Adaptive finite element methods in unbounded domains

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Robust adaptivity for nonlinear partial differential equations
March 23–25, 2026
Inria Paris

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Introduction

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$$(\boldsymbol{A} \boldsymbol{\nabla} u_\ell, \boldsymbol{\nabla} v_\ell)_\Omega = (f, v_\ell)_\Omega$$

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Solved questions (at least, partially)

Can we robustly estimate the error $\|\boldsymbol{\nabla}(u - u_\ell)\|_\Omega$?
Is there a way to optimally design/produce the mesh $\mathcal{T}_\ell$?

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These inequalities can be tightened to "$\leq$ and $\lesssim_{\not p}$" using an equilibrated estimator.

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If there exists a sequence of meshes $\widehat{\mathcal{T}}_\ell$ reachable from $\mathcal{T}_0$ by finite NVBs, s.t.

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Optimal rate

If f is piecewise smooth and no anisotropic refinements are needed, then

$$\|\boldsymbol{\nabla}(u - u_\ell)\|_\Omega \leq C|\mathcal{T}_\ell|^{-p/3},$$

which is the rate expected for smooth solutions, even if u is not smooth in general.

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Questions

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We can then use the standard toolbox to approximate u^L by u^L_ℓ.

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Another issue is that it is hard to "balance" between ε and η in adaptive algorithms.

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This does implicitely introduce a truncation boundary Γ_ℓ.
However, we will never reference nor use the associated truncated PDE.
The truncation boundary Γ_ℓ does not need to be Lipschitz, and often is not.

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Main results in this talk

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Questions (with positive answers)

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If there exists a sequence of meshes $\widehat{\mathcal{T}}_\ell$ reachable from $\mathcal{T}_0$ by finite NVBs, s.t.

$$\|\boldsymbol{\nabla}(u - \widehat{u}_\ell)\|_\Omega \leq \widehat{C}|\widehat{\mathcal{T}}_\ell \setminus \mathcal{T}_0|^{-s}$$

with $s \geq 0$, then the sequence produced by the adaptive algorithm satisfies

$$\|\boldsymbol{\nabla}(u - u_\ell)\|_\Omega \leq C|\mathcal{T}_\ell \setminus \mathcal{T}_0|^{-s}.$$

Optimal rate

Under natural assumptions on f, if no anisotropic refinements are needed, then

$$\|\boldsymbol{\nabla}(u - u_\ell)\|_\Omega \leq C|\mathcal{T}_\ell \setminus \mathcal{T}_0|^{-p/3},$$

which is the rate expected for smooth solutions, even if u is not smooth in general.

Outline

Precise setting

Domain and coefficient

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We assume that f and $\boldsymbol{A}$ are piecewise constant. This will be made precise later.

Precise setting

Variational formulation

Sobolev spaces

For smooth compactly supported functions, the application

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The bilinear form $(\boldsymbol{A}\nabla\cdot, \nabla\cdot)_\Omega$ is naturally continuous and coercive over $\widehat{H}_0^1(\Omega)$.

If $v \in \widehat{H}_0^1(\Omega)$ we do have $\nabla v \in \boldsymbol{L}^2(\Omega)$ and $v|_\Gamma = 0$.
We also have $v \to 0$ as $|\boldsymbol{x}| \to +\infty$ in an appropriate.
However, we do not need to have $v \in L^2(\Omega)$, so that $\widehat{H}_0^1(\Omega) \neq H_0^1(\Omega)$ in general.

Hardy's inequality

The following "Poincaré–like" inequality is paramount.

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For all $v \in \widehat{H}^1_0(\mathbb{R}^3)$, we have

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Then, using Cauchy–Schwarz inequality, we have

$$\int_0^{+\infty} v(\partial_r v r) dr \leq \||\boldsymbol{x}|^{-1} v\|_{\mathbb{R}^3} \|\partial_r v\|_{\mathbb{R}^3}.$$

Variational formulation

Variational formulation

Assume that $|\boldsymbol{x}|f \in L^2(\Omega)$. Then, there exists a unique $u \in \widehat{H}^1_0(\Omega)$ such that

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The proof follows from Lax–Milgram lemma.
Coercivity and continuity of the LHS are obvious by definition.
Continuity of the RHS follows from Hardy's inequality.

Precise setting

Infinite meshes

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Let $\mathcal{T}_\ell$ be a collection of simplices and $\mathcal{T}_\ell^\dagger \subset \mathcal{T}_\ell$.

Shape regular mesh

We say that $\mathcal{T}_\ell \setminus \mathcal{T}_\ell^\dagger$ is a shape regular mesh with constant $\kappa \geq 1$
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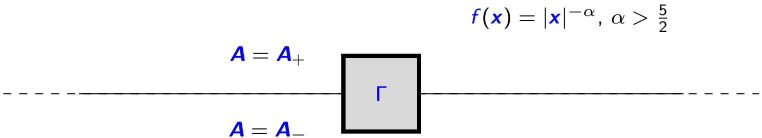
Throughout, we demand that f and $\boldsymbol{A}$ are constant on each $K \in \mathcal{T}_\ell$.

Precise setting

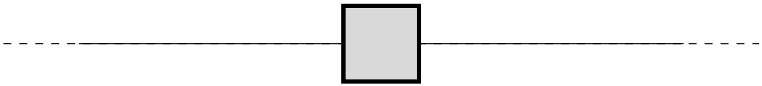
Example

Setting



$f(\boldsymbol{x}) = |\boldsymbol{x}|^{-\alpha}$, $\alpha > \frac{5}{2}$
$\boldsymbol{A} = \boldsymbol{A}_+$
Γ
$\boldsymbol{A} = \boldsymbol{A}_-$

Corresponding mesh



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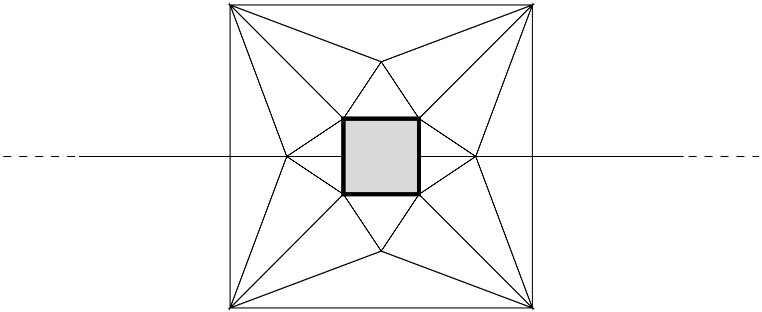

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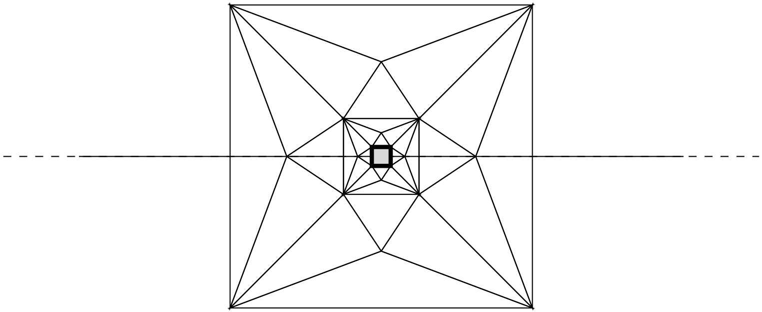

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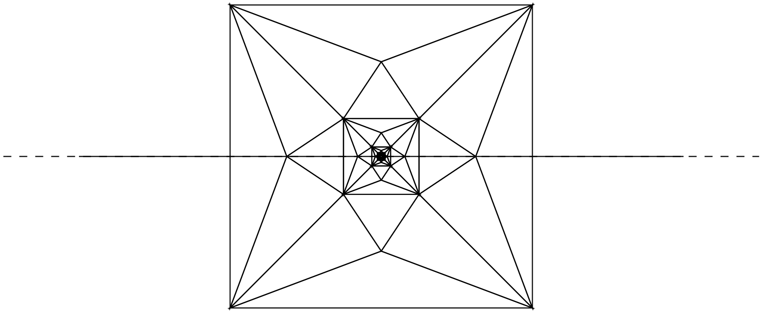
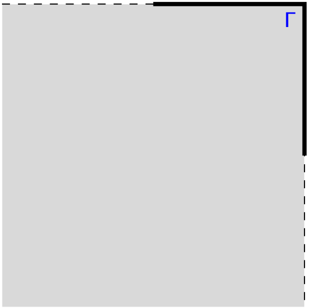


Precise setting

Another example

Setting



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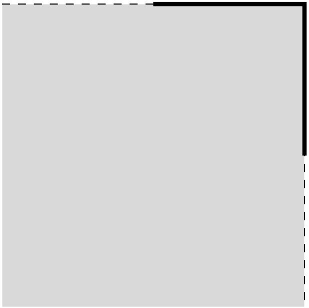

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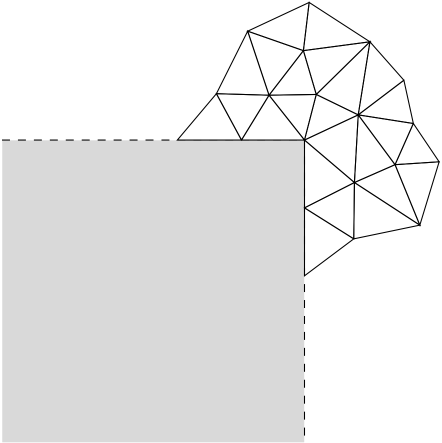

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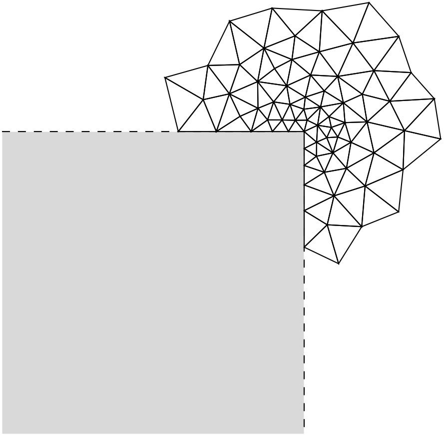

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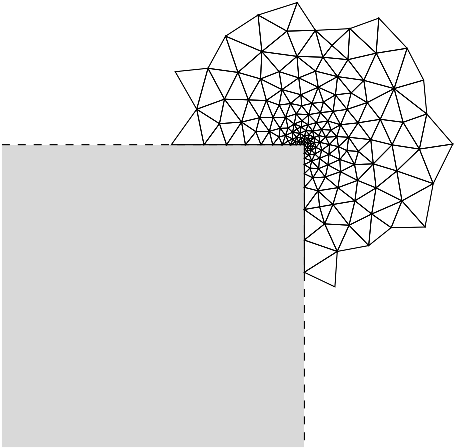

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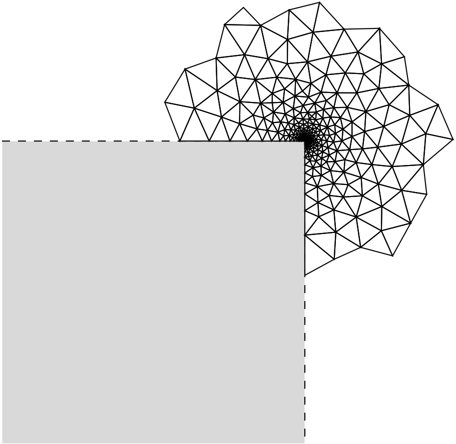

Residual based a posteriori estimator

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A posteriori error estimator

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A posteriori estimator

For all $K \in \mathcal{T}_\ell$ and $v_\ell \in \mathcal{P}_p(\mathcal{T}_\ell \setminus \mathcal{T}_\ell^\dagger) \cap H_0^1(\Omega)$, we introduce

$$\eta_\ell^2(v_\ell, K) := h_K^2 \|f + \boldsymbol{\nabla} \cdot (\boldsymbol{A} \boldsymbol{\nabla} v_\ell)\|_K^2 + h_K \| [\![\boldsymbol{A} \boldsymbol{\nabla} v_\ell]\!] \cdot \boldsymbol{n} \|_{\partial K \setminus \Gamma}^2.$$

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Here, are goal is to show the following properties.

Reliability and efficiency

$$\|\boldsymbol{\nabla}(u - u_\ell)\|_\Omega \lesssim \eta_\ell(u_\ell, \mathcal{T}_\ell) \qquad \eta_\ell(u_\ell, K) \lesssim_p \|\boldsymbol{\nabla}(u - u_\ell)\|_{\widetilde{K}}$$

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In fact, one can copy-paste the standard proof for the efficiency.
The interesting (and non-trivial) property is the reliability.

Residual based a posteriori estimator

Reliability

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The difficulty here is that J_ℓ maps into a submesh, not the whole mesh.
The solution hinge on
(a) a careful use of a Scott-Zhang type J_ℓ,
(b) the specific shape-regularity assumptions we made,
(c) Hardy's inequality.

Standard first step

For $v \in \widehat{H}^1_0(\Omega)$, using element-wise IBPs, we prove the following:

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&= (f-\boldsymbol{\nabla}_\ell\cdot(\boldsymbol{A}\boldsymbol{\nabla} u_\ell),v)_{\mathcal{T}_\ell} - \frac{1}{2}\langle [\![\boldsymbol{A}\boldsymbol{\nabla} u_\ell]\!]\cdot\boldsymbol{n},v\rangle_{\partial\mathcal{T}_\ell}
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$$\|\mathtt{h}^{-1}(v-J_\ell v)\|_\Omega + \|\boldsymbol{\nabla}(v-J_\ell v)\|_\Omega \lesssim \|\boldsymbol{\nabla} v\|_\Omega.$$

A nonstrandard second step

Quasi-interpolation estimate

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(a) On active elements K "far" from the truncation boundary, classicaly, we have

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(c) For the active elements touching Γ_ℓ, we combine the two above arguments.

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I now want to investigate the following adaptive algorithm.

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Given an initial mesh $\mathcal{T}_0$, for $\ell = 0, 1, \dots,$ do

1. Compute the discrete solution u_ℓ associated with the mesh $\mathcal{T}_\ell \setminus \mathcal{T}_0$.
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There are also concerns about wether NVB terminates for infinite meshes in Step 4.

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Let us first investigate Steps 2 and 3.

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Otherwise, we need the following (weak) extra assumption.

Extra load term decay

We assume that there exists $\varepsilon > 0$ such that

$$|\boldsymbol{x}|^{1+\varepsilon} f \in L^2(\Omega),$$

and that (an upper bound of) $\||\boldsymbol{x}|^{1+\varepsilon} f\|_\Omega$ is known.

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To do so, we observe that if $\mathcal{T}_\ell \setminus \mathcal{T}_0 \subset B_L$ for some $L \geq 0$, then

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As a result, we can find L large enough to exclude only negligeable elements far away.
That way, we can find a minimal set $\mathcal{M}_\ell$, which is sufficient for optimal convergence.

Newest vertex bisection

Our algorithm will always mark a finite set of elements $\mathcal{M}_\ell$.
However, these may lead to infinitely many refinements to enforce conformity.
In fact, we can build such counter examples, but there are “pathological”.

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All in all, Step 4 terminates with finitely many operations.

Adaptive algorithm

Optimal convergence

We now wish to establish the following result.

Optimal convergence

If there exists a sequence of meshes $\widehat{\mathcal{T}}_\ell$ reachable from $\mathcal{T}_0$ by finite NVBs, s.t.

$$\|\nabla(u - \widehat{u}_\ell)\|_\Omega \leq \widehat{C}|\widehat{\mathcal{T}}_\ell \setminus \mathcal{T}_0|^{-s}$$

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As it tunrs out, they "just work" for infinite meshes.

Axioms of adaptivity

We need to check the following "axioms".

Axioms of adaptivity

A1 Stability on non-refined elements:

$$|\eta_\ell(v_{\ell'}, \mathcal{T}_\ell \cap \mathcal{T}_{\ell'}) - \eta_\ell(v_\ell, \mathcal{T}_\ell \cap \mathcal{T}_{\ell'})| \lesssim \|\boldsymbol{\nabla}(v_{\ell'} - v_\ell)\|_\Omega.$$

A2 Reduction on refined elements: For some $q < 1$, we have

$$\eta_\ell(v_\ell, \mathcal{T}_{\ell+1} \setminus \mathcal{T}_\ell) \leq q\eta_\ell(v_\ell, \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}).$$

A3 Discrete reliability:
This is a stronger localized version of reliability.

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This is always satisfied for symmetric Lax–Milgram problems with nested spaces.

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Here, we are going to show the following.

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The next question to ask is how good is the best reachable mesh?

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We then obtain optimal rates under standard assumptions on the edges in Γ and $\boldsymbol{A}$.
These correspond to the absence of necessary anisotropic refinements.

Gist of the proof

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$$\|\nabla u\|_{B_L^c} \leq L^{-\varepsilon} \| |\boldsymbol{x}|^\varepsilon \nabla u\|_\Omega \lesssim L^{-\varepsilon}.$$

We pick $L = \tau^{-1/\varepsilon}$, so that

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We subdivide N times the elements of $\mathcal{T}_0 \cap B_L$. This is possible by NVB with $N = 2^n$.
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so that

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With $N = \tau^{-1/p}$, this solves the problem, since we then have

$$\|\boldsymbol{\nabla}(u - \widehat{I}_\ell u)\|_{\mathbb{R}^3} \lesssim \tau$$

and $|\widehat{\mathcal{T}}_\ell \setminus \mathcal{T}_0| \lesssim |\log \tau| \tau^{-3/p} \lesssim \tau^{-3/p-\nu}$ for any $\nu > 0$.

Numerical examples

Numerical examples

A quick discussion about 2D

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I will show 2D numerics though. I will only consider RHS with angular modes. For instance, for the first angular mode, the "fundamental" solution is

$$u(\boldsymbol{x}) = \frac{\sin(\theta)}{r},$$

which has similar decay properties that the 3D fundamental solution.

Numerical examples

Smooth solutions on structured meshes

Setting

We first consider $\Omega = \mathbb{R}^2 \setminus [-1,1]^2$, $\boldsymbol{A} = \boldsymbol{I}$, and

$$u(\boldsymbol{x}) = \chi(r)\frac{\sin(m\theta)}{r^m}$$

for $m = 1$ and 2, where χ is a smooth cutoff such that $u|_\Gamma = 0$.

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We employ the Dörfler parameter 0.3, meaning that we mark the element so that

$$\sum_{K \in \mathcal{M}_\ell} \eta_\ell^2(u_\ell, K) \geq 0.3^2 \sum_{K \in \mathcal{T}_\ell} \eta_\ell^2(u_\ell, K).$$

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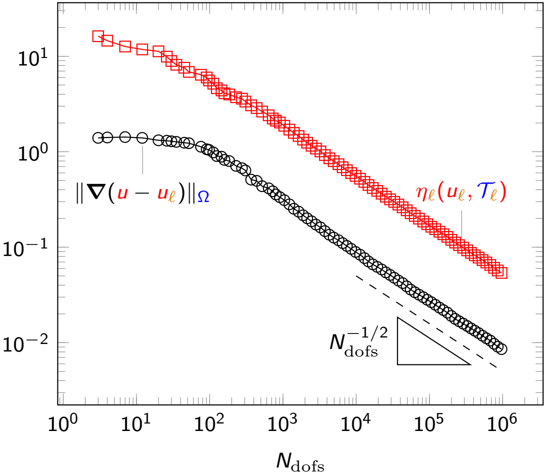
We consider $p = 1$, 2 and 3.

$m = 1$ and $p = 1$



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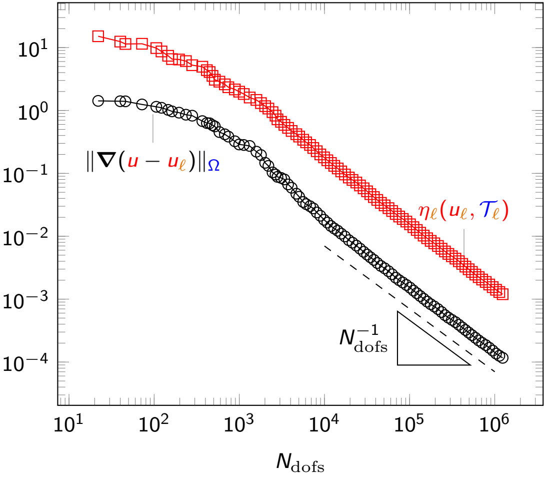
$\|\nabla(u - u_\ell)\|_\Omega$
$\eta_\ell(u_\ell, \mathcal{T}_\ell)$
$N_{\mathrm{dofs}}^{-1/2}$

$m = 1$ and $p = 2$



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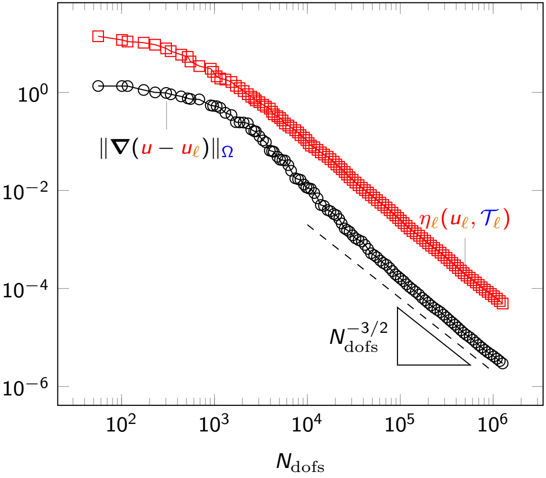

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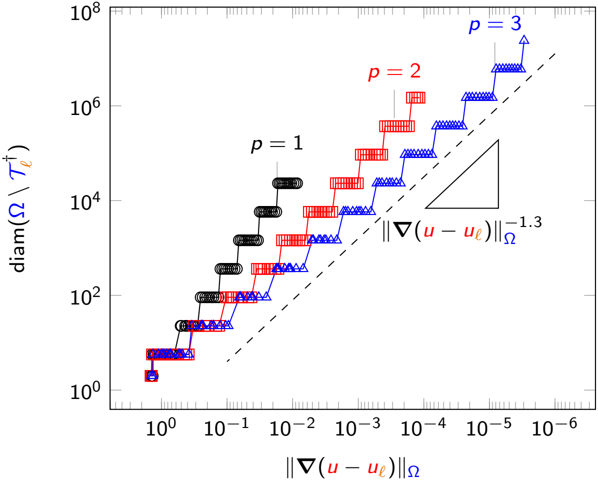
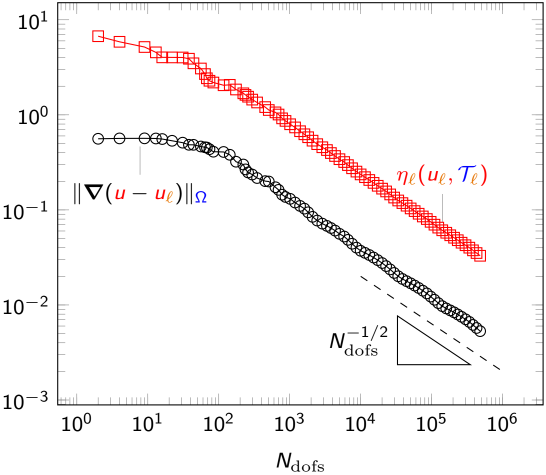
Increase in diameter for $m = 1$



$m = 2$ and $p = 1$



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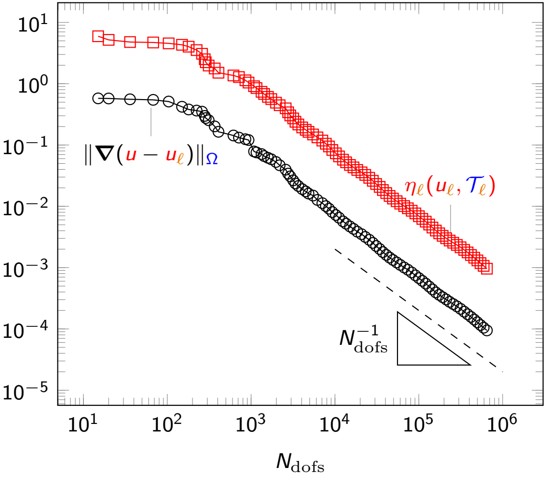

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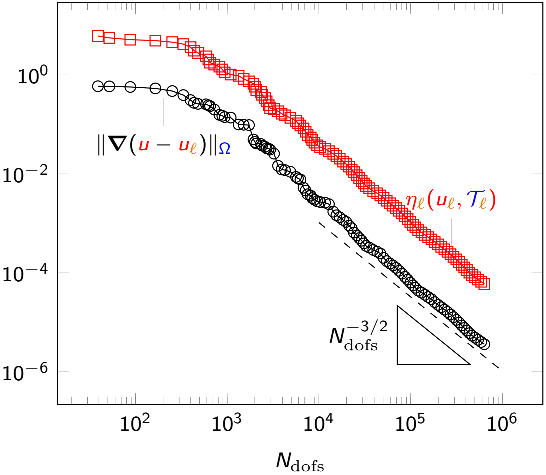

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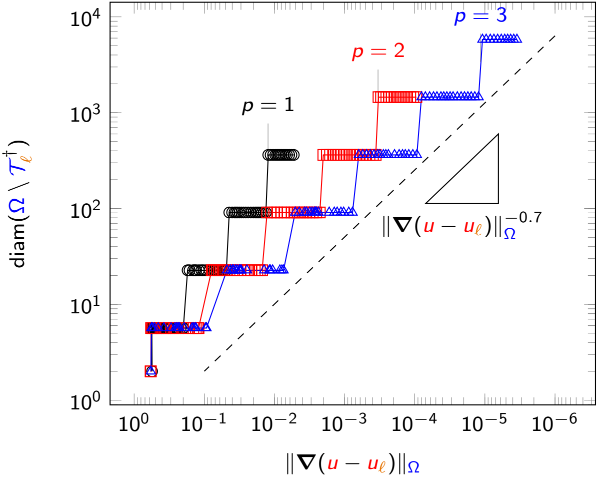
Increase in diameter for $m = 2$



Numerical examples

Corner singularities on structured meshes

Setting

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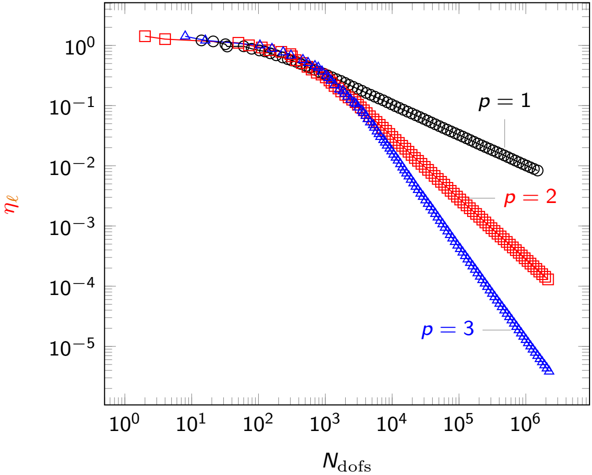
Convergence of the estimator



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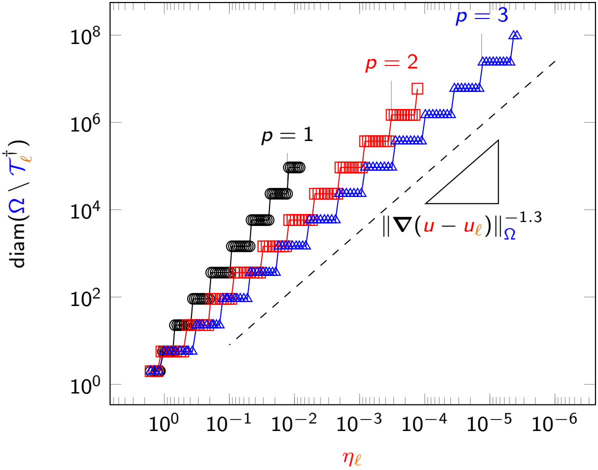

Numerical examples

Infinite L-shape with unstructured meshes

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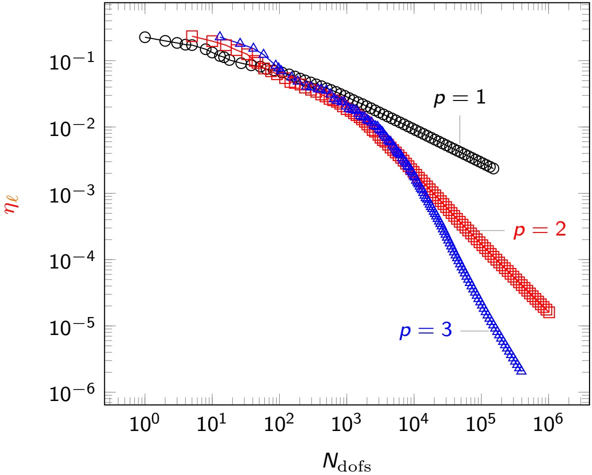
η_ℓ
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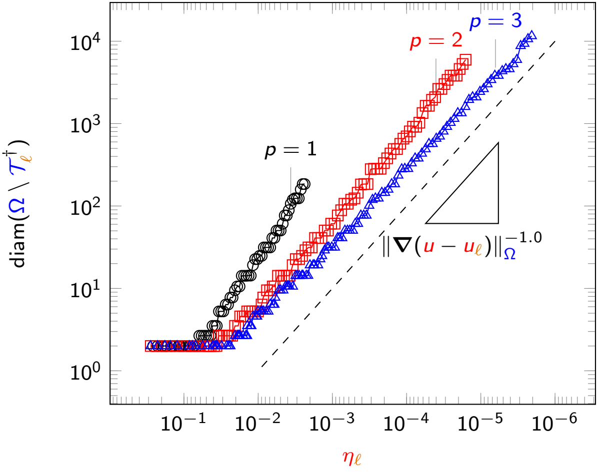

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This may be surprising at first. However, one can think of infinity as a point singularity.

Concluding remarks

The theory and implementation for AFEM naturally extend to unbounded domains.

The crucial point is to employ “growing” meshes towards infinity.

This may be surprising at first. However, one can think of infinity as a point singularity.

Related results in the simpler setting of reaction–diffusion equations are available:

T. Chaumont-Frelet, *arXiv*, 2024.

T. Chaumont-Frelet and G. Gantner, *arXiv*, 2025.