Estimation of the parameters of a diffusion with discontinuous coefficients

Antoine Lejay & Paolo Pigato

TOSCA, Inria Nancy-Grand Est

PDE & Probability Methods for Interactions

Sophia-Antipolis, March 2017

The Oscillating Brownian motion (OBM)

Terminology of Keilson & Wellner (1978)

$$X_t = x + \int_0^t \sigma(X_s)\,\mathrm{d}B_s,\ \sigma(x) = \begin{cases} \sigma_+ & \text{if } x \geq 0 \\ \sigma_- & \text{if } x < 0. \end{cases}$$

- Strong existence, uniqueness (Le Gall, 1978)
- Analytic formula of the density, occupation time (Keilson & Wellner, 1978)
- Convergence of the Euler scheme (Chan & Stramer, 1989, Yan, 2002, ...)
- Approximation by Random Walks (Keilson & Wellner, 1978; Helland 1982, ...)

The Oscillating Brownian motion (OBM)

Terminology of Keilson & Wellner (1978)

$$X_t = x + \int_0^t \sigma(X_s)\,\mathrm{d}B_s,\ \sigma(x) = \begin{cases} \sigma_+ & \text{if } x \geq 0 \\ \sigma_- & \text{if } x < 0. \end{cases}$$

Applications

- In a continuous time version of the Self-Exciting Threshold Auto-Regressive (SETAR) models (Tong, 1983).
- In finance, it mimicks a leverage effect of log-prices.

The realized volatility

Observations: $X_{kT/n}$, $k = 0, \dots, n$ (high-frequency data)

How to estimate σ_+ and σ_-?

Realized volatility type estimators

$$\sigma_\pm(n)^2 = \frac{\sum_{k=1}^n (X^\pm_{kT/n} - X^\pm_{(k-1)T/n})^2}{\frac{T}{n}\sum_{k=1}^n \mathbb{1}_{\pm X_{kT/n} \geq 0}}$$

Idea Itô-Tanaka formula $\Longrightarrow$

$$X_t^\pm = X_0^\pm + \sigma_\pm \int_0^t \mathbb{1}_{\pm X_s \geq 0}\, \mathrm{d}B_s + \frac{1}{2} L_t(X)$$

with

- $L_t(X)$ local time at 0 (finite variation process).
- $\left\langle \int_0^\cdot \mathbb{1}_{\pm X_s \geq 0}\, \mathrm{d}B_s \right\rangle_t = Q_t^\pm$ occupation time of $\mathbb{R}_\pm$

A convergence result

(i) $\sigma_\pm(n)$ is a consistent estimator of $\sigma_\pm$ as $n \to \infty$.

(ii) When $T = 1$,

$$\sqrt{n}(\sigma_\pm(n)^2 - \sigma_+^2)$$
$$\xrightarrow[n\to\infty]{\text{stable}} \frac{\sqrt{2}\sigma_\pm^2}{Q_1^\pm} \int_0^1 \mathbb{1}_{\pm X_s \geq 0} \,\mathrm{d}\widetilde{B}_s - \frac{1}{Q_1^\pm} \frac{2\sqrt{2}}{3\sqrt{\pi}} \frac{\sigma_-\sigma_+}{\sigma_- + \sigma_+} L_1(X),$$

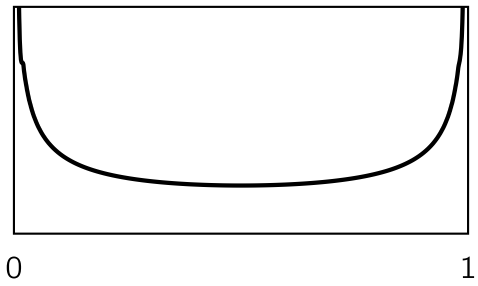
$\widetilde{B}$ is a BM indep. from B.

Remarks

- Joint convergence of $(\sigma_+(n), \sigma_-(n))$.
- Using Girsanov's theorem, we could consider the presence of drift (the limit laws are changed).
- By scaling, high-frequency estimation = long time estimation (not true in presence of drift).

Comments

• The limit depends on $Q_1^{\pm}$ which follows a law of ArcSine type



$\Longrightarrow$ either Q_1^{+} or Q_1^{-} is likely to be close to 1
$\Longrightarrow$ either σ_+ or σ_- is likely to be loosely estimated.

• The process X is null recurrent
$\Longrightarrow$ the limit law is a mixture of normal distribution.

• There is an asymptotic bias which is due to the discontinuity.

Some ingredients of the proof

We have to prove, in particular, convergences of type

- $\sqrt{n}[L(B), L(B)] \xrightarrow[n\to\infty]{\text{proba}} \frac{4\sqrt{2}}{3\sqrt{\pi}} L(B)$
- $\sqrt{n}[L(X), X] \xrightarrow[n\to\infty]{\text{proba}} 0$
- $\sqrt{n}[L(X), |X|] \xrightarrow[n\to\infty]{\text{proba}} 0$

with $[Y, Z] = \sum_{i=1}^{n} (Y_{i/n} - Y_{(i-1)/n})(Z_{i/n} - Z_{(i-1)/n})$
For this, we use that for a suitably decreasing function f,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} f(\sqrt{n} X_{i/n}) \xrightarrow[n\to\infty]{\text{proba}} c(f) L_1(X)$$

by adapting some results of J. Jacod (1998) to the OBM by reducing it to a Skew Brownian motion $Y_t = B_t + \gamma L_t(Y)$.
Computations are based on explicit expression of the density (this limits immediate generalizations).

Removing the asymptotic bias

Our estimator is changed to

$$\widehat{\sigma}_{\pm}(n)^2 = \frac{\sum_{k=1}^{n}(X^{\pm}_{k/n} - X^{\pm}_{(k-1)/n})(X_{k/n} - X_{(k-1)/n})}{\frac{1}{n}\sum_{k=1}^{n} \mathbb{1}_{X_{k/n}\geq 0}}$$

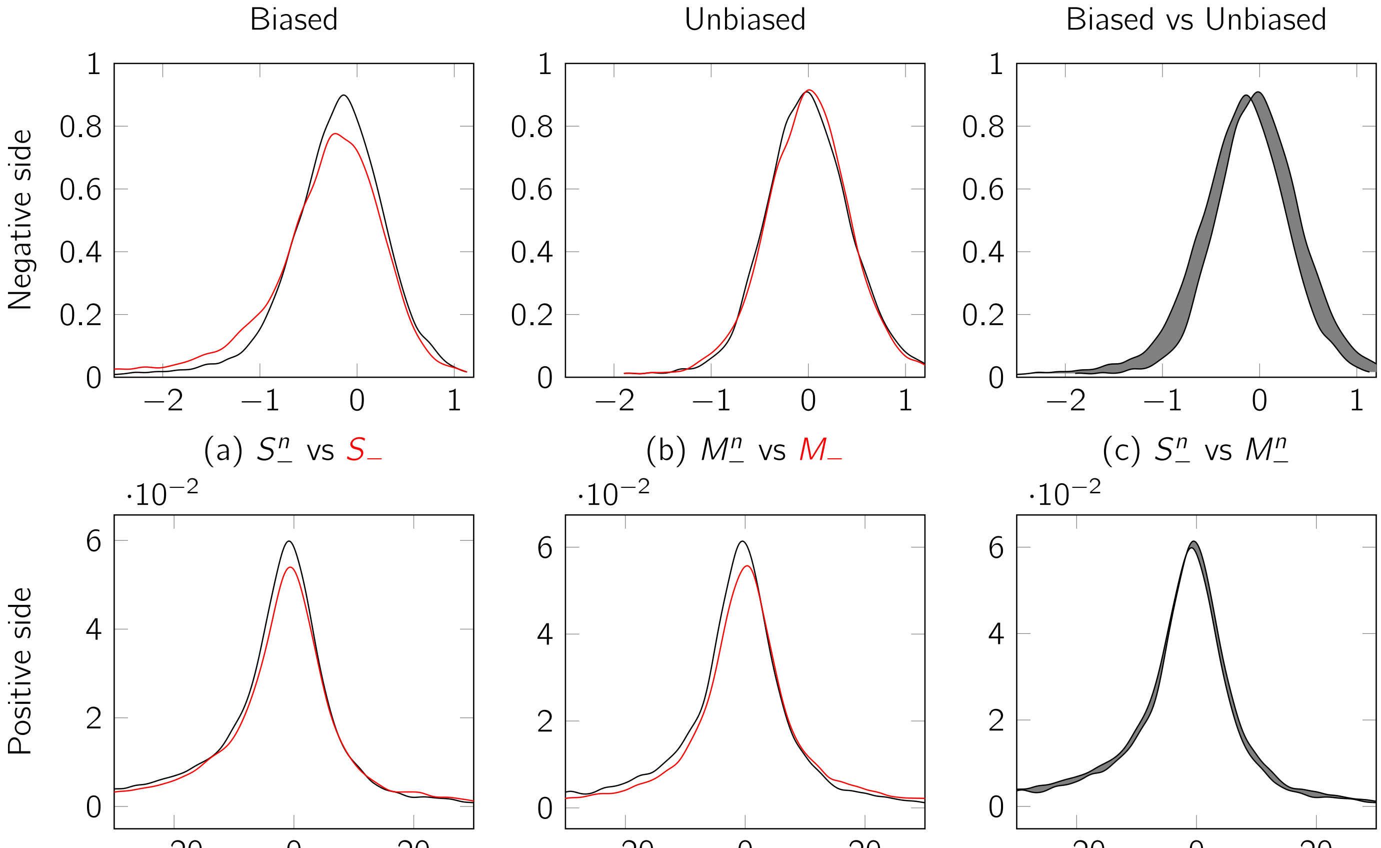
$$\boxed{\sqrt{n}(\widehat{\sigma}_{\pm}(n)^2 - \sigma_{\pm})^2 \xrightarrow[n\to\infty]{\text{stable}} \frac{\sqrt{2}\sigma_{\pm}^2}{Q_1^{\pm}}\int_0^1 \mathbb{1}_{\pm X_s\geq 0}\,\mathrm{d}\widetilde{B}_s.}$$

The reason is that

$$\sqrt{n}\sum_{k=1}^{n}(X^{+}_{k/n} - X^{+}_{(k-1)/n})(X^{-}_{k/n} - X^{-}_{(k-1)/n})$$

$$\xrightarrow[n\to\infty]{\text{stable}} \frac{2\sqrt{2}}{3\sqrt{\pi}}\frac{\sigma_-\sigma_+}{\sigma_- + \sigma_+}L_1(X).$$

Numerical illustration on $S^n_+ := \sqrt{n}(\sigma_\pm(n) - \sigma)$ $M^n_+ := \sqrt{n}(\widehat{\sigma}_\pm(n) - \sigma)$



(a) S^n_- vs S_- (b) M^n_- vs M_- (c) S^n_- vs M^n_-

(d) S^n_+ vs S_+ (e) M^n_+ vs M_+ (f) S^n_+ vs M^n_+

$\sigma_- = 1/2$, $\sigma_+ = 2$, $n = 500$, on 10 000 paths

Estimation of a two-valued drift

$$X_t = x + \int_0^t \sigma(X_s)\,\mathrm{d}B_s + \int_0^t b(X_s)\,\mathrm{d}s,$$

with

$$\sigma(x) = \begin{cases} \sigma_+ & \text{if } x \geq 0 \\ \sigma_- & \text{if } x < 0. \end{cases} \quad \text{and } b(x) = \begin{cases} b_+ & \text{if } x \geq 0 \\ b_- & \text{if } x < 0. \end{cases}$$

How to estimate (b_-, b_+)?

- We should consider long time estimation.
- The respective signs of b_+ and b_- are fundementals:

	$b_+ > 0$	$b_+ = 0$	$b_+ < 0$
$b_- > 0$	transient	null recurrent	ergodic
$b_- = 0$	transient	null recurrent	null recurrent
$b_- < 0$	transient	transient	transient

Estimation of a two-valued drift

Itô-Tanaka formula + Maximization of the Girsanov weight

$$\beta_\pm = \pm \frac{X_T^\pm - X_0^\pm - L_T/2}{Q_T^\pm} = b_\pm + \frac{M_t^\pm}{Q_T^\pm}$$

where $M^\pm = \int_0^t \mathbb{1}_{\pm X_s \geq 0} \, \mathrm{d}B_s$, $\langle M^\pm \rangle = Q^\pm$.

$\Longrightarrow$ Empirical estimator for large T

$$b_\pm = \pm \frac{X_T^\pm - X_0^\pm - \hat{L}_T/2}{\hat{Q}_T^\pm}$$

where $\hat{Q}^\pm$, $\hat{L}$ are empirical estimators of $Q^\pm$, L.

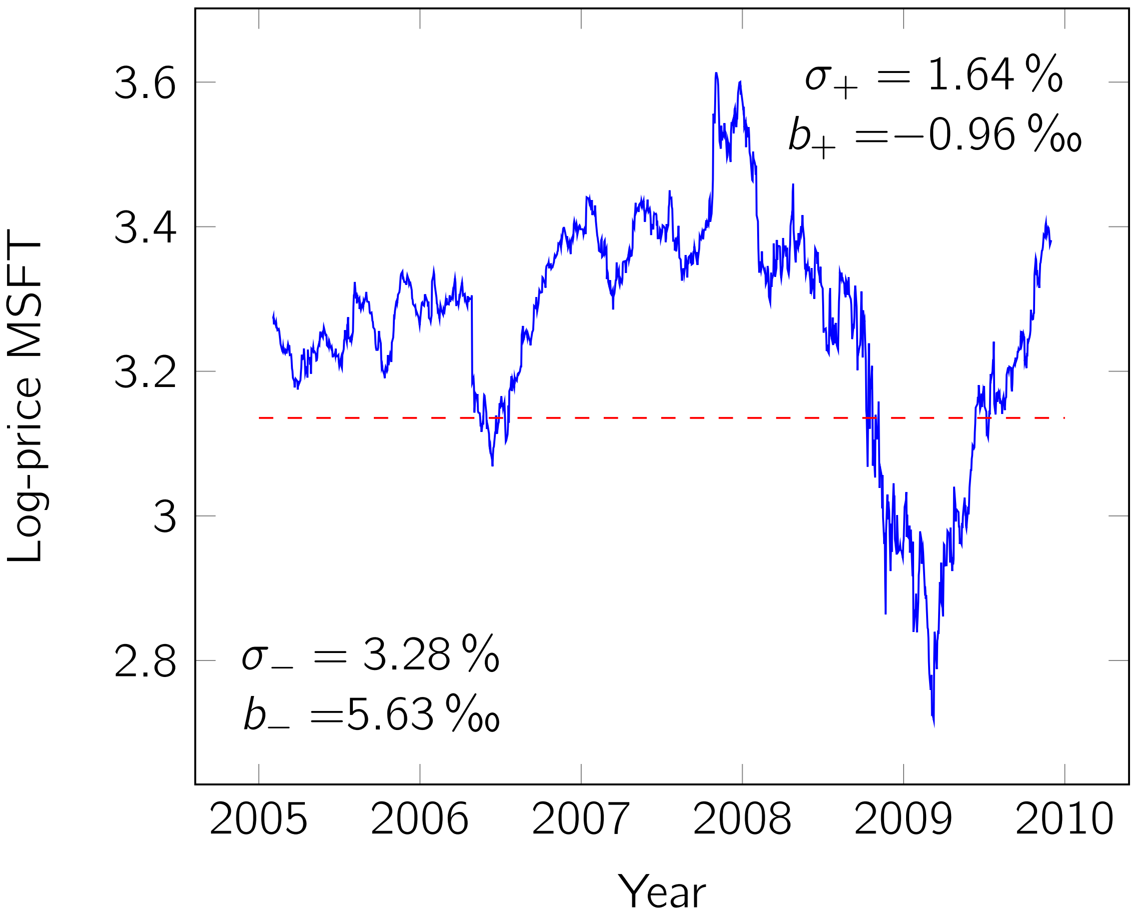
The convergence of the estimator depends on the long time behavior of $Q_T^\pm$, and of the regime of X (work in progress).

Application to financial data

- P. Mota & M. Esquível (2014) have proposed a continuous time version of a SETAR model with delay and threshold regime switching (DTRS).
- Their model uses an artifical thin layer for switching to avoid immediate switchings.
- They propose a least square estimation procedure (coming from time series).
- 21 stocks are analyzed (2005-2010): leverage and mean-reverting effects hold for most of them.

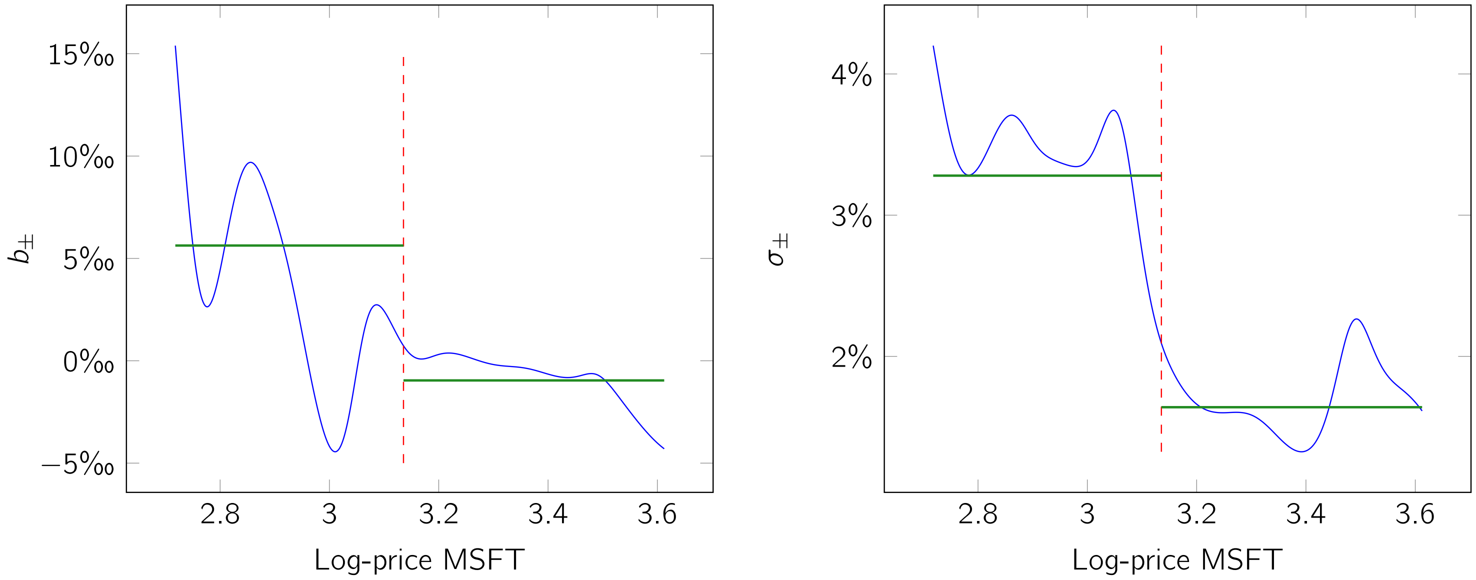
Our estimators gives consistent results with these ones.

Leverage and mean-reverting effects



The threshold is detected with the AIC model selection.

Comparison with a non-parametric estimator



Comparison with a Nadaraya-Watson non-parametric estimation.

Conclusion

- The problem of estimation of SDE with discontinuous coefficients is surprisingly open.
- Asymptotics of occupation and local times play a very important role.
- Heavily relies on the limits theorems contained in the book Jacod & Protter. However, they should be adapted to the Skew Brownian motion (some questions are left open).
- The presence of a drift really change the picture.

✠ AL & PP, Statistical estimation of the Oscillating Brownian Motion, arxiv:1701.02129 (2017).

✠ Estimation of drift, application to financial data: works in progress.