Martingales & Strict Local Martingales
PDE & Probability Methods
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- We begin with inspiration from Mathematical Finance
- What is a financial bubble?
- Let $S = (S_t)_{t \geq 0} \geq 0$ be the price process of a risky asset, with spot interest rate=0.
- When is S the correct price of a risky asset?
- We let S_t^* denote the "correct" price of a risky asset at time $t > 0$; economists call it the **fundamental price of the asset**
- Arbitrage considerations imply $S_t \geq S_t^*$ always
- Moreover, S^* must always be a martingale under a risk neutral measure; there is no such restriction on S; it need only be a local martingale

- Eugene Fama: $S = S^*$ always (prices are always correct)
- Robert Shiller: $S \geq S^*$ possible; market prices can exceed fundamental prices, in which case we have **bubble**
- We let $\beta_t = S_t - S_t^*$, the amount the market price exceeds the fundamental price
- **Definition:** When $\beta_t > 0$ the stock is undergoing bubble pricing
- On a compact time interval $[0, T]$ one can prove (**Jarrow, P^2, Shimbo; 2010**) that If β is not the zero process, then it is a **strict local martingale**
- Since S^* is *a fortiori* a martingale, **we have a bubble if and only if S is a strict local martingale**

- The question becomes: When is S a martingale, and when it is a strict local martingale (under the risk neutral measure)
- This is not easy to answer!
- The Delbaen-Shirakawa theory(2001) (extended by Mijatovic-Urusov): Suppose S follows an SDE:

$$dS_t = \sigma(S_t)dB_t + \mu(S_t, Y_t)dt \text{ with } S \geq 0; \quad S_0 = 1 \tag{1}$$

- Y is an external source of randomness, creating an incomplete market (no martingale representation)

- Under an equivalent local martingale measure ("risk neutral measure") we have (4) becomes

$$dS_t = \sigma(S_t)dB_t \tag{2}$$

- Assume the Engelbert-Schmidt necessary and sufficient conditions for weak uniques of (5), and we have the choice of the risk neutral measure is irrelevant(!)
- **Delbaen-Shirakawa:** S is a strict local martingale if and only if

$$\int_{\varepsilon}^{\infty} \frac{x}{\sigma(x)^2} ds < \infty \tag{3}$$

- In a stochastic volatility framework we have a result of Lions & Musiela (2007):
- Lions & Musiela studied SDEs with stochastic volatility (Heston type SDEs) to see when the solution S was a local martingale, and when it was a strict local martingale (2007)
- L. Andersen and V. Piterbarg simultaneously published a similar result in 2007
- **Lions-Musiela framework:**

$$dS_t = S_t v_t dB_t; \qquad S_0 = 1 \tag{4}$$

$$dv_t = \sigma(v_t)dW_t + b(v_t)dt; \quad v_0 = 1 \tag{5}$$

- B and W are correlated Brownian motions, with correlation coefficient ρ and our time interval is compact, $[0, T]$.
- Assume $\rho > 0$

The PL Lions-M Musiela Framework, Continued

- If

$$\limsup_{x \to +\infty} \frac{\rho\ x\sigma(x) + b(x)}{x} < \infty \tag{6}$$

holds, **then S is an integrable non negative martingale.**

- If

$$\liminf_{x \to +\infty} (\rho\ x\sigma(x) + b(x))\phi(x)^{-1} > 0 \tag{7}$$

holds, **then S is a strict local martingale.**

- $\phi(x)$ is an increasing positive smooth function that satisfies

$$\int_a^\infty \frac{1}{\phi(x)} dx < \infty$$

- The Lions-Musiela paradigm extends to processes driven by Lévy noise
- We assume that S and v follow SDEs of the form:

$$dS_t = S_{t_-} v_t^{\alpha} dM_t \tag{8}$$
$$dv_t = \sigma(v_t) dB_t + b(v_t) dt \tag{9}$$

- M is a Lévy martingale, with Lévy measure ν, such that $[M, M]$ is locally in L^1
- A sufficient condition for S to be a martingale on $[0, T]$ is that

$$E[e^{\int_0^T (\frac{1}{2}+\int_{\mathbb{R}} x^2 \nu(dx)) v_s^{2\alpha} ds}] < \infty \tag{10}$$

- The condition

$$\liminf_{x \to +\infty} (\rho \, x\sigma(x) + b(x)) \phi(x)^{-1} > 0$$

is sufficient for S to be a strict local martingale.
- A similar analysis applies for martingales M that are not necessarily Lévy, but are such that $d\langle M, M\rangle_t = \lambda_t dt$.

Our first question

- Suppose we are in the Lions-Musiela framework, and suppose S is a martingale; can S change to a strict local martingale?
- **The answer is two fold: Yes, but it's only minimally interesting**
- **Yes, and it's interesting from a math finance framework**

The uninteresting yes

- We first assume S **is a martingale under a risk neutral measure** Q, so that (6) holds under Q
- Under correct hypotheses, we can find another risk neutral measure Q^* such that under Q^* equation (7) holds. This gives that S **is a strict local martingale under a risk neutral measure** Q^*

The Interesting Yes

- If at a random time (a stopping time) we expand the underlying filtration (think of news arriving to the market), then the decompositions change, and S need no longer be even a local martingale
- To undo the damage of the decomposition wrought by the filtration enlargement, we change the risk neutral measure to a new one, $Q^{\otimes}$, which undoes the new drift from the filtration enlargement, so that S is again at least a local martingale

- For the volatility equation, the two changes (the filtration enlargement and the change to a risk neutral measure to undo it for S), combine to make the drift in the volatility equation for v such that instead of (6), we now have (7),
- Whereas we previously had the Lions-Musiela **martingale condition** (6) satisfied, now under the larger filtration $\mathbb{G}$ and the new risk neutral measure $Q^{\otimes}$, we have the Lions-Musiela **strict local martingale condition** (7) satisfied.

The Vector Case

- The idea is simple, but the technical hurdles to achieve it are somewhat formidable
- We now have the next question: what about a system of SDEs?
- In finance, **suppose we have a portfolio of n stocks, and their prices interact; could some be in bubbles, and some not be in bubbles?**

- Here the framework is a system of SDEs of the form:

$$
\begin{aligned}
dX^1 &= \sum_{i=1}^{d} \sigma_i^1(X_t^1, \ldots, X_t^n) dB_t^i \\
&\vdots \\
dX_t^k &= \sum_{i=1}^{d} \sigma_i^k(X_t^1, \ldots, X_t^n) dB_t^i \\
&\vdots \\
dX_t^m &= \sum_{i=1}^{d} \sigma_i^m(X_t^1, \ldots, X_t^n) dB_t^i
\end{aligned}
$$

- Let Γ denote the indices $\{1, 2, \ldots, d\}$. Let Λ be a subset of Γ.
- Can we have X^i be a martingale for all $i \in \Lambda$, and have X^i be a strict local martingale for all $i \in \Gamma \backslash \Lambda$?
- We can adapt the theory developed by Khasminskii, Narita, Stroock, and Varadhan on the explosions (or lack thereof) of systems of SDEs
- The conditions are a little complicated to give in a 20 minute talk, but if anyone is interested, Aditi Dandapani and I have a preprint we can share, "il n'y a que demander"

An Example

- As an example, let us consider equations of the form, with $S_0 = N_0 = v_0 = 1$:

$$\begin{aligned} dS_t &= S_t f(S_t, N_t, v_t) dB_t \\ dN_t &= N_t g(S_t, N_t, v_t) dZ_t \\ dv_t &= \sigma(S_t, N_t, v_t) dW_t + b(S_t, N_t, v_t) dt \end{aligned}$$

- with correlations $d[B, W]_t = \rho^1 t, \quad d[B, Z]_t = \rho^2 t, \quad d[W, Z]_t = \rho^3 t$

- If we take $\rho^1 = -1, \quad \rho^2 = \rho^3 = 1$, and for coefficients

$$
\begin{aligned}
f(x_1, x_2, x_3) &= \frac{1}{\sqrt{3}} \frac{1}{x_1^2} \left(\frac{\| x \|}{2} \right)^{1+\varepsilon} \\
g(x_1, x_2, x_3) &= \frac{1}{\sqrt{3}} \frac{1}{x_2^2} \left(\frac{\| x \|}{2} \right)^{1+\varepsilon} \\
\sigma(x_1, x_2, x_3) &= \frac{1}{\sqrt{3}} \frac{1}{x_3} \left(\frac{\| x \|}{2} \right)^{1+\varepsilon} \\
b(x_1, x_2, x_3) &= 2(\| x \|^2)^{1+\varepsilon}
\end{aligned}
$$

- Then we have that S **is a martingale** and N **is a strict local martingale**, while v is just a stochastic volatility process.

The End

Thank you for your attention