

Systemic Risk and Stochastic Games with Delay

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PDE and Probability Methods for Interactions
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A Toy Model of Liquidity

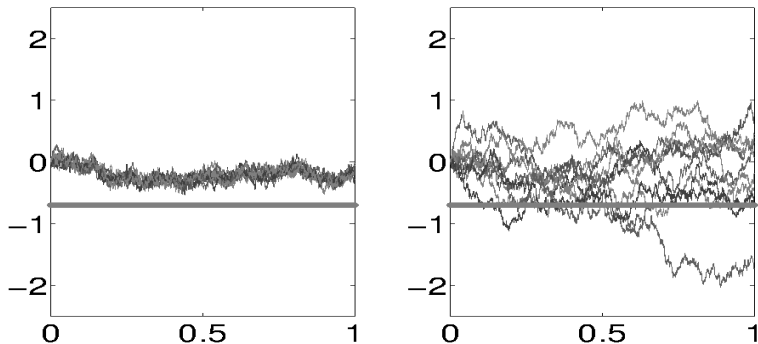
From “*Systemic Risk Illustrated*” J.-P. Fouque and L.-H. Sun (2013)

$$\begin{aligned}dX_t^{(i)} &= \frac{a}{N} \sum_{j=1}^N (X_t^{(j)} - X_t^{(i)}) dt + \sigma dW_t^{(i)} \\ &= a \left[\left(\frac{1}{N} \sum_{j=1}^N X_t^{(j)} \right) - X_t^{(i)} \right] dt + \sigma dW_t^{(i)}.\end{aligned}$$

The processes $X^{(i)}$'s are “OUs” **mean-reverting** to the **ensemble average** which satisfies

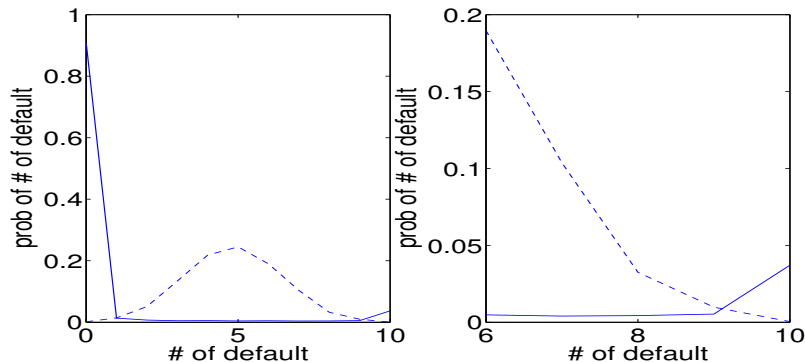
$$d \left(\frac{1}{N} \sum_{i=1}^N X_t^{(i)} \right) = d \left(\frac{\sigma}{N} \sum_{i=1}^N W_t^{(i)} \right).$$

Strong Coupling: $a = 100$



One realization of the trajectories of the coupled diffusions $X_t^{(i)}$ with **$a = 100$** (left plot) and trajectories of the independent Brownian motions ($a = 0$) (right plot) using the same Gaussian increments. Solid horizontal line: default level $D = -0.7$

Loss Distribution: strong coupling



On the left, we show plots of the loss distribution for the coupled diffusions with **$a = 100$** (solid line) and for the independent Brownian motions (dashed line). The plots on the right show the corresponding tail probabilities.

Systemic Risk vs Stability

Large deviation estimate:

$$\lim_{N \rightarrow \infty} -\frac{1}{N} \log \mathbf{P} \left(\min_{0 \leq t \leq T} \left(\frac{\sigma}{N} \sum_{i=1}^N W_t^{(i)} \right) \leq D \right) = \frac{D^2}{2\sigma^2 T}.$$

For a large number of banks, the probability that the ensemble average reaches the default barrier is of order $\exp\left(-\frac{D^2 N}{2\sigma^2 T}\right)$

Assuming $X_0^i = 0$, $\frac{1}{N} \sum_{i=1}^N X_t^{(i)} = \frac{\sigma}{N} \sum_{i=1}^N W_t^{(i)}$, and we identify

$$\left\{ \min_{0 \leq t \leq T} \left(\frac{1}{N} \sum_{i=1}^N \mathbf{x}_t^{(i)} \right) \leq D \right\} \text{ as a } \mathbf{systemic\ event}$$

This event does not depend on a . In fact, once in this event, increasing a creates more defaults by **“flocking to default”**.

So far we have seen that:

“Lending and borrowing improves stability but also contributes to systemic risk”

**But how about if the banks compete?
(minimizing costs, maximizing profits,...)**

- Can we find an equilibrium in which the previous analysis can still be performed?
- Can we find and characterize a Nash equilibrium?

What follows is from

Mean Field Games and Systemic Risk

by R. Carmona, J.-P. Fouque and L.-H. Sun (2015)

Stochastic Game/Mean Field Game

Banks are borrowing from and lending to a central bank:

$$dX_t^i = \alpha_t^i dt + \sigma dW_t^i, \quad i = 1, \dots, N$$

where α^i is the control of bank i which wants to **minimize**

$$J^i(\alpha^1, \dots, \alpha^N) = \mathbf{E} \left\{ \int_0^T f_i(X_t, \alpha_t^i) dt + g_i(X_T) \right\},$$

with **running cost**

$$f_i(x, \alpha^i) = \left[\frac{1}{2}(\alpha^i)^2 - q\alpha^i(\bar{x} - x^i) + \frac{\epsilon}{2}(\bar{x} - x^i)^2 \right], \quad q^2 < \epsilon,$$

and **terminal cost** $g_i(x) = \frac{c}{2}(\bar{x} - x^i)^2$.

This is an example of **Mean Field Game (MFG)** studied extensively by P.-L. Lions and collaborators, R. Carmona and F. Delarue, ...

Nash Equilibria (FBSDE Approach)

The Hamiltonian (with Markovian feedback strategies):

$$\begin{aligned} H^i(x, y^{i,1}, \dots, y^{i,N}, \alpha^1(t, x), \dots, \alpha_t^i, \dots, \alpha^N(t, x)) \\ = \sum_{k \neq i} \alpha^k(t, x) y^{i,k} + \alpha^i y^{i,i} \\ + \frac{1}{2} (\alpha^i)^2 - q \alpha^i (\bar{x} - x^i) + \frac{\epsilon}{2} (\bar{x} - x^i)^2, \end{aligned}$$

Minimizing H^i over α^i gives the choices:

$$\hat{\alpha}^i = -y^{i,i} + q(\bar{x} - x^i), \quad i = 1, \dots, N,$$

Ansatz:

$$Y_t^{i,j} = \eta_t \left(\frac{1}{N} - \delta_{i,j} \right) (\bar{X}_t - X_t^i),$$

where η_t is a deterministic function satisfying the terminal condition $\eta_T = c$.

Forward-Backward Equations

Forward Equation:

$$\begin{aligned}dX_t^i &= \partial_{y^i, i} H^i dt + \sigma dW_t^i \\ &= \left[q + \left(1 - \frac{1}{N}\right) \eta_t \right] (\bar{X}_t - X_t^i) dt + \sigma dW_t^i,\end{aligned}$$

with initial conditions $X_0^i = x_0^i$.

Backward Equation:

$$\begin{aligned}dY_t^{i,j} &= -\partial_{x^j} H^i dt + \sum_{k=1}^N Z_t^{i,j,k} dW_t^k \\ &= \left(\frac{1}{N} - \delta_{i,j} \right) (\bar{X}_t - X_t^i) \left[q \eta_t - \frac{1}{N} \left(\frac{1}{N} - 1 \right) \eta_t^2 + q^2 - \epsilon \right] dt \\ &\quad + \sum_{k=1}^N Z_t^{i,j,k} dW_t^k, \quad Y_T^{i,j} = c \left(\frac{1}{N} - \delta_{i,j} \right) (\bar{X}_T - X_T^i).\end{aligned}$$

Solution to the Forward-Backward Equations

By summation of the forward equations: $d\bar{X}_t = \frac{\sigma}{N} \sum_{k=1}^N dW_t^k$.

Differentiating the ansatz $Y_t^{i,j} = \eta_t \left(\frac{1}{N} - \delta_{i,j} \right) (\bar{X}_t - X_t^i)$, we get:

$$\begin{aligned} dY_t^{i,j} &= \left(\frac{1}{N} - \delta_{i,j} \right) (\bar{X}_t - X_t^i) \left[\dot{\eta}_t - \eta_t \left(q + \left(1 - \frac{1}{N} \right) \eta_t \right) \right] dt \\ &\quad + \eta_t \left(\frac{1}{N} - \delta_{i,j} \right) \sigma \sum_{k=1}^N \left(\frac{1}{N} - \delta_{i,k} \right) dW_t^k. \end{aligned}$$

Identifying with the backward equations:

$$Z_t^{i,j,k} = \eta_t \sigma \left(\frac{1}{N} - \delta_{i,j} \right) \left(\frac{1}{N} - \delta_{i,k} \right) \text{ for } k = 1, \dots, N,$$

and η_t **must satisfy the Riccati equation**

$$\dot{\eta}_t = 2q\eta_t + \left(1 - \frac{1}{N^2} \right) \eta_t^2 - (\epsilon - q^2),$$

with the terminal condition $\eta_T = c$.

Financial Implications

1. Once the function η_t has been obtained, bank i implements its strategy by using its control

$$\hat{\alpha}_t^i = -Y_t^{i,i} + q(\bar{X}_t - X_t^i) = \left[q + \left(1 - \frac{1}{N}\right)\eta_t \right] (\bar{X}_t - X_t^i),$$

It requires its own log-reserve X_t^i but also the average reserve \bar{X}_t which may or may not be known to the individual bank i .

Observe that the average \bar{X}_t is given by $d\bar{X}_t = \frac{\sigma}{N} \sum_{k=1}^N dW_t^k$, and is identical to the average found in the uncontrolled case.

Therefore, systemic risk occurs in the same manner as in the case of uncontrolled dynamics.

2. In fact, the controlled dynamics can be rewritten

$$dX_t^i = \left(q + \left(1 - \frac{1}{N}\right)\eta_t \right) \frac{1}{N} \sum_{j=1}^N (X_t^j - X_t^i) dt + \sigma dW_t^i.$$

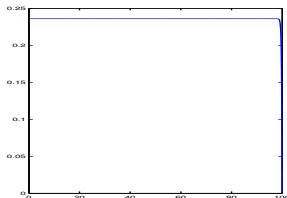
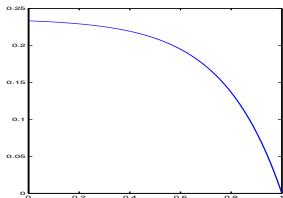
The effect of the banks using their optimal strategies corresponds to inter-bank borrowing and lending at the **effective rate**

$$A_t := q + \left(1 - \frac{1}{N}\right)\eta_t.$$

Under this equilibrium, the central bank is simply a **clearing house**, and the system is operating as if banks were borrowing from and lending to each other at the rate A_t , and the net effect is **creating liquidity** quantified by the rate of lending/borrowing.

Financial Implications

3. For T large (most of the time $T - t$ large), η_t is mainly constant. For instance, with $c = 0$, $\lim_{T \rightarrow \infty} \eta_t = \frac{\epsilon - q^2}{-\delta} := \bar{\eta}$.



Plots of η_t with $c = 0$, $q = 1$, $\epsilon = 2$ and $T = 1$ on the left, $T = 100$ on the right with $\bar{\eta} \sim 0.24$ (here we used $1/N \equiv 0$).

Therefore, in this infinite-horizon equilibrium, banks are borrowing and lending to each other at the constant rate

$$A := q + \left(1 - \frac{1}{N}\right)\bar{\eta} = q + \bar{\eta} \quad \text{in the Mean Field Limit.}$$

Stochastic Game/Mean Field Game with Delay

What follows is from: **Systemic Risk and Stochastic Games with Delay**

R. Carmona, J.-P. Fouque, M. Mousavi, and L.-H. Sun (submitted, 2016)

Stochastic Game/Mean Field Game with Delay

Banks are borrowing from and lending to a central bank and money is returned at maturity τ :

$$dX_t^i = [\alpha_t^i - \alpha_{t-\tau}^i] dt + \sigma dW_t^i, \quad i = 1, \dots, N$$

where α^i is the control of bank i which wants to **minimize**

$$J^i(\alpha^1, \dots, \alpha^N) = \mathbf{E} \left\{ \int_0^T f_i(X_t, \alpha_t^i) dt + g_i(X_T) \right\},$$

$$f_i(x, \alpha^i) = \left[\frac{1}{2}(\alpha^i)^2 - q\alpha^i(\bar{x} - x^i) + \frac{\epsilon}{2}(\bar{x} - x^i)^2 \right], \quad q^2 < \epsilon,$$

$$g_i(x) = \frac{c}{2}(\bar{x} - x^i)^2,$$

$$X_0^i = \xi^i, \quad \alpha_t^i = 0, \quad t \in [-\tau, 0).$$

Case $\tau = 0$: no lending/borrowing \longrightarrow no liquidity.

Case $\tau = T$: no return/delay \longrightarrow full liquidity.

Forward-Advanced-Backward SDEs

Theorem. The strategy $\hat{\alpha}$ given by


$$\hat{\alpha}_t^i = q(\bar{X}_t - X_t^i) - Y_t^{i,i} + \mathbf{E}^{\mathcal{F}_t}(Y_{t+\tau}^{i,i})$$

is an **open-loop Nash equilibrium** where (X, Y, Z) is the unique solution to the following system of **FABSDEs**:

$$X_t^i = \xi^i + \int_0^t (\hat{\alpha}_s^i - \hat{\alpha}_{s-\tau}^i) ds + \sigma W_t^i, \quad t \in [0, T],$$

$$Y_t^{i,j} = c \left(\frac{1}{N} - \delta_{i,j} \right) (\bar{X}_T - X_T^i) + \int_t^T \left(\frac{1}{N} - \delta_{i,j} \right) \left\{ (\epsilon - q^2) (\bar{X}_s - X_s^i) + qY_s^{i,j} - q\mathbf{E}^{\mathcal{F}_s}(Y_{s+\tau}^{i,j}) \right\} ds - \sum_{k=1}^N \int_t^T Z_s^{i,j,k} dW_s^k, \quad t \in [0, T],$$

$$Y_t^{i,j} = 0, \quad t \in (T, T + \tau], \quad i, j = 1, \dots, N,$$

where the processes $Z_t^{i,j,k}$, $k = 1, \dots, N$ are adapted and square integrable, and $\mathbf{E}^{\mathcal{F}_t}$ denotes the conditional expectation with respect to the filtration generated by the Brownian motions. 

Outline of the Proof

Denote by $\tilde{\alpha} = (\hat{\alpha}^{-i}, \tilde{\alpha}^i)$ the strategy obtained from the strategy $\hat{\alpha}$ by replacing the i th component by $\tilde{\alpha}^i$. Denote by \tilde{X} , the state generated by $\tilde{\alpha}$ and observe that $X^j = \tilde{X}^j$ for all $j \neq i$ since the dynamics of X^j is only driven by $\hat{\alpha}^j$. We have by **convexity** and **Itô's formula**

$$\begin{aligned} J^i(\hat{\alpha}) - J^i(\tilde{\alpha}) &= \mathbf{E} \left\{ \int_0^T \left(f_i(X_t, \hat{\alpha}_t^i) - f_i(\tilde{X}_t, \tilde{\alpha}_t^i) \right) dt + g_i(X_T) - g_i(\tilde{X}_T) \right\}. \\ &\leq \mathbf{E} \int_0^T \left(\partial_{x^i} f_i(X_t, \hat{\alpha}_t^i) (X_t^i - \tilde{X}_t^i) + \partial_{\alpha^i} f_i(X_t, \hat{\alpha}_t^i) (\hat{\alpha}_t^i - \tilde{\alpha}_t^i) \right) dt + \mathbf{E} (Y_T^{i,i} (X_T^i - \tilde{X}_T^i)) \\ &= \mathbf{E} \int_0^T \left(q \left(\frac{1}{N} - 1 \right) (X_t^i - \tilde{X}_t^i) - (\hat{\alpha}_t^i - \tilde{\alpha}_t^i) \right) \times \\ &\quad \left[-\hat{\alpha}_t^i + q(\bar{X}_t - X_t^i) - Y_t^{i,i} + \mathbf{E}^{\mathcal{F}_t} (Y_{t+\tau}^{i,i}) \right] dt \\ &= 0 \end{aligned}$$

Therefore, the strategy $\hat{\alpha}$ is a **Nash equilibrium for the open-loop game with delay** provided that the FABSDE system admits a solution.

This is shown by a **continuation argument** introduced by Shige Peng in the context of stochastic control problems. This is quite technical and we refer to the Appendix in the paper.

Existence, no Uniqueness

Therefore, the strategy $\hat{\alpha}$ is a **Nash equilibrium for the open-loop game with delay** provided that the **FABSDE system admits a solution**.

This is shown by a **continuation argument** introduced by Shige Peng in the context of stochastic control problems. This is quite technical and we refer to the Appendix in the paper.

In general, there is **no uniqueness** of Nash equilibrium for the open-loop game with delay.

We observe that in contrast with the case without delay, there is **no simple explicit formula** for the optimal strategy $\hat{\alpha}$.

Clearing House Property

Summing over $i = 1, \dots, N$ the equations for $Y^{i,i}$ and denoting

$$\bar{Y}_t = \frac{1}{N} \sum_{i=1}^N Y_t^{i,i}, \quad \bar{Z}_t^k = \frac{1}{N} \sum_{i=1}^N Z_t^{i,i,k}, \quad \text{gives}$$

$$d\bar{Y}_t = - \left(\frac{1}{N} - 1 \right) q(\bar{Y}_t - \mathbf{E}^{\mathcal{F}_t}(\bar{Y}_{t+\tau})) dt + \sum_{k=1}^N \bar{Z}_t^k dW_t^k, \quad t \in [0, T],$$

$$\bar{Y}_t = 0, \quad t \in [T, T + \tau],$$

which admits the unique solution

$$\bar{Y}_t = 0, \quad t \in [0, T + \tau], \quad \text{with} \quad \bar{Z}_t^k = 0, \quad k = 1, \dots, N, \quad t \in [0, T].$$

Summing over $i = 1, \dots, N$ the equations for $\hat{\alpha}_t^i$ gives

$$\sum_{i=1}^N \hat{\alpha}_t^i = \sum_{i=1}^N \left[q(\bar{X}_t - X_t^i) - Y_t^{i,i} + \mathbf{E}^{\mathcal{F}_t}(Y_{t+\tau}^{i,i}) \right] = 0.$$

Note that $\bar{X}_t = \bar{\xi} + \frac{\sigma}{N} \sum_{i=1}^N W_t^i$ as in the case with no delay.

Infinite-dimensional HJB Approach

Following Gozzi and Marinelli (2004). Let \mathbf{H}^N be the Hilbert space defined by $\mathbf{H}^N = \mathbf{R}^N \times L^2([-\tau, 0]; \mathbf{R}^N)$, with the inner product $\langle z, \tilde{z} \rangle = z_0 \tilde{z}_0 + \int_{-\tau}^0 z_1(\xi) \tilde{z}_1(\xi) d\xi$, where $z, \tilde{z} \in \mathbf{H}^N$, and z_0 and $z_1(\cdot)$ correspond respectively to the \mathbf{R}^N -valued and $L^2([-\tau, 0]; \mathbf{R}^N)$ -valued components (the states and the past of the strategies in our case). In order to use the **dynamic programming principle** for stochastic game in search of a **closed-loop Nash equilibrium**, at time $t \in [0, T]$, given the initial state $Z_t = z$, bank i chooses the control α^i to minimise its objective function $J^i(t, z, \alpha)$.

$$J^i(t, z, \alpha) = \mathbf{E} \left\{ \int_t^T f_i(Z_{0,s}, \alpha_s^i) dt + g_i(Z_{0,T}) \mid Z_t = z \right\},$$

See also *Stochastic Control and Differential Games with Path-Dependent Controls* by Yuri Saporito (2017) for a FITO (PPDE) approach.

Coupled HJB Equations

Bank i 's value function $V^i(t, z)$ is

$$V^i(t, z) = \inf_{\alpha} J^i(t, z, \alpha).$$

The set of **value functions** $V^i(t, z)$, $i = 1, \dots, N$ is the unique solution (in a suitable sense) of the following system of **coupled HJB equations**:

$$\partial_t V^i + \frac{1}{2} \text{Tr}(Q \partial_{zz} V^i) + \langle Az, \partial_z V^i \rangle + H_0^i(\partial_z V^i) = 0,$$

$$V^i(T) = g_i,$$

$$Q = G * G, \quad G : z_0 \rightarrow (\sigma z_0, 0),$$

$$A : (z_0, z_1(\gamma)) \rightarrow (z_1(0), -\frac{dz_1(\gamma)}{d\gamma}) \quad \text{a.e.}, \quad \gamma \in [-\tau, 0],$$

$$H_0^i(p^i) = \inf_{\alpha} [\langle B\alpha, p^i \rangle + f_i(z_0, \alpha^i)], \quad p^i \in \mathbf{H}^N,$$

$$B : u \rightarrow (u, -\delta_{-\tau}(\gamma)u), \quad \gamma \in [-\tau, 0].$$

Ansatz

By convexity of $f_i(z_0, \alpha^i)$ with respect to (z_0, α^i) ,

$$\hat{\alpha}^i = -\langle B, p^{i,i} \rangle - q(z_0^i - \bar{z}_0), \quad \text{and}$$

$$\begin{aligned} H_0^i(p^i) &= \langle B \hat{\alpha}, p^i \rangle + f_i(z_0, \hat{\alpha}^i), \\ &= \sum_{k=1}^N \langle B, p^{i,k} \rangle \left(-\langle B, p^{k,k} \rangle - q(z_0^k - \bar{z}_0) \right) \\ &\quad + \frac{1}{2} \langle B, p^{i,i} \rangle^2 + \frac{1}{2} (\epsilon - q^2) (\bar{z}_0 - z_0^i)^2. \end{aligned}$$

We then make the **ansatz**

$$\begin{aligned} V^i(t, z) &= E_0(t) (\bar{z}_0 - z_0^i)^2 - 2(\bar{z}_0 - z_0^i) \int_{-\tau}^0 E_1(t, -\tau - \theta) (\bar{z}_{1,\theta} - z_{1,\theta}^i) d\theta \\ &\quad + \int_{-\tau}^0 \int_{-\tau}^0 E_2(t, -\tau - \theta, -\tau - \gamma) (\bar{z}_{1,\theta} - z_{1,\theta}^i) (\bar{z}_{1,\gamma} - z_{1,\gamma}^i) d\theta d\gamma + E_3(t). \end{aligned}$$

$$\begin{aligned} \partial_t V^i &= \frac{dE_0(t)}{dt} (\bar{z}_0 - z_0^i)^2 - 2(\bar{z}_0 - z_0^i) \int_{-\tau}^0 \frac{\partial E_1(t, -\tau - \theta)}{\partial t} (\bar{z}_{1,\theta} - z_{1,\theta}^i) d\theta \\ &+ \int_{-\tau}^0 \int_{-\tau}^0 \frac{\partial E_2(t, -\tau - \theta, -\tau - \gamma)}{\partial t} (\bar{z}_{1,\theta} - z_{1,\theta}^i) (\bar{z}_{1,\gamma} - z_{1,\gamma}^i) d\theta d\gamma + \frac{dE_3(t)}{dt}, \end{aligned}$$

$$\partial_{z_j} V^i =$$

$$\left[\begin{array}{c} 2E_0(t)(\bar{z}_0 - z_0^i) - 2 \int_{-\tau}^0 E_1(t, -\tau - \theta) (\bar{z}_{1,\theta} - z_{1,\theta}^i) d\theta \\ -2(\bar{z}_0 - z_0^i) E_1(t, \theta) + 2 \int_{-\tau}^0 E_2(t, -\tau - \theta, -\tau - \gamma) (\bar{z}_{1,\gamma} - z_{1,\gamma}^i) d\gamma \end{array} \right] \left(\frac{1}{N} - \delta_{i,j} \right),$$

$$\partial_{z^j z^k} V^i = \left[\begin{array}{cc} 2E_0(t) & -2E_1(t, -\tau - \theta) \\ -2E_1(t, -\tau - \theta) & 2E_2(t, -\tau - \theta, -\tau - \gamma) \end{array} \right] \left(\frac{1}{N} - \delta_{i,j} \right) \left(\frac{1}{N} - \delta_{i,k} \right),$$

and plug in the HJB equation.

PDEs for the coefficients E_i , $i = 0, 1, 2, 3, 4$

The equation corresponding to the **constant terms** is

$$\frac{dE_3(t)}{dt} + \left(1 - \frac{1}{N}\right)\sigma^2 E_0(t) = 0,$$

The equation corresponding to the $(\bar{z}_0 - z_0^i)^2$ **terms** is

$$\frac{dE_0(t)}{dt} + \frac{\epsilon}{2} = 2\left(1 - \frac{1}{N^2}\right)(E_1(t, 0) + E_0(t))^2 + 2q(E_1(t, 0) + E_0(t)) + \frac{q^2}{2}.$$

The equation corresponding to the $(\bar{z}_0 - z_0^i)(\bar{z}_1 - z_1^i)$ **terms** is

$$\frac{\partial E_1(t, \theta)}{\partial t} - \frac{\partial E_1(t, \theta)}{\partial \theta} = \left[2\left(1 - \frac{1}{N^2}\right)(E_1(t, 0) + E_0(t)) + q\right] (E_2(t, \theta, 0) + E_1(t, \theta)).$$

The equation corresponding to the $(\bar{z}_1 - z_1^i)(\bar{z}_1 - z_1^i)$ **terms** is

$$\begin{aligned} & \frac{\partial E_2(t, \theta, \gamma)}{\partial t} - \frac{\partial E_2(t, \theta, \gamma)}{\partial \theta} - \frac{\partial E_2(t, \theta, \gamma)}{\partial \gamma} = \\ & 2\left(1 - \frac{1}{N^2}\right) (E_2(t, \theta, 0) + E_1(t, \theta)) (E_2(t, \gamma, 0) + E_1(t, \gamma)). \end{aligned}$$

$$E_0(T) = \frac{c}{2},$$

$$E_1(T, \theta) = 0,$$

$$E_2(T, \theta, \gamma) = 0,$$

$$E_2(t, \theta, \gamma) = E_2(t, \gamma, \theta),$$

$$E_1(t, -\tau) = -E_0(t), \quad \forall t \in [0, T),$$

$$E_2(t, \theta, -\tau) = -E_1(t, \theta), \quad \forall t \in [0, T),$$

$$E_3(T) = 0.$$

We have existence and uniqueness for this system of PDEs

$$\begin{aligned}\hat{\alpha}_t^i &= -\langle B, \partial_{z^i} V^i \rangle - q(z_0^i - \bar{z}_0), \\ &= 2 \left(1 - \frac{1}{N}\right) \left[\left(E_1(t, 0) + E_0(t) + \frac{q}{2 \left(1 - \frac{1}{N}\right)} \right) (\bar{z}_0 - z_0^i) \right. \\ &\quad \left. - \int_{-\tau}^0 (E_2(t, -\tau - \theta, 0) + E_1(t, -\tau - \theta)) (\bar{z}_{1,\theta} - z_{1,\theta}^i) d\theta \right].\end{aligned}$$

In terms of the original system of coupled diffusions, the **closed-loop Nash equilibrium** corresponds to

$$\begin{aligned}\hat{\alpha}_t^i &= \left[2 \left(1 - \frac{1}{N}\right) (E_1(t, 0) + E_0(t)) + q \right] (\bar{X}_t - X_t^i) \\ &\quad + 2 \int_{t-\tau}^t [E_2(t, \theta - t, 0) + E_1(t, \theta - t)] (\bar{\hat{\alpha}}_\theta - \hat{\alpha}_\theta^i) d\theta, \quad i = 1, \dots, N.\end{aligned}$$

Clearing house property: $\sum_{i=1}^N \hat{\alpha}_t^i = 0$.

Closed-loop Nash Equilibria: Verification Theorem

At time $t \in [0, T]$, given $X_t = x$ and $\alpha_{[t]} = (\alpha_\theta)_{\theta \in [t-\tau, t]}$, bank i chooses the strategy α^i to minimise its objective function

$$J^i(t, x, \alpha, \alpha^i) = \mathbf{E} \left\{ \int_t^T f_i(X_s, \alpha_s^i) ds + g_i(X_T) \mid X_t = x, \alpha_{[t]} = \alpha \right\}.$$

Bank i 's value function $V^i(t, x, \alpha)$ is

$$V^i(t, x, \alpha) = \inf_{\alpha^i} J^i(t, x, \alpha, \alpha^i).$$

Guessing that the value function should be quadratic in the state and in the past of the control, we make the following **ansatz** for the value function:

Ansatz (from HJB formal derivation)

$$\begin{aligned} V^i(t, x, \alpha) &= E_0(t)(\bar{x} - x^i)^2 + 2(\bar{x} - x^i) \int_{t-\tau}^t E_1(t, \theta - t)(\bar{\alpha}_\theta - \alpha_\theta^i) d\theta \\ &+ \int_{t-\tau}^t \int_{t-\tau}^t E_2(t, \theta - t, \gamma - t)(\bar{\alpha}_\theta - \alpha_\theta^i)(\bar{\alpha}_\gamma - \alpha_\gamma^i) d\theta d\gamma + E_3(t), \end{aligned}$$

where $E_0(t)$, $E_1(t, \theta)$, $E_2(t, \theta, \gamma)$, $E_3(t)$, are deterministic functions satisfying the particular system of partial differential equations for $t \in [0, T]$ and $\theta, \gamma \in [-\tau, 0]$ obtained before.

Itô's formula

Applying Itô's formula to $V^i(t, X_t, \alpha_{[t]})$, we obtain the following expression for the **nonnegative** quantity

$$\mathbf{E}V^i(T, X_T, \alpha_{[T]}) - V^i(0, \xi^i, \alpha_{[0]}) + \mathbf{E} \int_0^T f^i(X_s, \alpha_s^i) dt$$

A long computation and use of the system of PDEs for the E_i 's \longrightarrow

Outline of proof

$$\begin{aligned} & \mathbf{E} \int_0^T \left\{ \frac{1}{2} \left(\alpha_t^i - 2 \left(E_1(t, 0) + E_0(t) + \frac{q}{2} \right) (\bar{X}_t - X_t^i) \right. \right. \\ & \left. \left. - 2 \int_{t-\tau}^t [E_2(t, \theta - t, 0) + E_1(t, \theta - t)] (\bar{\alpha}_\theta - \alpha_\theta^i) d\theta \right)^2 \right. \\ & \left. + 2(\bar{\alpha}_t - \bar{\alpha}_{t-\tau}) \left[E_0(t)(\bar{X}_t - X_t^i) + \int_{t-\tau}^t E_1(t, \theta - t)(\bar{\alpha}_\theta - \alpha_\theta^i) d\theta \right] \right. \\ & \left. + 2\bar{\alpha}_t \left[\left(E_1(t, 0) - \frac{q}{2} \right) (\bar{X}_t - X_t^i) + \int_{t-\tau}^t E_2(t, \theta - t, 0)(\bar{\alpha}_\theta - \alpha_\theta^i) d\theta \right] \right. \\ & \left. - 2\bar{\alpha}_{t-\tau} \left[E_1(t, -\tau)(\bar{X}_t - X_t^i) + \int_{t-\tau}^t E_2(t, \theta - t, -\tau)(\bar{\alpha}_\theta - \alpha_\theta^i) d\theta \right] \right\} dt. \end{aligned}$$

An optimal strategy can be characterized as the strategy $\hat{\alpha}$ which makes the previous quantity equal to zero.

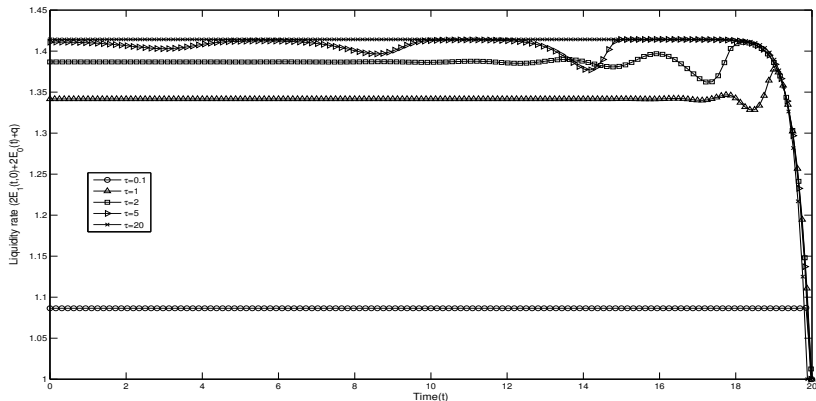
Therefore, if all the other banks choose their optimal strategies, bank i 's optimal strategy $\hat{\alpha}^i$ should satisfy

$$\begin{aligned}\hat{\alpha}_t^i &= 2 \left[E_1(t, 0) + E_0(t) + \frac{q}{2} \right] (\bar{X}_t - X_t^i) \\ &+ 2 \int_{t-\tau}^t [E_2(t, \theta - t, 0) + E_1(t, \theta - t)] (\bar{\alpha}_\theta - \hat{\alpha}_\theta^i) d\theta, \\ &\text{for } i = 1, \dots, N,\end{aligned}$$

since, with that choice, the square term in the integral is zero, and the three other terms vanish because $\bar{\alpha}_t = \bar{\alpha}_{t-\tau} = 0$ (by summing over i).

Effect of delay on liquidity

$$T = 20, q = 1, \varepsilon = 2, c = 0$$



The end

THANKS FOR YOUR ATTENTION