Robust adaptivity for nonlinear partial differential equations

A Newton method on subspaces for ODEs

Roland Becker, Pau

I FEMs for ODEs

II Inexact globalized Newton's method

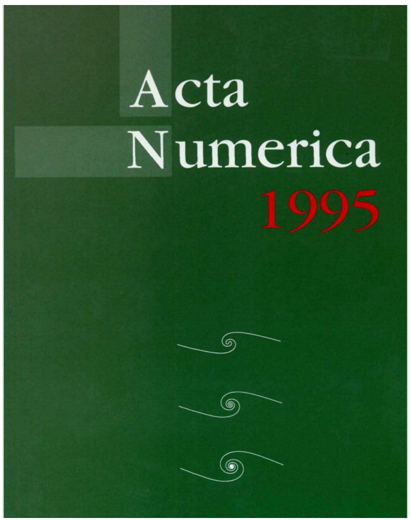
III Newton on FEM subspaces

IV Back to ODEs

... early results in adaptive time stepping and [12, 31, 34] current applications in geophysics). Particularly, adaptive step size control goes back to the 1890s, when C. Runge used computations with halved step sizes to find reliable digits in his calculations [22, Page 164]...

[4] M. Feischl and D. Niederkofler, *Optimal adaptive implicit time stepping*, 2025.

I FEMs for ODEs



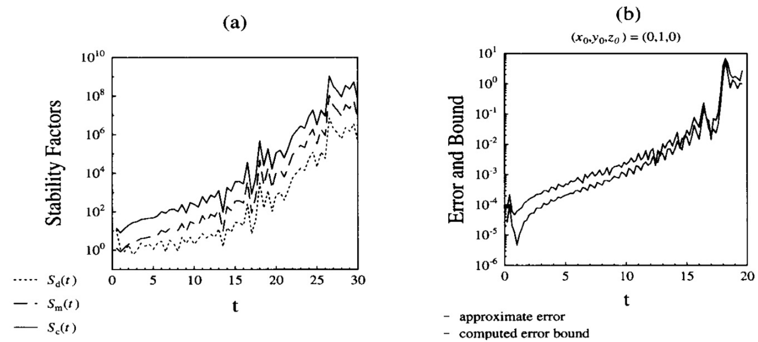


Fig. 11. Stability factors and error bound for the Lorentz system

2. Computability and predictability

Was man mit Fehlerkontrolle nicht berechnen kann, darüber muss mann schweigen (Wittgenstein).

$$\frac{dx}{dt}(t) = f(t, x(t)), \quad x(0) = x_0 \qquad J = [0;T]$$

$$f \in C^1(J \times V, \mathbb{R}^n),\ V \subset \mathbb{R}^n \text{ open},\ x_0 \in V$$

$$\left|\frac{\partial f}{\partial x}(t, x+p) - \frac{\partial f}{\partial x}(t,x)\right| \leq L_f(t,x)\,|p| \quad \forall p \in B_{r_f(t,x)} \quad \forall t \in J.$$

We want to write the ODE as $F(x) = 0$

$$F : \mathcal{U} \subset X \longrightarrow Y'$$

$$U := \{x \in L^\infty(J) \mid x(t) \in V\ \forall t \in J\}.$$

primal

$$\begin{cases} X := H^1(J), \quad Y := L^2(J) \times \mathbb{R}^n \\ \langle F(x), (y, y_0) \rangle_{Y' \times Y} = a(x)(y) + (x(0) - x_0) \cdot y_0, \\ a(x)(y) := \displaystyle\int_0^T y(t)^\mathsf{T} \left(\frac{dx}{dt}(t) - f(t, x(t)) \right) dt. \end{cases}$$

$$x(t) = x_0 + \int_0^t f(s, x(s))\, ds.$$

dual

$$\begin{cases} X := L^2(J), \quad Y := H^1_T(J) \\ \langle F(x), y \rangle_{Y' \times Y} = a(x)(y), \\ a(x)(y) := \displaystyle\int_0^T \left(x(t)^\mathsf{T} \frac{dy}{dt}(t) + y(t)^\mathsf{T} f(t, x(t)) \right) dt + {x_0}^\mathsf{T} y(0). \end{cases}$$

Linearized problem at $x \in \mathcal{U}$

$$\frac{dp}{dt} = Ap + b, \quad p(0) = 0, \quad A(t) := \frac{\partial f}{\partial x}(t, x)$$

$r := \frac{dp}{dt} - Ap$

$\frac{dp}{dt} = Ap + r$

$\|p\|_\infty \lesssim \|r\|_2$

$\|\frac{dp}{dt}\|_2 \lesssim \|r\|_2$

$$\|p\|_{L^\infty(J)} \le C_{\mathrm{gr}}(\|x\|_\infty)\,\|b\|_{L^1(J)}$$

$$\|\frac{dp}{dt}\|_{L^2(J)} \le M(\|x\|_\infty)\,\|\frac{dp}{dt} - Ap\|_{L^2(J)}.$$

$$0 < \gamma(x) := \inf_{p\in B_X} \sup_{y\in B_Y} a'(x)(p, y) = \inf_{y\in B_Y} \sup_{p\in B_X} a'(x)(p, y),$$

$\gamma(x) = M(\|x\|_\infty)^{-1}$

$\left(M(\|x\|_\infty) := \inf_{\substack{p \in H_0^1(J) \\ p \neq 0}} \frac{\|\frac{dp}{dt} - Ap\|_2}{\|\frac{dp}{dt}\|_2} \right)$

II Inexact globalized Newton's method

$F(x) = 0 \qquad F : X \to Y'$

Throughout we suppose F' locally Lipschitz: there exists $r > 0$ such that

(1) $\|F'(x+p) - F'(x)\|_{L(X,Y')} \leq L_F(\|x\|_X)\|p\|_X$ for all $p \in B_r$

We consider the update in the k-th iteration of Newton's method

$$x_k = x_{k-1} + \alpha_k p_k \in X_k, \quad R_k := F(x_{k-1}) + F'(x_{k-1})p_k, \tag{6}$$

Then the classical argument in convergence proofs is

$$\begin{cases} F(x_k) = F(x_{k-1}) + \displaystyle\int_0^{\alpha_k} F'(x_{k-1} + sp_k)(p_k)\,ds \\ \qquad\quad = (1-\alpha_k)F(x_{k-1}) + \alpha_k R_k + \xi_k, \\ \quad \xi_k := \displaystyle\int_0^{\alpha_k} \left(F'(x_{k-1} + sp_k) - F'(x_{k-1})\right)(p_k)\,ds. \end{cases} \tag{7}$$

By assumption **(1)** we have

$$\|p_k\|_X \leq r \quad \Rightarrow \quad \|\xi_k\|_{Y'} \leq \frac{L_k \alpha_k^2}{2}\|p_k\|_X^2, \quad L_k := L(\|x_k\|_X).$$

(**2**) There exists $0 \le \lambda < 1$ such that for all k

$$\|R_k\|_{Y'} \le \lambda \|F(x_{k-1})\|_{Y'}.$$

[1] R. S. Dembo, S. C. Eisenstat, and T. Steihaug, *Inexact Newton methods*, SIAM J. Numer. Anal., 19 (1982), pp. 400–408.

(**3**) There exists M_k such that

$$\|p_k\|_X \le M_k \|F(x_{k-1})\|_{Y'}.$$

Merit function: $\Phi(x) = \|F(x)\|_{Y'}$

Applying these assumptions we get

$$\Phi(x_k) = \|F(x_k)\|_{Y'} \le \left(1 - (1-\lambda)\alpha_k + \frac{L_k M_k \alpha_k^2}{2}\|p_k\|_X\right) \|F(x_{k-1})\|_{Y'} = \rho_k\, \Phi(x_{k-1}).$$

This leads to linear convergence, $\rho_k \le \rho < 1$, of the merit function under the condition

$$\alpha_k \le \alpha_k^* := \min\left\{\frac{\lambda}{L_k M_k \|p_k\|_X}, \frac{r}{\|p_k\|_X}, 1\right\}. \tag{8}$$

*) This motivates backtracking line-search.

**) Discretization: bound the residuals by estimators.

III Newton on FEM subspaces

Let $(X, \langle\cdot,\cdot\rangle_X)$ and $(Y, \langle\cdot,\cdot\rangle_Y)$ be two Hilbert spaces and $(Z, \|\cdot\|_Z)$ be a Banach space continuously embedded in X. We consider nested subspaces

$Z = L^\infty(\Omega)$

primal: $Z = X$!

$$(X_0, Y_0) \subset \ldots \subset (X_{k-1}, Y_{k-1}) \subset (X_k, Y_k) \subset \ldots \subset (X, Y).$$

with $\dim Y_k = \dim X_k = N_k$.

$$\|y'\|_{Y_k'} = \sup_{y \in B_{Y_k}} \langle y', y \rangle$$

$$P_k : Y' \to Y_k' \qquad \|y'\|_{Y'}^2 = \|P_k y'\|_{Y'}^2 + \|(I - P_k)y'\|_{Y'}^2 \quad \forall y' \in Y'.$$

$$x_k = x_{k-1} + \alpha_k p_k \in U \cap X_k, \qquad R_k := F(x_{k-1}) + F'(x_{k-1})p_k,$$

(H1) We suppose there exists an open set $U \subset Z$, such that $F \in C^1(U, Y')$, $F'(x) \in \mathcal{L}(X, Y')$ for all $x \in U$, and F' locally Lipschitz: there exist $L_F \in \mathcal{B}^+(\mathbb{R}^+)$ and $r \in \mathcal{B}^*(U)$, such that for $x \in U$ and all $p \in B_{r(x),Z}$ we have $x + p \in U$ and

$$\|F'(x+p) - F'(x)\|_{\mathcal{L}(X,Y')} \leq L_F(\|x\|_Z)\|p\|_Z.$$

(H2) There exist estimators $\eta_k : X_k \to \mathbb{R}$, $\zeta_k : X_k \times X_k \to \mathbb{R}$ and $C_R \in \mathbb{R}$ such that for all k and $x, p \in X_k$

$$\|(I - P_k)F(x)\|_{Y'} \leq C_R\, \eta_k(x), \quad \|(I - P_k)(F(x) + F'(x)p)\|_{Y'} \leq C_R\, \zeta_k(x, p).$$

(H3) There exists $\beta > 0$ such that for all k

$$\eta_k(x) + \beta\|F(x)\|_{Y'_k} \leq \eta_{k-1}(x) + \beta\|F(x)\|_{Y'_{k-1}} \quad \forall x \in X_k.$$

(H4) For all k, $0 \leq \alpha \leq 1$, $x \in X_k$

$$\eta_k(x + \alpha p) \leq (1 - \alpha)\eta_k(x) + \alpha\zeta_k(x, p) + \frac{L_F(\|x\|_Z)\alpha^2}{2}\|p\|_Z^2 \quad \forall p \in X_k \cap B_{r(x),Z}.$$

(H5) The discrete tangent equation

$$p \in X_k : \quad \langle F(x) + F'(x)p, y\rangle_{Y' \times Y} = 0 \quad \forall y \in Y_k$$

admits a solution for all $x \in X_k$, $k \in \mathbb{N}$ and there exist $M, C_D \in \mathcal{B}^+(\mathbb{R}^+)$ such that

$$\|p_k\|_Z \leq M(\|x\|_Z)\|F(x)\|_{Y'_k} + C_D(\|x\|_Z)\left(\eta_k(x_{k-1}) + \zeta_k(x_{k-1}, p_k)\right).$$

(H6) The initial guess $x_0 \in U$ has a bounded sublevel set

$$\sup\left\{\|x\|_Z \mid x \in U,\ \|F(x)\|_{Y'} \leq \|F(x_0)\|_{Y'}\right\}.$$

$$\Phi_k(x) := \eta_k(x) + \beta \|F(x)\|_{Y_k'}.$$

For $\alpha_k \|p_k\|_Z \le r(x_{k-1})$

$$\langle F(x_k), y_k\rangle_{Y'\times Y} \le (1-\alpha_k)\langle F(x_{k-1}), y_k\rangle_{Y'\times Y} + \underbrace{\langle \xi_k, y_k\rangle}_{\text{quadratic}}{}_{Y'\times Y}$$

implies

$$\|F(x_k)\|_{Y_k'} \le (1-\alpha_k)\|F(x_{k-1})\|_{Y_k'} + \frac{L_F(\|x_{k-1}\|_Z)\alpha_k^2}{2}\|p_k\|_Z^2,$$

(H3) and (H4):

$$\Phi_k(x_k) = \eta_k(x_k) + \beta\|F(x_k)\|_{Y_k'} \le (1-\alpha_k)\left(\eta_{k-1}(x_{k-1}) + \beta\|F(x_{k-1})\|_{Y_{k-1}'}\right)$$
$$+ \alpha_k\zeta_k(x_{k-1}, p_k) + \frac{L_k}{2}\alpha_k^2\|p_k\|_Z^2,$$

(H5): $\|p_k\|_Z \le M_k\Phi_{k-1}(x_{k-1})$

$\zeta_k(x_{k-1}, p_k) \le \lambda_k\Phi_{k-1}, \quad \lambda_k \le 1$

(stopping criterion)

Algorithm: Subspace Newton

Parameters: $\varepsilon > 0$, $\underline{\alpha} > 0$, $\beta > 0$, $0 < \lambda < 1$, $0 < c_0, \omega < 1$, $0 < \kappa \leq 1$.
Input: (X_0, Y_0) and $x_0 \in X_0$. Set $k = 0$ and $\Phi_{-1} := \lambda$.

while $\Phi_k(x_k) > \varepsilon$
  (A1) Solve the tangent equation.
  Set $\lambda_k := \min\{\lambda, \Phi_{k-1}^{\kappa}\}$ and $\delta_k := \lambda_k \Phi_{k-1}$.
  (X_k, Y_k, p_k) result of AFEM applied to $(\delta_k, X_{k-1}, Y_{k-1})$.

  (A2) Compute the step size and update.
  Set $\alpha_k^{(0)} = r(\|x_{k-1}\|)/\|p_k\|_Z$ and $j = 0$.
  while $\Phi_k(x_{k-1} + \alpha_k^{(j)} p_k) > (1 - c_0)\Phi_k(x_{k-1})$
    Set $\alpha_k^{(j+1)} = \omega\alpha_k^{(j)}$,
    if $\alpha_k^{(j+1)} < \underline{\alpha}$, stop the algorithm and report failure.
    Set $\ell \leftarrow \ell + 1$
  end while
  Set $\alpha_k := \alpha_k^{(j)}$.
  Set $x_k := x_{k-1} + \alpha_k p_k$.
end while

Theorem 5.2 (Convergence of Newton). *Suppose that Algorithm 5 generates a sequence of steps with*

$$\alpha_k \geq \underline{\alpha} > 0.$$

Then

i) *We have linear convergence of the merit function:*

$$\Phi_k \leq \rho_k \Phi_{k-1}, \qquad \rho_k \leq 1 - \frac{\underline{\alpha}(1-\lambda)}{2} < 1.$$

ii) *R-linear convergence of* $(\zeta_k(x_{k-1}, p_k))_{k\in\mathbb{N}}$ *and* $(\|p_k\|_Z)_{k\in\mathbb{N}}$.

iii) *There is* $k_0 \in \mathbb{N}$ *such that superlinear convergence holds*

$$\Phi_k \leq C_{\mathrm{sl}}\, \Phi_{k-1}^{1+\kappa} \qquad \forall k \geq k_0,$$

iv) *If the algorithms converges towards a solution* x^* *with* $F'(x^*)$ *regular,* $(\|x^* - x_k\|_Z)_{k\in\mathbb{N}}$ *converges R-linearly.*

Lemma 5.3. *Let δ_k be the accuracy in in step **(A1)**. If there exists an adaptive algorithm, which, given δ_k, the pair of subspaces (X_{k-1}, Y_{k-1}), and $x_{k-1} \in X_{k-1}$, produces a new pair of finite-dimensional subspaces $(X_k, Y_k) \subset (X, Y)$ and corresponding solution to the tangent equation $p_k \in X_k$, such that*

$$\zeta_k(x_{k-1}, p_k) \leq \delta_k, \quad N_k - N_{k-1} \leq C_A \delta_k^{-1/s}.$$

Then we have the complexity estimate

$$\sum_{k=1}^{k_*} N_k - N_0 \leq C\varepsilon^{-1/s},$$

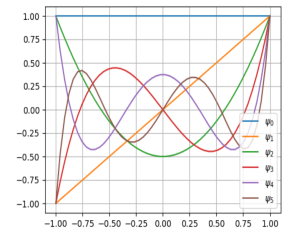
where k_* is the number of iterations performed by the algorithm.

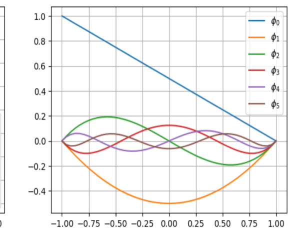
Since $\forall m \geq k$

$$\Phi_{m-1} \leq Cq^{m-k} \underbrace{\min\left\{\lambda, \Phi_{k-1}^{\kappa}\right\} \Phi_{k-1}}_{\delta_k}.$$

IV Back to ODEs

$$\begin{cases} \text{(primal)} & X_h := \mathcal{P}_h^{\mathrm{k}+1} \cap C(\overline{J}), \qquad Y_h := \mathcal{P}_h^{\mathrm{k}} \times \mathbb{R}^n, \\ \text{(dual)} & X_h := \mathcal{P}_h^{\mathrm{k}}, \qquad Y_h := \mathcal{P}_h^{\mathrm{k}+1} \cap H_T^1(J). \end{cases}$$





$$x_h \in X_h : \qquad a(x_h)(y_h) = 0 \quad \forall y_h \in Y_h .$$

(H1) $$\|F'(x+w) - F'(x)\|_{\mathcal{L}(X,Y')} \le \underbrace{C_S \,\|L_f(\cdot,x)\|_{L^2(J)}}_{L_f(\|x\|_\infty)} \|w\|_{L^\infty(J)} \quad \forall w \in B_{r(x)}$$

Error estimators

primal:

$$\begin{cases} \eta_h(x) := \left(\sum_{i=1}^{n} \frac{h_i^2}{\pi^2} \| \frac{d}{dt} \left(\frac{dx}{dt} - f(\cdot, x) \right) \|_{L^2(J_i)}^2 \right)^{1/2}, \\ \zeta_h(x,p) := \left(\sum_{i=1}^{n} \frac{h_i^2}{\pi^2} \| \frac{d}{dt} \left(\frac{dp}{dt} - f(\cdot, x) - \frac{\partial f}{\partial x}(\cdot, x) p \right) \|_{L^2(J_i)}^2 \right)^{1/2}, \\ \mu_h(x, p_h) := \left(\sum_{i=1}^{n} \frac{h_i^2}{\pi^2} \| \frac{d}{dt} \left(\frac{\partial f}{\partial x}(\cdot, x) p \right) \|_{L^2(J_i)}^2 \right)^{1/2}. \end{cases}$$

dual:

$$\begin{cases} \eta_h(x) := C_{\mathrm{I}} \left(\sum_{i=1}^{N_h} h_i^2 \| (I - \pi_i^{\mathrm{k}-1}) f(\cdot, x) \|_{L^2(J_i)}^2 \right)^{1/2}, \\ \zeta_h(x, p_h) := C_{\mathrm{I}} \left(\sum_{i=1}^{N_h} h_i^2 \| (I - \pi_i^{\mathrm{k}-1}) (f(\cdot, x) + \frac{\partial f}{\partial x}(\cdot, x) p_h) \|_{L^2(J_i)}^2 \right)^{1/2}, \\ \mu_h(x, p_h) := C_{\mathrm{I}} \left(\sum_{i=1}^{N_h} h_i^2 \| (I - \pi_i^{\mathrm{k}-1}) \frac{\partial f}{\partial x}(\cdot, x) p_h \|_{L^2(J_i)}^2 \right)^{1/2}, \end{cases}$$

[4] M. Feischl and D. Niederkofler, *Optimal adaptive implicit time stepping*, 2025.

(H2), (H3) and (H4) OK

Discrete stability

$$\gamma_h(x) := \inf_{p \in B_{X_h}} \sup_{y \in B_{Y_h}} a'(x)(p,y) \geq \gamma(x) - \sup_{p \in B_{x_h}} \mu_h(x,p).$$

implies uniform inf-sup for $\mu_{h_0}(x_{h_0}, p_{h_0}) \leq \frac{\gamma(x_{h_0})}{2}$

[3] M. Feischl, *Inf-sup stability implies quasi-orthogonality*, Math. Comp., 91 (2022), pp. 2059–2094.

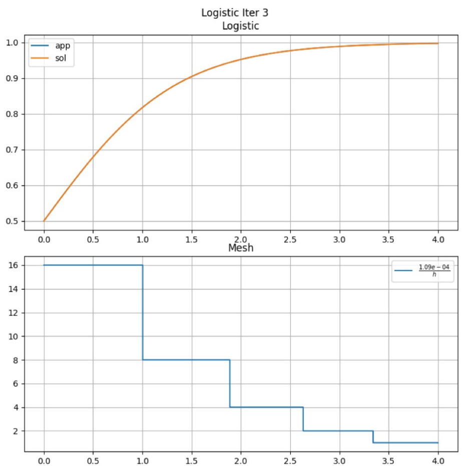
[4] M. Feischl and D. Niederkofler, *Optimal adaptive implicit time stepping*, 2025.

i) $\|p - p_h\|_{L^2(J)} \leq M(\|x\|_\infty)\zeta_h(x, p_h).$

ii) $\|p_h\|_{L^\infty(J)} \leq M(\|x\|_\infty)\left(\|F(x)\|_{Y'} + \|A\|_{L^2(J)}\,\mu_h(x, p_h)\right)$

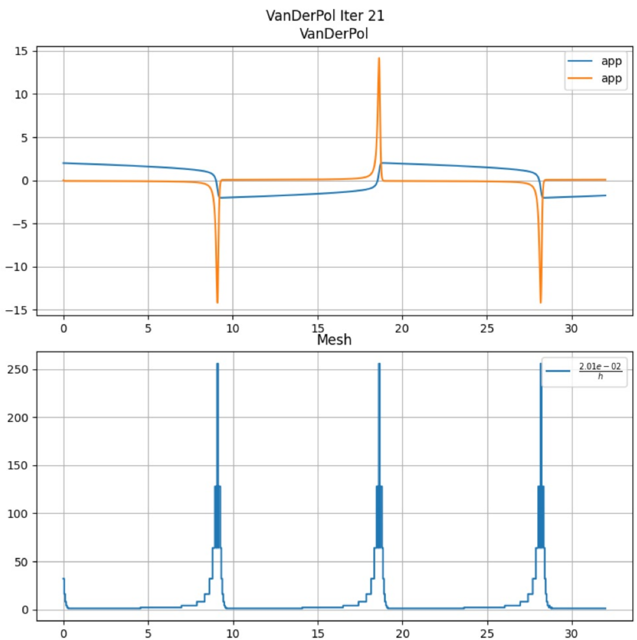
Three examples

1) Logistic equation



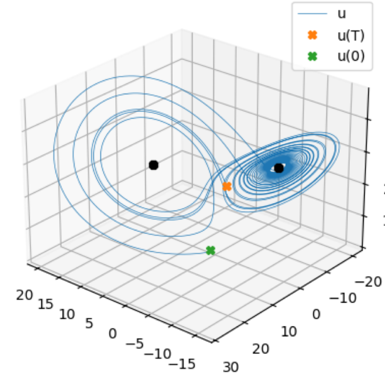
it	bt	mer	ρ_{mer}	$\|p\|$	$\rho_{\|p\|}$	η	$\|F(x)\|_{Y_k'}$	N	meshiter
0	0	4.652e−03	0.00	0.000e+00	0.00	2.201e−03	2.451e−03	10	0
1	0	1.223e−04	0.03	3.163e−03	0.00	1.202e−04	2.138e−06	35	3
2	0	1.712e−07	0.00	9.776e−05	0.03	1.711e−07	8.774e−11	835	7
3	0	4.171e−09	0.02	1.389e−07	0.00	4.171e−09	1.841e−16	5592	4

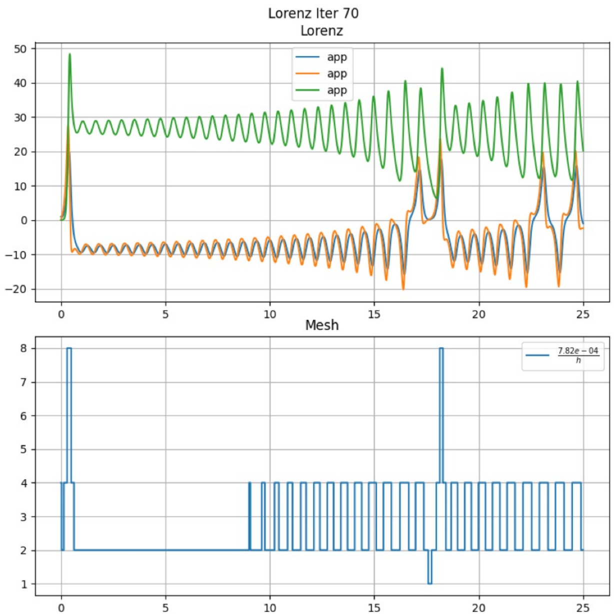
2) Van der Pol



it	bt	mer	ρ_{mer}	$\|p\|$	$\rho_{\|p\|}$	η	$\|F(x)\|_{Y_k'}$	N	meshiter
0	0	2.271e+00	0.00	0.000e+00	0.00	7.255e−01	1.546e+00	400	0
1	4	1.938e+00	0.85	2.858e+01	0.00	1.645e−01	1.774e+00	425	3
2	3	1.934e+00	1.00	2.881e+01	1.01	1.302e−01	1.804e+00	443	1
3	4	1.758e+00	0.91	3.700e+01	1.28	9.019e−02	1.668e+00	460	1
4	4	1.678e+00	0.95	3.684e+01	1.00	8.937e−02	1.589e+00	460	0
5	4	1.550e+00	0.92	3.362e+01	0.91	8.573e−02	1.464e+00	472	1
6	4	1.470e+00	0.95	2.830e+01	0.84	8.514e−02	1.385e+00	472	0
7	4	1.403e+00	0.95	2.284e+01	0.81	8.565e−02	1.317e+00	472	0
8	3	1.345e+00	0.96	1.850e+01	0.81	8.572e−02	1.260e+00	472	0
9	2	1.254e+00	0.93	1.451e+01	0.78	6.666e−02	1.188e+00	483	1
10	3	1.094e+00	0.87	1.504e+01	1.04	6.604e−02	1.028e+00	485	1
11	3	1.012e+00	0.93	1.337e+01	0.89	6.878e−02	9.432e−01	485	0
12	2	8.442e−01	0.83	1.152e+01	0.86	4.413e−02	8.001e−01	496	1
13	2	6.232e−01	0.74	8.767e+00	0.76	3.781e−02	5.854e−01	508	1
14	2	5.391e−01	0.87	6.564e+00	0.75	4.130e−02	4.978e−01	508	0
15	2	4.519e−01	0.84	4.886e+00	0.74	4.364e−02	4.083e−01	508	0
16	1	3.937e−01	0.87	3.620e+00	0.74	4.669e−02	3.470e−01	508	0
17	1	2.414e−01	0.61	1.914e+00	0.53	4.765e−02	1.938e−01	508	0
18	0	8.174e−02	0.34	1.006e+00	0.53	1.922e−02	6.252e−02	533	1
19	0	8.206e−03	0.10	1.261e−01	0.13	7.823e−03	3.839e−04	580	1
20	0	4.873e−04	0.06	7.226e−03	0.06	4.866e−04	7.406e−07	1071	3
21	0	1.858e−06	0.00	4.005e−04	0.06	1.857e−06	7.098e−10	13767	6

3) Lorenz





it	bt	mer	ρ_{mer}	$\|p\|$	$\rho_{\|p\|}$	η	$\|F(x)\|_{Y_k'}$	N	meshiter
0	0	1.000e-02	0.00	0.000e+00	0.00	1.038e-03	8.961e-03	2000	0
1	6	9.679e-03	0.97	3.177e+01	0.00	6.562e-04	9.023e-03	2053	1
2	7	9.674e-03	1.00	2.992e+01	0.94	6.563e-04	9.018e-03	2053	0
3	7	9.671e-03	1.00	2.906e+01	0.97	6.563e-04	9.014e-03	2053	0
4	7	9.667e-03	1.00	2.825e+01	0.97	6.564e-04	9.011e-03	2053	0
5	7	9.664e-03	1.00	2.746e+01	0.97	6.565e-04	9.008e-03	2053	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
61	3	6.050e-03	0.94	2.381e+00	0.90	6.765e-04	5.373e-03	2053	0
62	2	5.967e-03	0.99	2.134e+00	0.90	6.780e-04	5.289e-03	2053	0
63	2	5.546e-03	0.93	1.686e+00	0.79	6.792e-04	4.866e-03	2053	0
64	2	4.928e-03	0.89	1.322e+00	0.78	6.801e-04	4.248e-03	2053	0
65	1	4.399e-03	0.89	1.027e+00	0.78	6.817e-04	3.718e-03	2053	0
66	0	1.433e-03	0.33	5.526e-01	0.54	2.305e-04	1.203e-03	2388	1
67	0	7.965e-05	0.06	2.533e-02	0.05	7.783e-05	1.826e-06	3388	1
68	0	8.873e-06	0.11	6.836e-05	0.00	8.873e-06	2.567e-12	7014	2
69	0	3.417e-07	0.04	8.579e-06	0.13	3.417e-07	2.349e-14	20351	3
70	0	4.445e-09	0.01	2.901e-07	0.03	4.445e-09	1.832e-14	86118	4

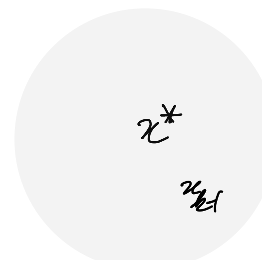
Approximation classes

Suppose we have a unique solution on $X_h \longrightarrow x_h^*$

$$\mathcal{H}_N := \{ h \in \mathcal{H} \mid N_h \leq N \}$$

$$\sup_{N \geq N_0} N^s \inf_{h \in \mathcal{H}_N} \begin{cases} \|F(x_h^*)\|_Y \\ \eta_h(x_h^*) \\ \|x^* - x_h^*\| \end{cases} < +\infty$$

for tangent equation $\begin{cases} \eta_h(x_{h-1}, P_h) \\ \|P_h^* - P_h\| \end{cases}$



Kantorovich

$$\mathcal{M}^e$$

$$\begin{array}{ccc} X \;-\;\cdot & X \cdots\to -\;\cdot & X \\ h_{k-1} & h^e_{k-1} & h_k = h^e_{k-1} \\ & H & \end{array}$$

$\exists\, h = \mathrm{REF}(H, \mathcal{M}_*)$ $\quad \eta_h(x^*_h) \lesssim \varepsilon \quad \#\mathcal{M}_* \lesssim \varepsilon^{-1/s}$

$(\varepsilon = \beta\, \xi_H(x_{k-1}, P_H))$

we want $\boxed{\xi_H(x_{k-1}, P_H, \mathcal{M}_*) \geq \theta\, \xi_H(x_{k-1}, P_H)}$

$$\xi_h(x_{k-1}, P_h) \leq \eta_h(x_{k-1} + P_h) + C\|P_h\|^2$$

$$\leq \eta_h(x^*_h) + C\Big(\underbrace{\|x^*_h - (x_{k-1} + P_h)\|}_{\lesssim \|x^*_h - x_{k-1}\|^2 + \|P_{k-1}\|^2 \;\lesssim\; \phi^2_{k-1}} + \|P_H\|^2\Big)$$

Stopping: $\xi_H(x_{k-1}, P_H) \geq \delta_k$

$\delta_k =\colon \phi^{1+\kappa}_{k-1}$

$$\xi_h(x_{k-1}, P_h) \leq \beta\, \xi_H(x_{k-1}, P_H) + \phi^{1-\kappa}_{k-1}\, \xi_H(x_{k-1}, P_H)$$

small for k big... $\quad (\kappa < 1)$