Obstacle problem arising from Moral Hazard

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Introduction

- Develop a structural model of optimal exit with dynamic contracting in a Brownian framework.
- Dynamic principal-agent relationship where a firm's assets owned by a risk-neutral principal is contracted out to a risk-neutral agent to manage.
- The model builds on recent papers by De Marzo-Sannikov (2006), Sannikov (2008), Biais-Mariotti-Rochet-Villeneuve (2010), Zhu (2012) and Cvitanic-Possamai-Touzi (2016) but introduces hidden actions that are persistent.

Model

- Firm's cash flow X is observable, Action or effort $a_t \in [0,1]$ is unobservable and

$$dX_t = -a_t\,dt + dZ_t^a$$

- Z^a is a brownian motion under $\mathbb{P}^a$.
- The principal owns the cash flow thus $a_t = 0$ is optimal for the principal.
- Shirking $(a \neq 0)$ induces a private benefit λa_t.
- Principal and agent are risk-neutral with the same discount factor **but** agent has limited liability.
- A contract is a pair $\Gamma = ((C_t)_t, \tau_L)$ where C is $\mathcal{F}^X$ adapted and increasing and τ_L is a stopping time with respect to $\mathcal{F}^X$.

Moral hazard as a Stackelberg leadership model

- Principal is the leader by offering a contract Γ.
- Agent gives a best response in terms of effort, *Incentive compatibility constraint*

$$\sup_a \mathbb{E}^a\left(\int_0^{\tau_L} e^{-rt}(\lambda a_t\, dt + dC_t)\right) \to a^*(\Gamma).$$

- Principal anticipates the agent's best response to offer the optimal contract

$$\sup_\Gamma \mathbb{E}^{a^*(\Gamma)}\left(\int_0^{\tau_L} e^{-rt}\Big(\underbrace{X_t}_{\text{Firm cash flows}} dt - \underbrace{dC_t}_{\text{Agent compensation}}\Big)\right)$$

such that, *Participation constraint*

$$\underbrace{\mathbb{E}^{a^*(\Gamma)}\left(\int_0^{\tau_L} e^{-rt}(\lambda a_t\, dt + dC_t)\right)}_{\text{Utility from accepting the contract}} \geq \underbrace{y_0}_{\text{reservation utility}}.$$

Moral hazard as a Markovian control problem

- We focus on *Full-Effort contracts* Γ such that $a^*(\Gamma) = 0$.
- Introducing the agent's continuation value

$$Y_t^a = \mathbb{E}^a \left(\int_t^{\tau_L} e^{-r(s-t)} (\lambda a_s \, dt + dC_s) | \mathcal{F}_t^X \right)$$

- The process

$$M_t = e^{-rt} Y_t^0 + \int_0^t e^{-rs} dC_s = \mathbb{E}^0 \left[\int_0^{\tau_L} e^{-rs} dC_s | \mathcal{F}_t^X \right]$$

is an uniformly integrable martingale under $\mathbb{P}^0$.

- Representation theorem for martingales and integration by part lead to: $dY_t^0 = rY_t^0 \, dt + \beta_t dZ_t - dC_t$ for $t \leq \tau_L$.

Proposition

A full-effort contract is incentive-compatible if and only if $\beta_t \geq \lambda$ for all $t \leq \tau_L$.

Moral hazard as a Markovian control problem

- Principal's value

$$V(x_0, y_0) = \sup_{\Gamma, \beta_t \geq \lambda} \mathbb{E}^0 \left(\int_0^{\tau_L \wedge \tau_0^\beta} e^{-rs}((x_0 + Z_s)\, ds - dC_s) \right)$$

with

$$dY_t^\beta = rY_t^\beta\, dt + \beta_t dZ_t - dC_t, \quad Y_0^\beta = y_0,$$

and

$$\tau_0^\beta = \inf\{t \geq 0\,,\ Y_t^\beta = 0\}.$$

because the agent has limited liability.

- Because,

$$y_0 = \mathbb{E}^0 \left(\int_0^{\tau_L} e^{-rs} dC_s \right),$$

we get, $V(x_0, y_0) = v(x_0, y_0) - y_0$ with

$$v(x, y) = \sup_{\tau, \beta_t \geq \lambda} \mathbb{E}^0 \left(\int_0^{\tau \wedge \tau_0^\beta} e^{-rs} X_s\, ds \right).$$

Optimal Full-Effort Contract

- Same discount factor thus it is optimal to postpone the compensation and to deliver $Y^{\beta}_{\tau_L}$ at τ_L^-.
- Principal' s value

$$v(x,y) = \sup_{\tau,\beta\geq\lambda} \mathbb{E}^0 \left(\int_0^{\tau\wedge\tau_0} e^{-rs}(x+Z_s)\,ds \right)$$

- It is a constrained optimal stopping problem related to the solution in some sense to

$$\max(\sup_{\beta\geq\lambda} \mathcal{L}(\beta)V, -V) = 0,$$

with the degenerate operator

$$\mathcal{L}(\beta)V \equiv -rV(x,w) + rwV_w(x,w) + \frac{1}{2}V_{xx}(x,w) + \frac{1}{2}\beta^2 V_{ww}(x,w) + \beta V_{xw}(x,w) + x$$

Optimal Full-Effort Contract

- Consider the constrained optimal stopping problem

$$u(x,y) = \sup_{\tau} \mathbb{E}^0 \left(\int_0^{\tau \wedge \tau_0^\lambda} e^{-rs} X_s \, ds \right)$$

with $dY_t^\lambda = rY_t^\lambda \, dt + \lambda dZ_t$.

- We have

Proposition

The stopping time $\tau^* = \inf\{t \geq 0\,,\, X_t \leq \frac{-\sigma}{\sqrt{2r}}\}$ is optimal for u. Moreover, u is concave in y and satisfies $\lambda u_{yy} + u_{xy} \leq 0$ therefore $u = v$.

Conclusion

- Studying a simple optimal stopping for Brownian motion under moral hazard leads to difficult constrained optimal stopping problem.
- When focusing on contracts implementing full effort, the principal value reduces to an obstacle problem with degenerate HJB.