Reading group: Mathematical Control Theory by Eduardo D. Sontag

Alexandre Vieira

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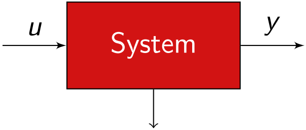
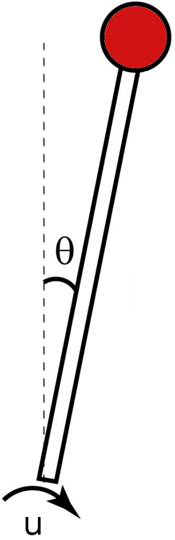
Control Theory



$$\dot{x} = Ax + Bu$$

Plenty application in engineering: automobile, microprocessors, robotics, airplanes, drones...

Toy example: Inverted Pendulum

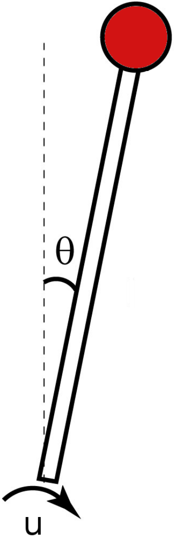


We assume small variations of θ, and we end up with equation :

$$\ddot{\theta} - \theta = u$$

How to bring, from any point $\theta(0)$ and $\dot{\theta}(0)$ small enough, the system to $(\theta, \dot{\theta}) = (0, 0)$?

Toy example: Inverted Pendulum

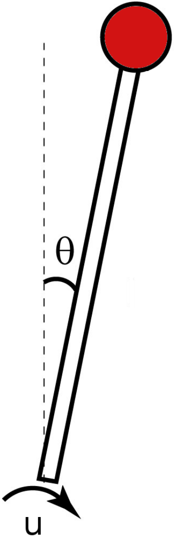


$$\ddot{\theta}(t) - \theta(t) = u(t) = 3e^{-2t}$$

with $\theta(0) = 1$ and $\dot{\theta}(0) = -2$: stabilizes the system to (0,0).

But if we add a little disturbance to the initial conditions: $\dot{\theta}(0) = -2 + \varepsilon$, then the system $\theta(t) \xrightarrow[t \to +\infty]{} +\infty$.

Toy example: Inverted Pendulum



$$\ddot{\theta} - \theta = u$$

If we choose a control proportional to the angle: $u = \alpha\theta$, then we will never approach (0,0). This can be shown for the whole system: left as an exercice!

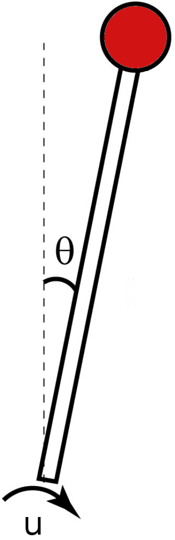
However, there exist positive scalars α and β such that the control

$$u = -\alpha\theta - \beta\dot{\theta}$$

stabilizes the system at (0,0).

Only problem : how to know $\theta(t)$ and $\dot{\theta}(t)$?

Toy example: Inverted Pendulum



$$\ddot{\theta} - \theta = u$$

Writing $x = (\theta, \dot{\theta})^\intercal$, we rewrite the system as:

$$\dot{x} = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{=A} x + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{=B} u$$

We apply a linear feedback $u = Kx$:

$$\dot{x} = (A + BK)x$$

How to choose K? Such that $A + BK$ has negative eigenvalues!

Description of a system

Definition

A **system** or **machine** consists of:

- A time set $\mathcal{T}$, which is a subgroup of $(\mathbb{R}, +)$
- A non empty-set $\mathcal{X}$ called the state-space
- A non empty-set $\mathcal{U}$ called a control-value space
- A map $\phi : \mathcal{D}_\phi \to \mathcal{X}$ called the transition map, which is defined on a subset $\mathcal{D}_\phi$ of

$$\left\{(\tau, \sigma, x, \omega) | \sigma, \tau \in \mathcal{T},\ \sigma \leq \tau,\ x \in \mathcal{X}, \omega \in \mathcal{U}^{[\sigma,\tau[}\right\}$$

Description of a linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \text{ for } t \in \mathbb{R}$$

For a given $u(\cdot) \in \mathbb{R}^m$ and $x(0) \in \mathbb{R}^n$, we have a unique solution:

$$x(t) = e^{At}x(0) + e^{At}\int_0^t e^{-As}Bu(s)ds$$

- Time set: $\mathcal{T} = \mathbb{R}$
- State space: $\mathcal{X} = \mathbb{R}^n$
- Control-value space: $\mathcal{U} = \mathbb{R}^m$
- Transition map: $\phi(\tau, \sigma, x, \omega) = e^{A(\tau-\sigma)}x + e^{A\tau}\int_\sigma^\tau e^{-As}B\omega(s)ds$

Description of a system

Definition

A **system** $\Sigma(\mathcal{T}, \mathcal{X}, \mathcal{U}, \phi)$ must comply with:

- Nontriviality: for all state $x \in \mathcal{X}$, there exist at least one pair $(\sigma, \tau) \in \mathcal{T}^2$, $\sigma < \tau$ and $\omega \in \mathcal{U}^{[\sigma,\tau[}$ such that ω is admissibile for x, i.e. $(\tau, \sigma, x, \omega) \in \mathcal{D}_\phi$.
- Restriction: If $\omega \in \mathcal{U}^{[\sigma,\mu[}$ is admissible for x, then for each $\tau \in [\sigma, \mu[$, $\omega_1 = \omega_{|[\sigma,\tau[}$ is also admissible for x and $\omega_{|[\tau,\mu[}$ is admissible for $\phi(\tau, \sigma, x, \omega_1)$.

Description of a system

Definition

A **system** $\Sigma(\mathcal{T}, \mathcal{X}, \mathcal{U}, \phi)$ must comply with:

- Semigroup: If σ, τ, μ are any three elements of $\mathcal{T}$ so that $\sigma < \tau < \mu$, if $\omega_1 \in \mathcal{U}^{[\sigma,\tau[}$ and $\omega_2 \in \mathcal{U}^{[\tau,\mu[}$, and if x is a state so that

$$\phi(\tau, \sigma, x, \omega_1) = x_1 \text{ and } \phi(\mu, \tau, x_1, \omega_2) = x_2$$

  then $\omega = \omega_1\omega_2$ is also admissible for x and $\phi(\mu, \tau, x, \omega) = x_2$.
- Identity: For each $\sigma \in \mathcal{T}$ and each $x \in \mathcal{X}$, the empty sequence $\diamond \in \mathcal{U}^{[\sigma,\sigma[}$ is admissible for x and $\phi(\sigma, \sigma, x, \diamond) = x$.

Output

Definition

A system with output is given by a system Σ together with:

- A set $\mathcal{Y}$ called the measurement-value or output-value space
- A map $h : \mathcal{T} \times \mathcal{X} \to \mathcal{Y}$ called the readout or measurement map

Trajectory

Definition

A trajectory Γ for the system Σ on the interval $\mathcal{I} \subseteq \mathcal{T}$ is a pair of function $(\xi, \omega) \in \mathcal{X}^{\mathcal{I}} \times \mathcal{U}^{\mathcal{I}}$ such that

$$\xi(\tau) = \phi(\tau, \sigma, \xi(\sigma), \omega_{|[\omega,\tau[})$$

holds for each pair $\sigma, \tau \in \mathcal{I}$, $\sigma < \tau$.
For a given ω, we call such ξ a path. We call $\xi(\sigma)$ the initial state, $\xi(\tau)$ the terminal state.

Definition

Σ is said complete if every input is admissible for every state:

$$\mathcal{D}_\phi = \left\{(\tau, \sigma, x, \omega) | \sigma, \tau \in \mathcal{T},\ \sigma \leq \tau,\ x \in \mathcal{X}, \omega \in \mathcal{U}^{[\sigma,\tau[}\right\}$$

Discrete systems

Definition

A system is said discrete if $\mathcal{T} = \mathbb{Z}$.

We define the transition mapping by

$$\begin{aligned} \mathcal{P} : \quad \mathcal{E} &\to \mathcal{X} \\ (t, x, u) &\mapsto \mathcal{P}(t, x, u) \end{aligned}$$

where $\mathcal{E}$ is a subset of $\mathbb{Z} \times \mathcal{X} \times \mathcal{U}$. For all t in $\mathbb{Z}$:

$$x(t+1) = \mathcal{P}(t, x(t), \omega(t))$$

Linear discrete systems

Definition

A discrete time system Σ is linear (over field $\mathbb{K}$) if:

- Σ is complete
- $\mathcal{X}$ and $\mathcal{U}$ are vector spaces
- $\mathcal{P}(t, \cdot, \cdot)$ is linear for each $t \in \mathbb{Z}$

If it has output in addition:

- $\mathcal{Y}$ is a vector space
- $h(t, \cdot)$ is linear for each $t \in \mathbb{Z}$.

The system is finite dimensional if $\mathcal{X}$ and $\mathcal{U}$ (and, if it exists, $\mathcal{Y}$) are finite dimensional. In that case, we call dimension of Σ the dimension of $\mathcal{X}$.

Linear discrete systems

If Σ is linear, there exists linear maps $A(t) : \mathcal{X} \to \mathcal{X}$, $B(t) : \mathcal{U} \to \mathcal{X}$ and $C(t) : \mathcal{X} \to \mathcal{Y}$ such that :

$$\mathcal{P}(t, x, u) = A(t)x + B(t)u$$

$$h(t, x) = C(t)x$$

Smooth discrete systems

We denote $\mathcal{E}_t = \{(x,u)|(t,x,u) \in \mathcal{E}\}$ the domain of $\mathcal{P}(t,\cdot,\cdot)$.

Definition

A discrete time system (over field $\mathbb{K}$) Σ is $\mathcal{C}^k$ if, for some intergers n and m:

- $\mathcal{X}$ is an open subset of $\mathbb{K}^n$
- $\mathcal{U}$ is an open subset of $\mathbb{K}^m$
- For each $t \in \mathbb{Z}$, the set $\mathcal{E}_t$ is open and the map $\mathcal{P}(t,\cdot,\cdot)$ is of class $\mathcal{C}^k$ there.

If it has output in addition:

- $\mathcal{Y}$ is an open subset of $\mathbb{K}^p$
- $h(t,\cdot)$ is of class $\mathcal{C}^k$.

Smooth discrete systems

Definition

Let Σ be a $\mathcal{C}^1$ discrete time system over $\mathbb{R}$, and assume that $\Gamma = (\overline{\xi}, \overline{\omega})$ is a trajectory on an interval $\mathcal{I}$. The linearization of Σ along Γ is the discrete-time linear system $\Sigma_*[\Gamma]$ with description $(\mathbb{R}^n, \mathbb{R}^m, \mathcal{P}_*)$ where :

$$\forall (x,u) \in \mathbb{R}^n \times \mathbb{R}^m,\ \mathcal{P}_*(t,x,u) = A(t)x + B(t)u$$

where $A(t) = \mathcal{P}_x(t, \overline{\xi}(t), \overline{\omega}(t))$ and $B(t) = \mathcal{P}_u(t, \overline{\xi}(t), \overline{\omega}(t))$ for all $t \in \mathcal{I}$ and $A(t) = B(t) = 0$ for all $t \notin \mathcal{I}$.
If Σ is a system with outputs, then the discrete-time linear system $\Sigma_*[\Gamma]$ admits as readout map:

$$h_*(t,x) = C(t)x$$

where $C(t) = h_x(t, \overline{\xi}(t))$ if $t \in \mathcal{I}$, $C(t) = 0$ otherwise.

Exercices

- Exercice 1.2.1, 1.4.2, 2.4.2 seem pretty cool!
- Another one to test if you got it well: start with the transition mapping of a linear discrete-time system

$$x(t+1) = A(t)x(t) + B(t)u(t)$$

and check if it actually defines a system (meaning, make sure that the transition map is:

$$\begin{aligned} x(\tau) &= \phi(\tau, \sigma, x(\sigma), u) \\ &= A^{(\tau-\sigma)}x(\sigma) + \sum_{k=\sigma}^{\tau-1} A^{k-\sigma}Bu(\tau-\sigma-1-k) \end{aligned}$$

and check if it complies with all the axioms in the definition of a system).