Mean field games with nondifferentiable Hamiltonians

Iain Smears

Department of Mathematics
University College London

March 2026

Supported by Engineering and Physical Sciences Research Council [grant number EP/Y008758/1]

Outline

Yohance A. P. Osborne
(Durham, Royal Society Fellowship)

Harry Wells
(UCL)

Mean field games (MFG):

- introduced independently by Lasry & Lions '06+'07; Huang, Malhamé and Caines, '06;
- models of **Nash equilibria** for **dynamic games** with infinitely many players;
- each player **optimally controls** their personal evolving state governed by a controlled SDE;
- the player population is described by its overall density over state space of the game;
- mean field interaction: players' objective functions depend on density of players.

Example applications in the literature:

- ecology: *animal habitat selection*[1];
- urban planning: *evolution of cities*[2];
- economics / finance: *saving/borrowing and household wealth*[3];
- energy: *power grids w/ dynamic energy pricing*
- multi-agent RL; ...

1. Cantrell, *J. Math. Biol.*, 2025. | 2. Barilla *et al.*, *J. Dyn. Games*, 2021. | 3. Achdou *et al.*, *Rev. Econ. Stud*, 2022

NB: there is a wide range of different MFG models (continuous/discrete state space / time; finite/infinite horizon; games of controls; ...). It cannot be expected to represent all possibilities in one go.

Introduction: Mean field games

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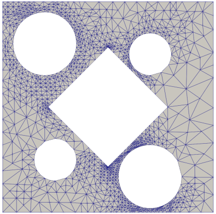
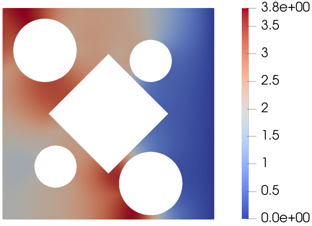
Example (Crowd motion)

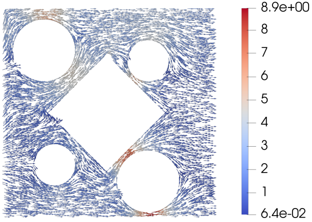
Toy model of a crowd moving from left to right around obstacles [Osborne, S. & Wells, IMA J. Numer. Anal. 2025].



Mesh



Density



Flux

Time-dependent MFG system [Lasry & Lions 2006, 2007]

$$-\partial_t u - \nu\Delta u + H(t,x,\nabla u) = F[m](t,x) \qquad \text{in } (0,T)\times\Omega, \tag{HJB}$$

$$\partial_t m - \nu\Delta m - \operatorname{div}\left(m\frac{\partial H}{\partial p}(t,x,\nabla u)\right) = G(t,x) \qquad \text{in } (0,T)\times\Omega, \tag{FP}$$

$$m(0,x) = m_0(x), \quad u(T,x) = S[m(T,\cdot)](x) \qquad \text{on } \Omega,$$

- $\Omega \subset \mathbb{R}^d$, $d \geq 1$, open, connected, bounded with Lipschitz boundary. Diffusion $\nu > 0$ constant.
- Some boundary conditions for (u,m) on $(0,T)\times\partial\Omega$, e.g. Dirichet, Neumann,...
- Hamiltonian is obtained from underlying optimal control problem:

$$H(t,x,p) := \sup_{\alpha\in\mathcal{A}} \left[b(t,x,\alpha)\cdot p - f(t,x,\alpha)\right] \qquad \forall (t,x,p) \in (0,T)\times\Omega\times\mathbb{R}^d,$$

  with control set $\mathcal{A}$ (compact metric space) and $(b,f) \in C\big([0,T]\times\overline{\Omega}\times\mathcal{A};\mathbb{R}^d\times\mathbb{R}\big)$.
- Detailed assumptions on other data later...

Prior work

- *PDE Analysis*: Lasry & Lions '06, '07, Cardaliaguet '10, Gomes & Saude '14, Gomes, Pimentel, Sanchez '15, , Porretta '15, Bardi & Fischer '19, book by Achdou, Cardaliaguet, Delarue, Porretta, Santambrogio '20, Ducasse, Mazanti & Santambrogio '22,...
- *Numerical Analysis*: Achdou & Cappuzzo-Dolcetta '10, Achdou, Camilli & Cappuzzo-Dolcetta '13, Carlini & Silva '14, '15, Achdou & Poretta '15
  - finite difference / semi-Lagrangian methods;
  - convergence proofs without rates
- Also *stochastic* literature: Lacker '15, Carmona & Delarue '18, ...

Contemporary / recent works: Bonnans, Liu, Pfeiffer '23, Berry, Ley & Silva '25

$$-\partial_t u - \nu \Delta u + H(t,x,\nabla u) = F[m](t,x) \quad \text{in } (0,T)\times\Omega, \tag{HJB}$$

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In many applications, H is not C^1 / differentiable: discrete control sets, *bang-bang* controls,...

Example. Minimal exit time problems / eikonal:

$$H(t,x,p) = |p|$$

Closely related to the issue of possible nonuniqueness of the optimal feedback controls

$$\operatorname{argmax}_{\alpha\in\mathcal{A}} \left[b(t,x,\alpha)\cdot\nabla u - f(t,x,\alpha) \right] \subset \mathcal{A}$$

If optimal controls are nonunique, which controls do players choose?

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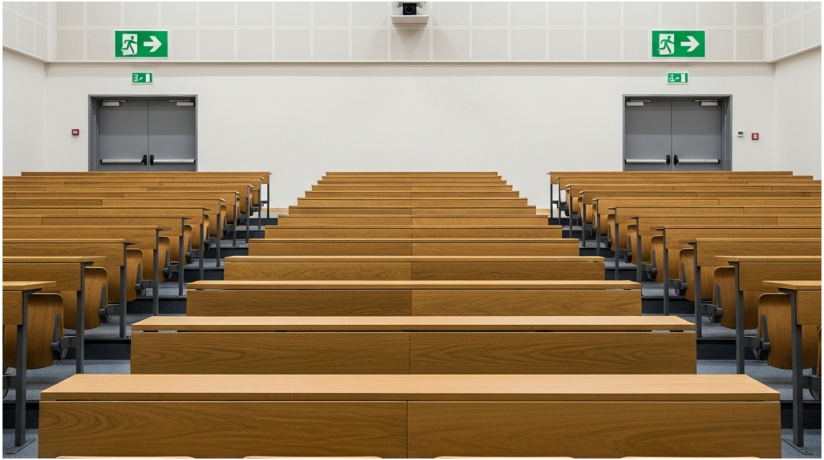

How do players choose among nonunique optimal controls?

The underlying controlled stochastic process for each player is of the form

$$\mathrm{d}x(t) = -b\left(t, x(t), \alpha\right)\mathrm{d}t + \sqrt{2\nu}\,\mathrm{d}B(t)$$

Players at location (t, x) use optimal feedback controls in the set

$$\Lambda(t,x) := \operatorname{argmax}_{\alpha\in\mathcal{A}}\left[b(t,x,\alpha)\cdot\nabla u(t,x) - f(t,x,\alpha)\right]$$

Describe the player choices of controls by a probability measure $\rho_{(t,x)} \in \mathcal{P}(\mathcal{A})$

$$\operatorname{supp}\rho_{t,x} \subset \Lambda(t,x), \qquad \int_{\mathcal{A}} \rho_{(t,x)}(\mathrm{d}\alpha) = 1.$$

The *effective mean drift* for the density m is

$$\tilde{b}_*(t,x) := \int_{\mathcal{A}} b(t,x,\alpha)\rho_{t,x}(\mathrm{d}\alpha) \in \operatorname{conv}\{b(t,x,\alpha) : \alpha \in \Lambda(t,x)\}$$

KFP equation becomes:

$$\partial_t m - \nu\Delta m - \operatorname{div}\left(\tilde{b}_* m\right) = G$$

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Proposition

If $\mathcal{A}$ is a compact metric space, and if $(b, f) : [0, T] \times \overline{\Omega} \times \mathcal{A} \to \mathbb{R}^d \times \mathbb{R}$ continuous, then

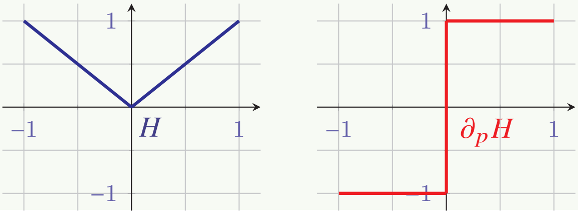
$$\operatorname{conv}\{b(t, x, \alpha) : \alpha \in \Lambda(t, x)\} = \partial_p H(t, x, \nabla u),$$

where $\partial_p H$ is the subdifferential of H defined by

$$\partial_p H(t, x, p) := \left\{\tilde{b} \in \mathbb{R}^d : H(t, x, q) \geq H(t, x, p) + \tilde{b} \cdot (q - p)\ \forall q \in \mathbb{R}^d\right\}.$$

Example

$$H(t, x, p) = |p| = \sup_{\alpha \in \overline{B_1(0)}} [\alpha \cdot p]$$



MFG Partial Differential Inclusion

Therefore we obtain formally the Fokker–Planck *partial differential inclusion* (PDI)

$$\begin{cases} \partial_t m - \nu\Delta m - \operatorname{div}\left(\tilde{b}_* m\right) = G \\ \tilde{b}_*(t,x) \in \partial_p H(t,x,\nabla u) \end{cases} \iff \partial_t m - \nu\Delta m - G \in \operatorname{div}\left(m\partial_p H(t,x,\nabla u)\right)$$

We propose the PDI system

$$\begin{aligned} -\partial_t u - \nu\Delta u + H(t,x,\nabla u) &= F[m](t,x) && \text{in } (0,T)\times\Omega, \\ \partial_t m - \nu\Delta m - G &\in \operatorname{div}\left(m\partial_p H(t,x,\nabla u)\right) && \text{in } (0,T)\times\Omega, \\ m(0,x) = m_0(x), \quad u(T,x) &= S[m(T,\cdot)](x) && \text{on } \Omega, \\ m = 0, \qquad u &= 0 && \text{on } (0,T)\times\partial\Omega, \end{aligned}$$

- Coincides with usual PDE when H is differentiable:
- In general finding the *effective drift* term of the subdifferential becomes part of the problem to be solved!

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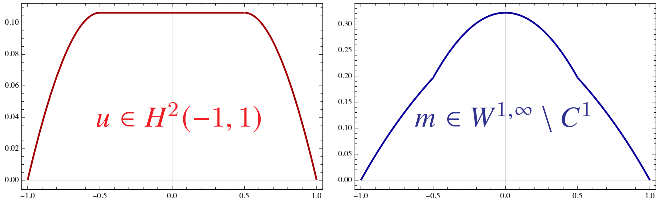
Example (Osborne & S., SINUM 2024)

Steady-state MFG PDI with

$$\Omega = (-1,1), \quad H(p) = |p| = \sup_{\alpha\in\{-1,1\}} \{\alpha p\}, \quad F[m] = m - F_0.$$

We can choose the problem data F_0, G, etc, s.t.

- Solution (u,m) is unique and can be solved analytically (plots below)



- Note $\partial_x u = 0$ on $(-1/2, 1/2)$ and H is nondifferentiable at 0.
- Drift $\tilde{b}_* \in L^\infty \setminus C^0$ is unique in this example and satisfies $\tilde{b}_* = 0$ on $(-1/2,1/2)$: i.e. $\tilde{b}_* \notin \mathcal{A}$.

Outline

Basic assumptions for the analysis

- $\Omega \subset \mathbb{R}^d$ bounded, open, connected, with Lipschitz boundary $\partial\Omega$,
- Initial data $m_0 \in L^2(\Omega)$, source term $G \in L^2(0,T;H^{-1}(\Omega))$
- Diffusion $\nu > 0$ constant.
- Coupling term $F : L^2(0,T;L^2(\Omega)) \to L^2(0,T;H^{-1}(\Omega))$ continuous operator with at most linear growth:

$$\|F[m]\|_{L^2(H^{-1})} \leq C\left(1 + \|m\|_{L^2(L^2)}\right).$$

- Final time coupling $S : L^2(\Omega) \to L^2(\Omega)$ is continuous and linear growth.

Note: not assuming that F or S are regularizing; e.g. $F[m] = m$ and $S[m] = m$ are allowed.

- Since $\mathcal{A}$ is compact and (b, f) continuous, it follows that H is real-valued, convex and Lipschitz in p: for all $(t,x) \in [0,T] \times \overline{\Omega}$,

$$|H(t,x,p_1) - H(t,x,p_2)| \le L_H |p_1 - p_2| \quad \forall p_1, p_2 \in \mathbb{R}^d; \qquad L_H = \|b\|_{C([0,T]\times\overline{\Omega}\times\mathcal{A};\mathbb{R}^d)}$$

- For all (t,x,p), the subdifferential $\partial_p H(t,x,p)$ is nonempty, closed, convex and contained in the closed ball $\overline{B(0,L_H)}$ in $\mathbb{R}^d$.
- Define the set-valued map $\mathcal{D}_p H : L^2(0,T;H_0^1(\Omega)) \rightrightarrows L^\infty((0,T)\times\Omega;\mathbb{R}^d)$ defined by

$$\mathcal{D}_p H[v] := \left\{\tilde{b} \in L^\infty((0,T)\times\Omega;\mathbb{R}^d) : \tilde{b}(t,x) \in \partial_p H(t,x,\nabla v(t,x)) \text{ for a.e. } (t,x) \in (0,T)\times\Omega\right\}$$

for each $v \in L^2(0,T;H_0^1(\Omega))$.
- By a measurable selection theorem (Kuratowski–Ryll-Nardzewski): for any $v \in L^2(0,T;H_0^1(\Omega))$, the set $\mathcal{D}_p H[v]$ is nonempty, convex and contained in

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Weak formulation: seek $u \in L^2(0,T;H_0^1(\Omega))$ and $m \in L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ s.t.

- Initial condition $m(0,\cdot) = m_0(\cdot)$ in Ω;
- there exists a $\tilde{b}_* \in \mathcal{D}_p H[u]$ s.t.

$$\int_0^T \left[\langle \partial_t \psi, u\rangle_{H^{-1}\times H_0^1} + \nu(\nabla u, \nabla\psi)_\Omega + (H[\nabla u], \psi)_\Omega\right] \mathrm{d}t = \int_0^T \langle F[m], \psi\rangle_{H^{-1}\times H_0^1} \mathrm{d}t + (S[m(T,\cdot)], \psi(T))_\Omega$$

$$\int_0^T \left[\langle \partial_t m, \phi\rangle_{H^{-1}\times H_0^1} + \nu(\nabla m, \nabla\phi)_\Omega + (m\tilde{b}_*, \nabla\phi)_\Omega\right] \mathrm{d}t = \int_0^T \langle G(t), \phi\rangle_{H^{-1}\times H_0^1} \mathrm{d}t,$$

for all test functions $\psi \in L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$, with $\psi(0) = 0$, and all $\phi \in L^2(0,T;H_0^1(\Omega))$.

Note: final time condition $u(T,\cdot) = S[m(T,\cdot)]$ appears as natural condition.

Existence Theorem [Osborne & S., Num. Math. 2025] + [Osborne, PhD Thesis 2024]

Under above assumptions, there exists a weak solution (u, m) of the MFG PDI.

Look for fixed point of the **set-valued map**

$$\mathcal{V} : \tilde{b} \mapsto \underbrace{\tilde{m} := FP[\tilde{b}]}_{\text{solves FP for } \tilde{b}} \mapsto \underbrace{\tilde{u} := HJB[\tilde{m}]}_{\text{solves HJB with data } F[\tilde{m}] \text{ and } S[\tilde{m}(T)]} \mapsto \mathcal{D}_p H[\tilde{u}].$$

$$(u, m) \text{ is a solution of MFG PDI} \iff \tilde{b}_* \in \mathcal{V}[\tilde{b}_*], \quad m = FP[\tilde{b}_*], \quad u = HJB[m].$$

Apply Kakutani–Fan–Glicksberg Fixed Point Theorem (1952) by verifying that:

1. $\mathcal{V}$ maps $\mathcal{B} = \{\tilde{b} : \|\tilde{b}\|_\infty \leq L_H\}$ to subsets of $\mathcal{B}$ (viewed as a compact convex set in weak-$*$ topology);
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In general MFG can have nonunique solutions. Uniqueness holds only under additional structural assumptions, e.g. Lasry–Lions monotonicity conditions.

Uniqueness Theorem [Osborne & S., Num. Math. 2025]

Assume that

- Initial density $m_0 \geq 0$ in Ω. Source term G is nonnegative in sense of distributions.
- Coupling F is strictly monotone: for all $w, v \in L^2(0,T;H_0^1(\Omega))$:

$$\int_0^T \langle F[w] - F[v], w - v \rangle_{H^{-1}\times H_0^1} \mathrm{d}t \leq 0 \quad \Longrightarrow \quad w = v.$$

- Final time cost is monotone

$$(S[w] - S[v], w - v)_\Omega \geq 0 \quad \forall w, v \in L^2(\Omega).$$

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Then the solution (u, m) is unique.

Lasry–Lions argument: assume (u_1, m_1) and (u_2, m_2) are both solutions,

- test the HJB equation with $\psi = m_1 - m_2$;
- test the KFP equation with $\phi = u_1 - u_2$;

take difference of the two equations:

$$\int_0^T \langle F[m_1] - F[m_2], m_1 - m_2 \rangle \, \mathrm{d}t + (S[m_1(T)] - S[m_2(T)], m_1(T) - m_2(T))_\Omega$$
$$= \int_0^T \underbrace{\Big(m_1, H[\nabla u_1] + \widetilde{b}_1 \cdot \nabla(u_2 - u_1) - H[\nabla u_2]\Big)_\Omega}_{\leq 0} \mathrm{d}t$$
$$+ \int_0^T \underbrace{\Big(m_2, H[\nabla u_2] + \widetilde{b}_2 \cdot \nabla(u_1 - u_2) - H[\nabla u_1]\Big)_\Omega}_{\leq 0} \mathrm{d}t \leq 0$$

since $\widetilde{b}_i \in \mathcal{D}_p H[u_i]$ and $m_i \geq 0$ for $i \in \{1, 2\}$. Strict Monotonicity implies $m_1 = m_2$ and then uniqueness of solutions of HJB equations implies $u_1 = u_2$.

Outline

Fully discrete scheme for time-dependent MFG PDI in [Osborne & S. Numer. Math., 2025]

- Conforming P1 FEM in space on simplicial meshes under Xu–Zikatanov condition.
- Implicit Euler discretization in time (Backward Euler for KFP, Forward Euler for HJB)
- Mass-lumping in time-derivative terms
- Stabilization to enforce a discrete maximum principle and nonnegativity of the density approximations
- Discrete fully space-time coupled inclusion problem to be solved.

Summary of main results

- Existence / uniqueness of solutions of discrete problems under same conditions as the continuous analysis.
- Plain convergence $u_k \to u$ in $L^2(H^1)$ and $m_k \to m$ in $L^p(L^2)$ for all $p < \infty$, and $m_k \rightharpoonup m$ in $L^2(H^1)$ as mesh size $h_k \to 0$ and timestep $\tau_k \to 0$.

Consider the elliptic problem

$$\begin{aligned} -\nu\Delta u + H(x,\nabla u) &= F[m](x) && \text{in } \Omega,\\ -\nu\Delta m - G &\in \operatorname{div}\left(m\partial_p H(x,\nabla u)\right) && \text{in } \Omega,\\ u = 0 \quad \text{and} \quad m &= 0 && \text{on } \partial\Omega, \end{aligned}$$

Hypotheses on the data

- Ω bounded Lipschitz domain.
- $H(x,p) := \sup_{\alpha\in\mathcal{A}} [b(x,\alpha)\cdot p - f(x,\alpha)]$
- $F : L^2(\Omega) \to L^2(\Omega)$ Lipschitz continuous, and also strongly monotone: exists $c_F > 0$ s.t.

$$\langle F[w]-F[v], w-v\rangle_{H^{-1}\times H^1_0} \geq c_F \|w-v\|^2_\Omega \quad \forall w,v\in H^1_0(\Omega)$$

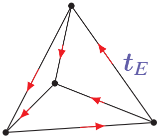
- Assume G is a nonnegative distribution with $G = g_0 - \nabla\cdot g_1$ with $g_0 \in L^{q/2}(\Omega)$ and $g_1 \in L^q(\Omega)$ for some $q > d$.

- Shape-regular sequence of conforming simplicial meshes $\mathcal{T}_k$ with mesh size $h_k \to 0$ as $k \to 0$. Assume Xu–Zikatanov condition.
- Corresponding P1 FEM spaces $V_k := \{v_k \in H_0^1(\Omega) : v_k|_K \in P_1(K) \quad \forall K \in \mathcal{T}_k\}$.
- FEM for MFG PDI: find $(u_k, m_k) \in V_k \times V_k$ such that there exists a $\tilde{b}_k \in D_p H[u_k]$ that satisfies

$$((\nu I + D_k)\nabla u_k, \nabla \psi_k)_\Omega + (H[\nabla u_k], \psi_k)_\Omega = (F[m_k], \psi_k)_\Omega \qquad \forall \psi_k \in V_k,$$
$$((\nu I + D_k)\nabla m_k, \nabla \phi_k)_\Omega + (m_k \tilde{b}_k, \nabla \phi_k)_\Omega = \langle G, \phi_k \rangle_{H^{-1}\times H_0^1} \quad \forall \phi_k \in V_k.$$

- Discrete Maximum Principle via stabilization, e.g. edge-weighted volume stabilization

$$D_k|_K := \sum_{E \in \mathcal{E}_K} \omega_{k,E} t_E \otimes t_E \quad \forall K \in \mathcal{T}_k,$$



where t_E chosen tangential vectors to (interior) edges $\mathcal{E}_K$ of each simplex element K, with weights $\omega_{k,E} \simeq L_H \operatorname{diam} E$.
- DMP ensures $m_k \geq 0$ in Ω if G is nonnegative.

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$$
\begin{aligned}
((\nu I + D_k)\nabla u_k, \nabla \psi_k)_\Omega + (H[\nabla u_k], \psi_k)_\Omega &= (F[m_k], \psi_k)_\Omega \qquad \forall \psi_k \in V_k, \\
((\nu I + D_k)\nabla m_k, \nabla \phi_k)_\Omega + (m_k \tilde{b}_k, \nabla \phi_k)_\Omega &= \langle G, \phi_k \rangle_{H^{-1}\times H_0^1} \quad \forall \phi_k \in V_k.
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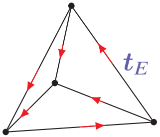


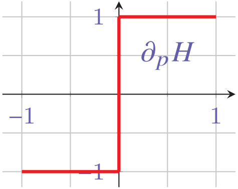
where $\boldsymbol{t}_E$ chosen tangential vectors to (interior) edges $\mathcal{E}_K$ of each simplex element K, with weights $\omega_{k,E} \simeq L_H \operatorname{diam} E$.
- DMP ensures $m_k \geq 0$ in Ω if G is nonnegative.

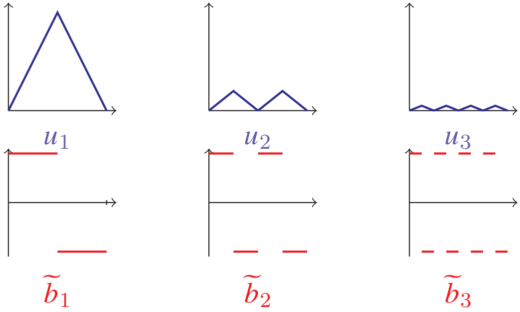
Problem: discontinuity of subdifferential:

$$\nabla u_k \to \nabla u \text{ in } L^2(\Omega;\mathbb{R}^d) \;\not\Longrightarrow\; \widetilde{b}_k \to \widetilde{b}_* \text{ in } L^p(\Omega;\mathbb{R}^d), \quad p \geq 1.$$

How to prove convergence rates if $\|\widetilde{b}_k - \widetilde{b}_*\| \not\to 0$?

Example





Elliptic regularity for Lipschitz domain Ω: there exists a $\gamma \in [1/2, 1]$ such that solutions of Poisson's equation with source term in L^2 have regularity in $H^{1+\gamma}(\Omega)$, see Jerison & Kenig 95.

Theorem: rate of convergence for MFG PDI [Osborne & S., Math. Comp. 2026]

There exists a $k_* \in \mathbb{N}$ such that

$$\|u - u_k\|_{H^1(\Omega)} + \|m - m_k\|_{L^2(\Omega)} \lesssim h_k^{\gamma/3} \quad \forall k \geq k_*$$

- Does not assume any higher regularity on the solution (u, m).
- Rate at least $h^{1/3}$ in convex domains, at least $h^{1/6}$ in general Lipschitz domains.
- *Quantitative convergence rates despite discontinuity in nonlinear inclusion terms!*

Strategy for the analysis

- Obtain rates for problems with $C^{1,1}$ Hamiltonians

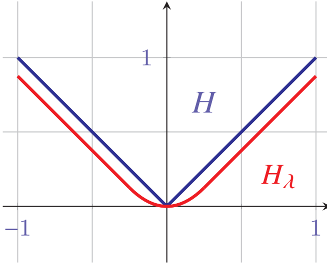
  [Osborne & S. Math. Comp. 2025]

- Regularize H by $C^{1,1}$ Hamiltonian and control regularization error

  [Osborne & S., SIAM J. Math. Anal. 2025]

- Combine the above

  [Osborne & S., Math. Comp. 2026]



Theorem: Asymptotic near quasi-optimality for $C^{1,1}$ Hamiltonians [Osborne & S., Math. Comp., 2025]

If H is $C^{1,1}$ w.r.t p:

$$\left|\frac{\partial H}{\partial p}(x,p) - \frac{\partial H}{\partial p}(x,q)\right| \leq L_{H_p}|p-q| \quad \forall (x,p,q) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^d,$$

then for $u,\ m$ with minimal regularity in $H^1(\Omega)$: for all k large enough,

$$\begin{aligned}\|m-m_k\|_{H^1(\Omega)} &+ \|u-u_k\|_{H^1(\Omega)} \\ &\leq C\left(\inf_{\overline{m}_k\in V_k}\|m-\overline{m}_k\|_{H^1(\Omega)} + \inf_{\overline{u}_k\in V_k}\|u-\overline{u}_k\|_{H^1(\Omega)} + h_k\|\nabla m\|_{L^2(\Omega)} + h_k\|\nabla u\|_{L^2(\Omega)}\right).\end{aligned}$$

for a constant C independent of k, m and u.

Therefore $\|u-u_k\|_{H^1} + \|m-m_k\|_{H^1} \lesssim h^s$ if u and m in $H^{1+s}(\Omega)$ for some $s \in [0,1]$.

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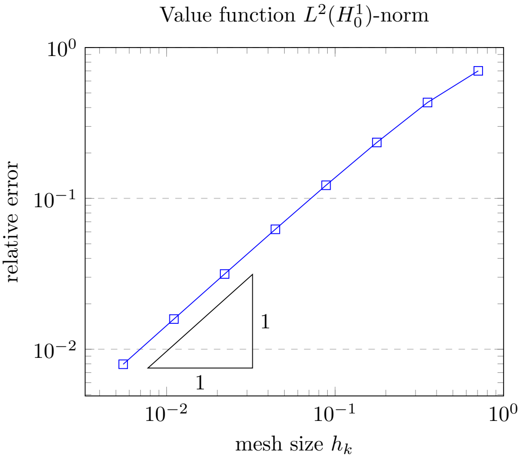
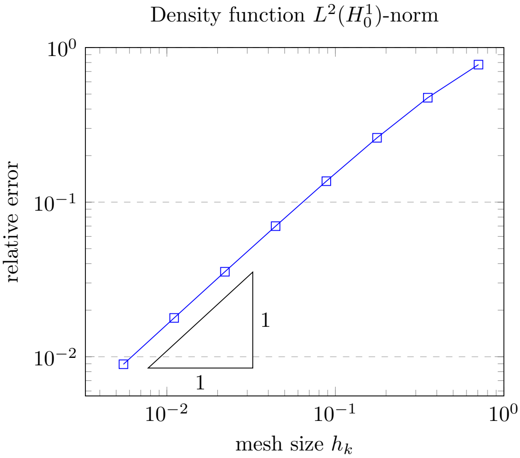
Outline

Parabolic problem on $(0,T)\times\Omega=(0,1)^3$; $\nu=1$, $H=|p|$, $F[m]=m+F_0$; $S[m]=\tanh(m)+S_0$, smooth (u,m), $\tau_k\approx h_k$.

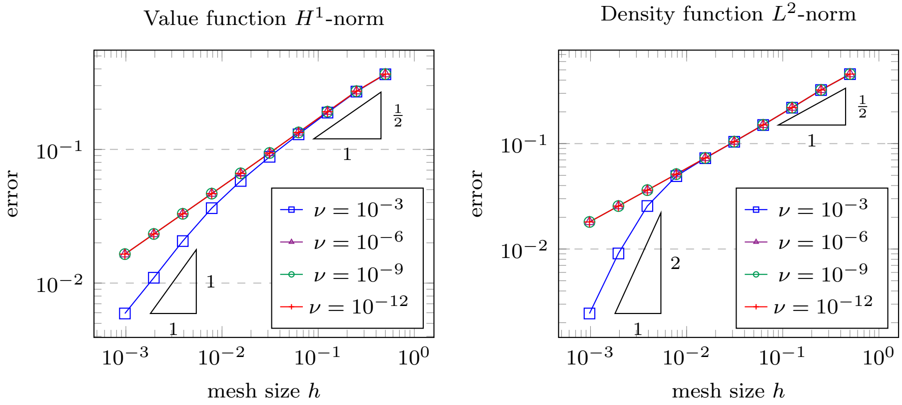




Computation on UCL Myriad cluster. Many more experiments in our papers+PhD Thesis of Yohance Osborne: vanishing ν, nonsmooth solutions; etc.

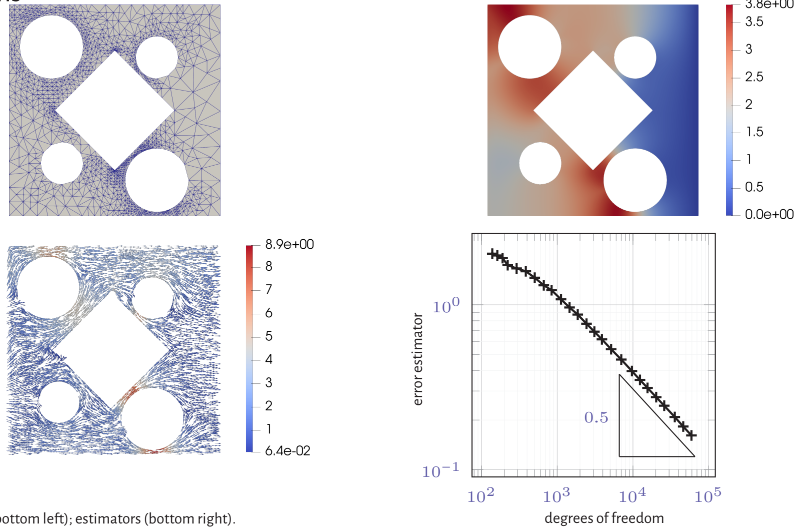
Numerical experiments

Elliptic problem in [Osborne & S., SINUM, 2024]: $\nu \to 0$ with $m = m_\nu \to m_* \notin H_0^1$ (boundary layers from Dirichlet condition).



Order $h^{1/2}$ typical for monotone / stabilized methods in the pre-asymptotic regime $\nu \ll h$.
This is optimal in this example since $\inf_{v_k \in V_k} \|m_* - v_k\|_{L^2} \gtrsim h_k^{1/2}$.

[Osborne, S. & Wells, IMA J. Numer. Anal. 2025]: a posteriori error bounds for $C^{1,1}$ Hamiltonians and adaptive computations



mesh (top left); density (top right); flux (bottom left); estimators (bottom right).

Conclusions

Summary:

- MFG PDI to handle nondifferentiable Hamiltonians & nonuniqueness of optimal controls;
- theorems on existence and uniqueness of solutions (steady-state + time-dependent);
- convergence of stabilized P1 FEM (steady-state + time-dependent);
- asymptotic near quasi-optimality of FEM for $C^{1,1}$ Hamiltonians;
- convergence rates of FEM for nondifferentiable Hamiltonians.

Additional work not shown today:

- A posteriori error analysis;
- *Fully nonlinear* MFG, with Tom Sales.

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- Osborne, S. & Wells, *A posteriori error bounds for finite element approximations of steady-state mean field games*, IMA Journal of Numerical Analysis, 2025.
- Sales & S., *Fully nonlinear second-order mean field games with nondifferentiable Hamiltonians*, arXiv preprint 2511.12780, 2025.

Thank you for your attention!