Notations"

- $V^{th} = 1 \qquad V^r = 0 \qquad \sigma = 1$

Theorem

Let $\epsilon > 0$ and $V_0 \leq 1 - \epsilon$.
$\exists \alpha_0 \in (0, 1]$ such that for any $0 \leq \alpha \leq \alpha_0$ and any $T > 0$, there exists a unique solution

$$\begin{cases} V_t = V_0 + \int_0^t b(V_s) ds + W_t - M_t + \alpha \mathbb{E}(M_t) \\ M_t = \sum_{k \geq 1} \mathbb{1}_{[0,t]}(\tau_k) \qquad \tau_{k+1} = \inf\{t \geq \tau_k, \ V_{t-} \geq 1\} \end{cases}$$

with $e(t) = \mathbb{E}(M_t) \in \mathcal{C}^1[0, T]$.

A non-existence result with continuous e

Blow up phenomenon

For any fixed α_0, there exist initial conditions V_0 such that any solution blows up in finite time.
[Cáceres, Carrillo, and Perthame. *J. Math. Neurosci.* (2011)]

Discontinuity of $t \mapsto e(t)$

When a blow up occurs at time t_1, a macroscopic proportion of neurons spikes and $e(t_1) = \mathbb{E}(M_{t_1}) \neq e(t_1-)$

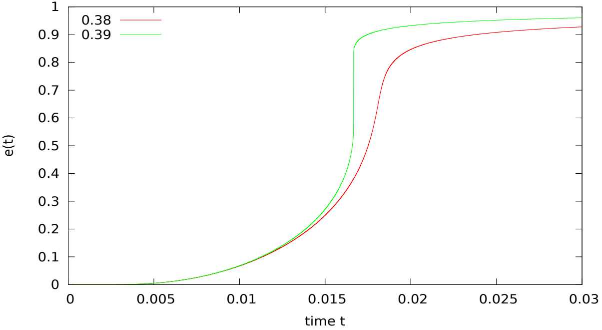


Figure: Plot of $t \mapsto e(t)$ for $v_0 = 0.8$, $b(v) \equiv 0$, $\alpha = 0.38$ and $\alpha = 0.39$.

More general solutions including discontinuous e

$$
\begin{cases}
V_t = V_0 + \int_0^t b(V_s)ds + W_t - M_t + \alpha\mathbb{E}(M_t) \\
M_t = \sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k) \\
\tau_{k+1} = \inf\{t \geq \tau_k,\ V_{t-} + \alpha\Delta e(t) \geq 1\}
\end{cases}
$$

Proposition

Assume that the pair $(V_t, M_t)_{t\geq 0}$ of càdlàg processes satisfies

1. $(M_t)_{t\geq 0}$ has integrable marginal distributions;
2. for all $t \geq 0$, $\mathbb{P}(\Delta M_t \leq 1) = 1$;
3. $\mathbb{P}$-almost surely, the previous equations hold true.

Then, for any time $t \geq 0$, the jump $\Delta e(t)$ satisfies

$$\Delta e(t) = \mathbb{P}\big(V_{t-} + \alpha \Delta e(t) \geq 1\big).$$

Definition of "physical" solutions

A pair $(V_t, M_t)_{t\geq 0}$ of càdlàg adapted processes such that

1. $(M_t)_{t\geq 0}$ has integrable marginal distributions;
2. for all $t \geq 0$, $\mathbb{P}(\Delta M_t \leq 1) = 1$;
3. $\mathbb{P}$-almost surely, hold true;

$$\begin{cases} V_t = V_0 + \int_0^t b(V_s)ds + W_t - M_t + \alpha\mathbb{E}(M_t) \\ M_t = \sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k) \\ \tau_{k+1} = \inf\{t \geq \tau_k,\ V_{t-} + \alpha\Delta e(t) \geq 1\} \end{cases}$$

4. the discontinuity points of $e(t) = \mathbb{E}(M_t)$ satisfy

$$\begin{aligned} \Delta e(t) &= \sup\left\{\eta \geq 0 : \forall \eta' \leq \eta, \mathbb{P}\big(V_{t-} + \alpha\eta' \geq 1\big) \geq \eta'\right\} \\ &= \inf\left\{\eta \geq 0 : \mathbb{P}\big(V_{t-} + \alpha\eta \geq 1\big) < \eta\right\}. \end{aligned}$$

A first approximation by a particle system

$$\begin{cases} V_t^{i,N} = V_0^{i,N} + \int_0^t b\big(V_s^{i,N}\big)\,ds + W_t^i - M_t^{i,N} + \frac{\alpha}{N}\sum_{j=1}^N M_t^{j,N} \\ V_0^{i,N} \overset{d}{=} V_0 \text{ independent and identically distributed,} \end{cases}$$

$$M_t^{i,N} := \sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k^{i,N}),$$

where $\tau_0^{i,N} = 0$ and

$$\tau_k^{i,N} := \inf\left\{ t > \tau_{k-1}^{i,N} : V_{t-}^{i,N} + \frac{\alpha}{N}\sum_{j=1}^N \big(M_t^{j,N} - M_{t-}^{j,N}\big) \geq 1 \right\}, \quad k \geq 1,$$

Again: non uniqueness!

A precise definition of the cascade of spikes

- First, consider $\Gamma_0 := \{i \in \{1, N\} : V^i_{t-} = 1\}$.

  t has to be a spike time for any neuron in Γ_0.
- $\Gamma_1 := \left\{ i \in \{1, N\} \backslash \Gamma_0 : V^i_{t-} + \alpha \frac{|\Gamma_0|}{N} \geq 1 \right\}$,

  t has to be a spike time for any neuron in Γ_1.
- $\Gamma_{k+1} := \left\{ i \in \{1, N\} \backslash \Gamma_0 \cup \cdots \cup \Gamma_k : V^i_{t-} + \alpha \frac{|\Gamma_0 \cup \cdots \cup \Gamma_k|}{N} \geq 1 \right\}$,
- $\Gamma := \bigcup_{0 \leq k \leq N-1} \Gamma_k$

i spikes iff $i \in \Gamma$

$$V_t = V_0 + \int_0^t b(V_s)ds + W_t - M_t + \alpha\mathbb{E}(M_t).$$

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A "technical" reformulation

$$Z_t := V_t + M_t.$$

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Property

- M_t can be expressed in terms of $(Z_s)_{0 \le s \le t}$

$$M_t = \left\lfloor \left(\sup_{0 \le s \le t} Z_s \right)_+ \right\rfloor = \sup_{0 \le s \le t} \lfloor (Z_s)_+ \rfloor .$$

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Dynamics of Z

$$Z_t = Z_0 + \int_0^t b(Z_s - M_s)ds + \alpha\mathbb{E}(M_t) + W_t.$$

Convergence of the particles system

- Let $\mu_N = \frac{1}{N} \sum_i \delta_{Z^{i,N}}$
- μ_N is a random variable with values in $\mathcal{P}(\hat{\mathcal{D}}([0,T],\mathbb{R}))$
- Let Π_N denotes the law of μ_N

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Theorem

- *The family $(\Pi_N)_{N\geq 1}$ is tight in $\mathcal{P}(\mathcal{P}(\hat{\mathcal{D}}([0,T],\mathbb{R})))$*
- *Let Π_∞ be a weak limit. For Π_∞-a.e. measure $\mu \in \mathcal{P}(\hat{\mathcal{D}}([0,T],\mathbb{R}))$, the canonical process $(z_t)_{t\in[0,T]}$ on $\hat{\mathcal{D}}([0,T],\mathbb{R})$ generates, under μ, a physical solution, i.e.*

$$\left(z_t - z_0 - \int_0^t b(z_s - m_s)ds - \alpha\langle \mu, m_t\rangle\right)_{t\in[0,T)}$$

is a Brownian motion, where $m_t = \lfloor (\sup_{0\leq s\leq t} z_s)_+ \rfloor$

Propagation of chaos

Theorem

- *Assume there exists a unique physical solution.*
- *Denote by J the (at most countable) set of discontinuity points of $t \mapsto \mathbb{E}(M_t)$.*
- *For any $k \geq 1$ and $S \in \mathbb{R}^+ \setminus J$*

$$\left((Z_s^{1,N}, M_s^{1,N}), \dots, (Z_s^{k,N}, M_s^{k,N})\right)_{s\in[0,S]} \Rightarrow \mathbb{P}^{\otimes k}_{(Z_s,M_s)_{s\in[0,S]}}.$$

Another approximation by a delayed interaction

$$V_t^\delta = V_0 + \int_0^t b(V_s^\delta)ds + \alpha e_\delta(t) + W_t - M_t^\delta, \quad t \geq 0.$$

where

$$e_\delta(t) := \begin{cases} 0 & \text{if } t \leq \delta \\ \mathbb{E}(M_{t-\delta}^\delta) & \text{if } t > \delta. \end{cases}$$

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Proposition

Let $T > 0$ and $\alpha \in (0, 1)$.
There exists a unique càdlàg process $(V_t^\delta, M_t^\delta)_{t \in [0,T]}$, solution of the delayed system.
The resulting map e_δ is continuously differentiable.

Convergence of the delayed system

Theorem

Consider the family of solutions $((V_t^\delta)_{t\in[0,T]})_{\delta\in(0,1)}$ *to the delayed equation and the associated* $((Z_t^\delta)_{t\in[0,T]})_{\delta\in(0,1)}$.
Define by μ^δ *the law of* $(Z_t^\delta)_{t\in[0,T]}$ *on* $\hat{\mathcal{D}}([0,T],\mathbb{R})$. *Then, the family* $(\mu^\delta)_{\delta\in(0,1)}$ *is tight in* $\mathcal{P}(\hat{\mathcal{D}}([0,T],\mathbb{R}))$.
Under any weak limit μ *as* δ *tends to* 0, *the canonical process* $(z_t)_{t\in[0,T]}$ *on* $\hat{\mathcal{D}}([0,T],\mathbb{R})$ *generates a physical solution.*

A more precise result

Theorem

- *Assume there exists a unique physical solution.*
- *Denote by J the (at most countable) set of discontinuity points of $t \mapsto \mathbb{E}(M_t)$.*
- *For any $S \in \mathbb{R}^+ \setminus J$*

$$(Z_s^\delta, M_s^\delta)_{s\in[0,S]} \Rightarrow \mathbb{P}_{(Z_s,M_s)_{s\in[0,S]}}$$