PDE strategies for the existence of McKean Nonlinear diffusion models

Mireille Bossy

Inria Tosca Laboratory

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Motivating problems

► McKean SDEs for fluid turbulent subscale models [Pope 95, 03; Durbin Speziale 94, Dreeben Pope 98, Waclawczyk Pozorski Minier 04]

$$\begin{cases} X_t = X_0 + \int_0^t U_s ds \\ (U_t, \Theta_t) = (U_0, \Theta_0) + \int_0^t \mathbb{E}_{\mathbb{P}}\left[\ell(U_s, \Theta_s) \,|\, X_s\right] ds + \int_0^t \mathbb{E}_{\mathbb{P}}\left[\gamma(U_s, \Theta_s) \,|\, X_s\right] dW_s, \end{cases}$$

Ingredient of the problem :

- Singular interaction (mean field) kernel in the diffusion term.
- degenerate diffusion coefficient

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► Calibrated Local and Stochastic Volatility (LSV) models [Gyöngy 86; Guyon Henry-Labordére 12]

$$\begin{cases} \dfrac{dS_t}{S_t} = rdt + \dfrac{a(Y_t)}{\sqrt{\mathbb{E}[a^2(Y_t)|S_t]}} \sigma_{\text{Dup}}(t, S_t) S_t dW_t \\ dY_t = \alpha(t, Y_t) dB_t + \xi(t) dt \end{cases}$$

where $\sigma_{\text{Dup}}(t, y)$ is the Dupire's local volatility function
[Abergel Tachet 2010 , Jourdain Zhou 2017]

Generic form

Find (X, Y, ρ) such that $\rho_t = \mathbb{P} \circ (X_t, Y_t)^{-1}$ satisfying

$$\begin{cases} X_t = X_0 + \int_0^t b(X_s, Y_s)ds + \int_0^t \sigma(X_s)dB_s \\ Y_t = Y_0 + \int_0^t \Lambda[X_s; \rho_s]ds + \int_0^t \Gamma[X_s; \rho_s]dW_s \end{cases}$$

(X_0, Y_0) is μ_0-distributed
$(W_t;\, t \geq 0)$,$(B_t;\, t \geq 0)$ are two independent $\mathbb{R}^d$ standard Brownian motions.
Λ and Γ defined for $(x, f) \in \mathbb{R}^d \times L^1(\mathbb{R}^d \times \mathbb{R}^d)$, as

$$\Lambda[x; f] = \frac{\int_{\mathbb{R}^d} \ell(y) f(x, y)dy}{\int_{\mathbb{R}^d} f(x, y)dy} \mathbb{1}_{\{\int_{\mathbb{R}^d} f(x,y)dy \neq 0\}}$$

$$\text{and} \quad \Gamma[x; f] = \frac{\int_{\mathbb{R}^d} \gamma(y) f(x, y)dy}{\int_{\mathbb{R}^d} f(x, y)dy} \mathbb{1}_{\{\int_{\mathbb{R}^d} f(x,y)dy \neq 0\}}.$$

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Toy model number one

Let's put $b = 0$ and go back this drift term with Girsanov transform.
This impose that strong ellipticity is assumed for σ: $\exists\, a_*, a* > 0$,for all $x \in \mathbb{R}^d$,

$$0 < a_* \textbf{\textit{Id}} \le \sigma(x)\sigma(x)^t \le a^* \textbf{\textit{Id}}.$$

Let's put $\ell = 0$, for simplicity

Theorem B. & Jabir preprint

In addition σ is such that X exists.
γ is Lipschitz and bounded on $\mathbb{R}^d$, and satisfies the strong ellipticity constraint : $\exists\, \alpha_*, \alpha^* > 0$, for all $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$,

$$0 < \alpha_* \textbf{\textit{Id}} < \gamma(x,y)\gamma(x,y)^t < \alpha^* \textbf{\textit{Id}}.$$

ρ_0 is $L^2(\mathbb{R}^{2d})$ and such that $\rho_0^X(x) = \int_{\mathbb{R}^d} \rho_0(x,y)dy \ge m > 0$.

Then there exists a unique strong solution to

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t \sigma(X_s)dB_s \\ Y_t = Y_0 + \displaystyle\int_0^t \mathbb{E}[\gamma(Y_s)|X_s]dW_s \end{cases}$$

Main argument number one : Linear/Nonlinear Fokker Planck equation

Given $f \in \mathcal{C}((0,T);L^2(\mathbb{R}^{2d})) \cap L^2((0,T);H^1_{x,y}(\mathbb{R}^{2d}))$,

Lemma

There exists a unique solution in $\mathcal{C}((0,T);L^2(\mathbb{R}^{2d})) \cap L^2((0,T);H^1_{x,y}(\mathbb{R}^{2d}))$ to

$$\begin{cases} \partial_t \rho(t,x,y) - \frac{1}{2}\text{trace}(\nabla_x^2 \times (\sigma(x)\sigma^t(x)\rho(t,x,y)) \\ \qquad -\frac{1}{2}\text{trace}(\nabla_y^2 \times (\Gamma[x,f]\Gamma^t[x,f]\rho) = 0, \\ \qquad \text{for all } (t,x,y) \in (0,T) \times \mathbb{R}^{2d}, \\ \rho(0,y,u) = \rho_0(x,y), \ \text{for all } (x,y) \in \times\mathbb{R}^{2d}. \end{cases}$$

and

$$\sup_{0\le t\le T} \|\rho(t)\|^2_{L^2(\mathbb{R}^{2d})} + \int_0^T \left(\|\nabla_x \rho(t)\|^2_{L^2(\mathbb{R}^{2d})} + \|\nabla_u \rho(t)\|^2_{L^2(\mathbb{R}^{2d})}\right) dt \le M$$

gives the existence of a solution ρ in $\mathcal{C}((0,T);L^2(\mathbb{R}^d)) \cap L^2((0,T);H^1_{x,u}(\mathbb{R}^{2d}))$ to the nonlinear Fokker Planck equation.

Main argument number two : iterative construction of the process

Theorem – (Zhang 2005)

Assume that S is continuous function of (t,x), strongly elliptic uniformly in time. Assume that $\nabla_x S(t,\cdot) \in L^{(2d+1)}_{\text{loc}}(\mathbb{R}^d)$, uniformly with respect to $t \in [0,T]$, $b \in L^{(2d+1)}_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$.
Then there exists a unique strong solution up to the explosion time for equation

$$dX_t = S(t, X_t)dW_t.$$

(In our case, S is will be also bounded so, there is no explosion).
Set $(Y^0_t, 0 \leq t \leq T) = Y_0$, and, for $n \geq 1$, given $(Y^n_t; t \geq 0)$ in $(\Omega, \mathcal{F}, \mathbb{P})$,

$$Y^{n+1}_t = Y_0 + \int_0^t \mathbb{E}_{\mathbb{P}}\left[\gamma(Y^n_s) \mid X_s\right] dW_s.$$

Given $\rho^n = \text{Law}(Y^n) \in \mathcal{C}((0,T); L^2(\mathbb{R}^{2d})) \cap L^2((0,T); H^1_{x,u}(\mathbb{R}^{2d}))$, we can prove that $\Gamma[x;\rho^n] = \mathbb{E}_{\mathbb{P}}\left[\gamma(Y^n_t) \mid X_t = x\right] \in L^{(2d+1)}_{\text{loc}}$ if $\int_{\mathbb{R}^d} \rho_0(x,u)du > 0$

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left[\max_{t\in[0,T]} |Y^{n+1}_t - Y^n_t|^2\right] &\leq T \int_0^T \mathbb{E}_{\mathbb{P}}\left[\left|\mathbb{E}_{\mathbb{P}}\left[\gamma(Y^n_s) \mid X_s\right] - \mathbb{E}_{\mathbb{P}}\left[\gamma(Y^{n-1}_s) \mid X_s\right]\right|^2\right] ds \\ &\leq \|\gamma\|^2_{Lip} \int_0^T \mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}\left[|Y^n_s - Y^{n-1}_s|^2 \mid X_s\right]\right] ds\end{aligned}$$

And what when σ could degenerate ?

Strategy one

$$\begin{cases} X_t = X_0 + \int_0^t U_s ds + \varepsilon B_t \\ Y_t = Y_0 + \int_0^t \mathbb{E}_{\mathbb{P}}\left[\ell(Y_s) \middle| X_s\right] ds + \int_0^t \mathbb{E}_{\mathbb{P}}\left[\gamma(Y_s) \,|\, X_s\right] dW_s, \end{cases}$$

Strategy two

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+ use recent advances on the flow regularity of McKean SDE's to get the regularity of ρ ? [Crisan, McMurray 2017]

Toy model number two

The moderated local McKean SDE :

$$X_t = X_0 + \int_0^t \sigma(p(s, X_s))dW_s, \quad 0 \le t \le T$$

$p \in C_b^{1,2}([0,T] \times \mathbb{R}^d)$ is such that the law of X_t is $p(t,x)dx$.

Hypotheses :

- σ Lipschitz, C^3 mapping on $\mathbb{R}$
- Strong ellipticity $\forall x \in \mathbb{R}^d, \forall y \in \mathbb{R}, x^*\sigma(y)x \ge m_\sigma |x|^2$.
- p_0 in the Hölder space $H^{2+\alpha}$ with $0 < \alpha < 1$.
- non negativity for the diffusion matrix leading the Fokker-Planck PDE written on divergence form :

$$\forall x \in \mathbb{R}^d, \forall y \in \mathbb{R}, x^* \left((\sigma\sigma^*)'(y)y + (\sigma\sigma^*)(y)\right) x \ge 0.$$

  is used to obtain uniqueness of the Fokker-Planck equation.
- Strong ellipticity on the leading matrix

  there exists $m_{\text{div}} > 0,\ \forall x \in \mathbb{R}^d, \forall y \in \mathbb{R}, x^* \left((\sigma\sigma^*)'(y)y + (\sigma\sigma^*)(y)\right) x \ge m_{\text{div}} |x|^2$.

Theorem – (Jourdain Méléard 98)

The McKean Vlasov Fokker-Planck equation has a solution in $H^{1+\frac{\alpha}{2}, 2+\alpha}$, and the nonlinear SDE admits a unique strong solution.

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Toy model number two

The moderated local McKean SDE revisited : $d = 1$

$$X_t = X_0 + \int_0^t \sigma(p(s, X_s))dW_s, \quad 0 \le t \le T$$

$p \in L^2([0,T]; L^2(\mathbb{R}))$ is such that the law of X_t is $p(t,x)dx$.

Hypotheses :

- $u \mapsto \sigma(u)$ is in $C_b^1(\mathbb{R})$.
- $u \mapsto \alpha(u) := (\sigma^2(u)u)' = 2\sigma'(u)\sigma(u)u + \sigma^2(u)$ is also bounded continuous on $\mathbb{R}$, and

$$\alpha(u) = 2\sigma'(u)\sigma(u)u + \sigma^2(u) \ge \eta > 0, \text{ uniformy in } u$$

- $u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, and $\int |x|^2 u_0(x)dt < \infty$

Theorem –(B. Jabir preprint)

Under the above hypotheses, there exist a solution in law to the above SDE.

Main argument number one : PDE analysis of the smoothed FK equation

We denote $\sigma_\varepsilon^2(p) := \sigma^2(p) + \varepsilon, \forall p \in \mathbb{R}$.
that maintains the strict positivity hypothesis on σ_ε^2 : $(\sigma_\varepsilon^2(p)p)' = (\sigma^2(p))' + \varepsilon,\ \forall u \in \mathbb{R}$.

$$\frac{\partial u^\varepsilon}{\partial t} - \frac{1}{2}\triangle_x(\sigma_\varepsilon^2(u^\varepsilon)u^\varepsilon) = 0 \qquad (1)$$
$$u_0 \quad \text{given in } L^2$$

Lemma

Under the previous hypotheses equation (1) admits a unique solution in $L^2([0,T];H^1(\mathbb{R}))$ satisfying the energy inequality

$$\sup_{t\in[0,T]} \|u(t)\|^2_{L^2(\mathbb{R})} + \varepsilon \int_0^T \|\partial_x u(t)\|^2_{L^2(\mathbb{R})} dt \le \|u_0\|^2_{L^2(\mathbb{R})}$$
$$\sup_\varepsilon \left\| \frac{\partial}{\partial x}\left(\sigma_\varepsilon^2(u^\varepsilon)u^\varepsilon\right)\right\|_{L^2([0,T]\times\mathbb{R})} < +\infty.$$

$(u^\varepsilon, \varepsilon)$ is a Cauchy sequence in $L^2((0,T)\times\mathbb{R})$.
We have also the uniqueness of the solution of the limit equation in $L^2((0,T)\times\mathbb{R}))$ such that $\left(\sigma_\varepsilon^2(u^0)u^0\right)$ is in $L^2([0,T];H^1(\mathbb{R}))$.

Proof : mainly adapted from Vasquez 06 book on porous media equation (Chapter 5).

Main argument number two : analysis of the smoothed SDE

Step 1) From u^ε solution in $L^2([0,T];H^1(\mathbb{R}))$ of the smoothed FP equation, we construct (by mean of smoothing and martingale problem) a weak solution to

$$X_t^\varepsilon = X_0 + \int_0^t \sqrt{\sigma_\varepsilon^2(u^\varepsilon(s, X_s^\varepsilon)}dW_s$$

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Step 2) $\sqrt{2\sigma_\varepsilon^2(u^\varepsilon(t,x))} > \varepsilon$ for all $(t,x)\in[0,T]\times\mathbb{R}$. Law$(X_t^\varepsilon)$ admits a density h^ε satisfying

$$h^\varepsilon(t,x) = G_t^\gamma(u_0) + \frac{1}{2}\int_0^t \triangle_x G_{t-s}^\gamma(\sigma_\varepsilon^2(u^\varepsilon) - \gamma^2)h_s^\varepsilon)(x)ds$$

for G_t^γ the Gaussian semigroup with variance $\gamma^2 t$. For a good choice of γ, this allows to prove that h^ε is in L^2, so $h^\varepsilon = u^\varepsilon$, and we have obtain a unique weak solution to the smoothed nonlinear SDE.

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Step 3) Tightness of the sequence $(X^\varepsilon, \varepsilon)$ and convergence in L^2 of the densities. Identification of the limit with rather classical arguments

This is joint work with Jean Francois Jabir (University of Valpareiso)

Bibliography

B., Fontbona, Jabin, Jabir. *Local existence of analytical solutions to an incompressible Lagrangian stochastic model in a periodic domain.CoPDE*, 2013.

B. and Jabir. *On confined McKean Langevin processes satisfying the mean no-permeability boundary condition. SPA 2011*

B. and Jabir. *Lagrangian stochastic models with specular boundary condition. JFA 2015*

B., J. Espina, etall. *Modeling the wind circulation around mills with a Lagrangian stochastic approach.* SMAI-JCM 2016.

B. and Jabir. *Some McKean models with non smooth nonlinear diffusion coefficients* Preprint.

References

Crisan and McMurray *Smoothing properties of McKean-Vlasov SDEs* 2017

Durbin and Speziale *Realizability of second-moment closure via stochastic analysis. J. Fluid Mech.* 1994

Guyon and Henry-Labordere *The Smile Calibration Problem Solved* 2011.

Jourdain, and Méléard. *Propagation of chaos and fluctuations for a moderate model with smooth initial data*, *Ann. Inst. H. P Probab. Statist.*, 1998.

Jourdain, Zhou. *Existence of a calibrated regime switching local volatility model and new fake Brownian motions.* Preprint 2016.

Pope. *Lagrangian pdf methods for turbulent flows Annu. Rev. Fluid Mech.* 1994

Minier and Pozorski *Wall-boundary conditions in probability density function methods and application to a turbulent channel flow* Physics of Fluids, 1999.

Vasquez, J. L. *The Porous Medium Equation, Oxford University Publications, 2006.*

Thank you for your attention