Reading group: Mathematical Control Theory by Eduardo D. Sontag

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Reminder

A **system** or **machine** consists of: a time set $\mathcal{T}$, $\mathcal{X}$ called the state-space, $\mathcal{U}$ called a control-value space, $\phi : \mathcal{D}_\phi \to \mathcal{X}$ called the transition map, which is defined on a subset $\mathcal{D}_\phi$ of

$$\left\{(\tau, \sigma, x, \omega) | \sigma, \tau \in \mathcal{T},\ \sigma \leq \tau,\ x \in \mathcal{X}, \omega \in \mathcal{U}^{[\sigma,\tau[}\right\}$$

and it complies with various properties (nontriviality, restriction, semigroup, identity).

Right-hand side

For continuous time system, we will describe the state's evolution through an ODE :

$$\dot{x}(t) = f(t, x(t), u(t))$$

Right-hand side

Definition

Let $\mathcal{X}$ be an open subset of $\mathbb{R}^n$, and let $\mathcal{U}$ be a metric space. A **right-hand side (rhs)** with respect to $\mathcal{X}$ and $\mathcal{U}$ is a function $f : \mathbb{R} \times \mathcal{X} \times \mathcal{U} \to \mathbb{R}^n$ which can be obtained in the following way: There must exist another metric space $\mathcal{S}$ as well as maps $\tilde{f} : \mathcal{S} \times \mathcal{U} \to \mathbb{R}^n$ and $\pi : \mathbb{R} \to \mathcal{S}$ so that

$$f(t, x, u) = \tilde{f}(\pi(t), x, u)$$

and the following properties hold:

1. $\tilde{f}(s, \cdot, u)$ is of class $\mathscr{C}^1$ for each fixed s, u
2. both $\tilde{f}$ and $\tilde{f}_x$ are continuous on $\mathcal{S} \times \mathcal{X} \times \mathcal{U}$
3. π is a measurable locally essentially bounded function.

Right-hand side

Lemma

Let f be a rhs. For any number $\sigma < \tau$, any measurable essentially bounded $\omega \in \mathcal{U}^{[\sigma,\tau[}$, and any $x^0 \in \mathcal{X}$, there is some nonempty subinterval $J \subseteq \mathcal{I} = [\sigma, \tau]$, open relative to $\mathcal{I}$ and containing σ, and there exists a unique solution of

$$\dot{\xi}(t) = f(t, \xi(t), \omega(t)) \tag{1}$$

$$\xi(\sigma) = x^0$$

on J.

If $J = \mathcal{I}$, then ω is said to be admissible for x^0.

Continuous time system

Definition

Let f be a rhs. We define a **continuous-time system** as the system $\Sigma_f = (\mathbb{R}, \mathcal{X}, \mathcal{U}, \phi)$, where ϕ is defined on

$$\mathcal{D} = \{(\tau, \sigma, x, \omega \,|\, \sigma < \tau,\ x \in \mathcal{X},\ \omega \in \mathcal{U}^{[\sigma,\tau[} \text{ is admissible for } x^0\}$$

by

$$\phi(\tau, \sigma, x, \omega) = \xi(\tau)$$

where $\xi(t)$ is the unique solution of (1) on $[\sigma, \tau]$.
A continuous-time system with outputs is a system Σ_f with outputs whose underlying system is a continuous-time system, $\mathcal{Y}$ is a metric space, and for which $h : \mathcal{T} \times \mathcal{X} \to \mathcal{Y}$ is continuous.

Continuous time system

From now on, we simply write Σ for Σ_f.

Definition

A continuous-time system of class $\mathscr{C}^k$ is one for which $\mathcal{U}$ is an open subset of $\mathbb{R}^m$ for some nonnegative integer m, and for which $\mathcal{S}$ and $\tilde{f}$ can be chosen so that $\mathcal{S}$ is an open subset of an Euclidian space and $\tilde{f}$ is of class $\mathscr{C}^k$.
If Σ is a system with outputs, it is required in addition that $\mathcal{Y}$ be an open subset of some Euclidian space $\mathbb{R}^p$ and that $h(t, \cdot)$ be of class $\mathscr{C}^k$ for each t.

Linear continuous-time system

Definition

A continuous-time system is **linear** (and finite dimensional on $\mathbb{R}$) if it is of class $\mathscr{C}^1$ and :

1. $\mathcal{U} = \mathbb{R}^m$ and $\mathcal{X} = \mathbb{R}^n$
2. $f(t, \cdot, \cdot)$ is linear for each $t \in \mathbb{R}$

If Σ is a system with outputs, it is required in addition that $\mathcal{Y} = \mathbb{R}^p$ and that $h(t, \cdot)$ be linear for each t.
We call **dimension of Σ** the dimension of $\mathcal{X}$.

Linear continuous-time system

Property

Every linear continuous-time system is complete (with respect to the class of all measurable essentially bounded controls).

Recall: Σ is said complete if every input is admissible for every state:

$$\mathcal{D}_\phi = \left\{ (\tau, \sigma, x, \omega) | \sigma, \tau \in \mathcal{T},\ \sigma \leq \tau,\ x \in \mathcal{X}, \omega \in \underbrace{\mathcal{U}^{[\sigma,\tau[}}_{L^\infty_{\mathcal{U}}} \right\}$$

Linearization

Definition

Let Σ be a $\mathcal{C}^1$ continuous-time system over $\mathbb{R}$, and assume that $\Gamma = (\overline{\xi}, \overline{\omega})$ is a trajectory on an interval $\mathcal{I}$. The linearization of Σ along Γ is the continuous-time linear system $\Sigma_*[\Gamma]$ with local-in-time description f_* where :

$$\forall (x,u) \in \mathbb{R}^n \times \mathbb{R}^m,\ f_*(t,x,u) = A(t)x + B(t)u$$

where $A(t) = f_x(t, \overline{\xi}(t), \overline{\omega}(t))$ and $B(t) = f_u(t, \overline{\xi}(t), \overline{\omega}(t))$ for all $t \in \mathcal{I}$ and $A(t) = B(t) = 0$ for all $t \notin \mathcal{I}$.
If Σ is a system with outputs, then the continuous-time linear system $\Sigma_*[\Gamma]$ admits as readout map:

$$h_*(t,x) = C(t)x$$

where $C(t) = h_x(t, \overline{\xi}(t))$ if $t \in \mathcal{I}$, $C(t) = 0$ otherwise.

Linearization always presented (for discrete and continuous-time case). But is it really useful ?
Suppose $(0, 0)$ is an equilibrium point of the smooth invariant system $\dot{x} = f(x, u)$. Then, for $(x, u) \in V(0, 0)$:

$$f(x, u) = f_x(0, 0)x + f_u(0, 0)u + h.o.t.$$

Linearization

Theorem

Let Σ be a (time-independant, continuous-time) $\mathscr{C}^1$ system, and pick any interval $[\sigma, \tau]$. Consider the map that gives the final state given an initial state and control

$$\alpha : \mathcal{D}_{\sigma,\tau} \to \mathcal{X} \ : \ (x, \omega) \mapsto \xi(\tau)$$

as well as the mapping describing the entire path

$$\psi : \mathcal{D}_{\sigma,\tau} \to \mathscr{C}_n^0 \ : \ (x, \omega) \mapsto \xi$$

where $\xi(t) = \phi(t, \sigma, x, \omega_{|[\sigma,t[})$.

Linearization

Theorem

Consider also (ξ, ω) any trajectory of this system, $\mu \in L_m^\infty$, and $\lambda : [\sigma, \tau] \to \mathbb{R}^n$ the solution of the following ODE

$$\dot{\lambda}(t) = A(t)\lambda(t) + B(t)\mu(t)$$

with initial condition $\lambda(\sigma) = \lambda_0 \in \mathbb{R}^n$, where

$$A(t) = f_x(\xi(t), \omega(t)), \; B(t) = f_u(\xi(t), \omega(t))$$

The following conclusions then hold :

- The set $\mathcal{D}_{\sigma,\tau}$ is an open subset of $\mathcal{X} \times L_{\mathcal{U}}^\infty$, and both ψ and α are continuous.

Linearization

Theorem

- α is of class $\mathcal{C}^1$, and

$$\frac{\partial \alpha}{\partial x}\Big|_{x,\omega} \lambda_0 + \frac{\partial \alpha}{\partial \omega}\Big|_{x,\omega} \mu = \lambda(\tau)$$

That is, $\alpha_*[x, \omega]$ is the same as the final state map corresponding to the linearization $\Sigma_*[\xi, \omega]$.
In particular, for systems of class $\mathscr{C}^1$, $\alpha(x, \cdot)$ has full rank at ω if and only if the linear map

$$\mu \mapsto \int_\sigma^\tau \Phi(\tau, s) B(s) \mu(s) ds$$

is onto, where $\Phi(\tau, \sigma)$ is the solution to the matrix ode $\dot{X}(t) = f_x(\xi(t), \omega(t)) X(t)$, $X(\sigma) = I$.

Linearization

Theorem

- Take any x and any ω admissible for x, and denote $\xi = \psi(x, \omega)$. Let $\{\omega_j\}_{j=1}^{\infty}$ be an equibounded sequence of controls, and $\lim x_j = x$. If either one of the following conditions hold :

  1. $\omega_j \rightarrow \omega$ pointwise almost everywhere
  2. $\omega_j \rightharpoonup \omega$ and Σ is affine in controls

  then $\xi_j = \psi(x_j, \omega_j)$ is defined for all large j and

$$\lim \|\xi_j - \xi\|_\infty = 0$$

Sampling

Definition

Let $\Sigma = (\mathbb{R}, \mathcal{X}, \mathcal{U}, \phi)$ be a continuous-time system, and pick any real number $\delta > 0$. The δ-**sampled system associated to** Σ is the discrete-time system $\Sigma_{[\delta]}$ defined as follows.
Let $\mathcal{E}_\delta$ be the subset of $\mathbb{Z} \times \mathcal{X} \times \mathcal{U}$ consisting of those triples (k, x, u) that satisfy that ω is admissible for x, where $\omega \in \mathcal{U}^{[k\delta,(k+1)\delta[}$ is defined by

$$\omega(t) \equiv u \text{ for all } t \in [k\delta, (k+1)\delta[$$

The state space of $\Sigma_{[\delta]}$ is then the set of all $x \in \mathcal{X}$ for which there is some k, ω with $(k, x, \omega) \in \mathcal{E}_\delta$ and the local-in-time dynamics are given on $\mathcal{E}_\delta$ by

$$P(k, x, u) = \phi((k+1)\delta, k\delta, x, \omega)$$

Basic reachability notions

Definition

An **event** is a pair $(x, t) \in \mathcal{X} \times \mathcal{T}$.

- (z, τ) **can be reached from** (x, σ) iff there is a path of Σ on $[\sigma, \tau]$ whose initial state is x and final state is z. One also says that (x, σ) **can be controlled to** (z, τ).
- If $x, z \in \mathcal{X}$, $T \geq 0 \in \mathcal{T}$, and there exist $\sigma, \tau \in \mathcal{T}$ with $\tau - \sigma = T$ such that (z, τ) can be reached from (x, σ), then z **can be reached from** x **in time** T. Equivalently, x **can be controlled to** z **in time** T.
- z **can be reached from** x (or x **can be controlled to** z) if this happens for at least one T.

Basic reachability notions

For each case, one writes:

- $(x, \sigma) \rightsquigarrow (z, \tau)$ for (x, σ) can be controlled to (z, τ)
- $x \underset{T}{\rightsquigarrow} z$ for x can be controlled to z in time T
- $x \rightsquigarrow z$ for x can be controlled to z

Various properties can be proved easily (see Lemma 3.1.2, and 3.1.5 for the linear case)

Basic reachability notions

Definition

The system Σ is **(completely) controllable on the interval** $[\sigma, \tau]$ if for each $x, z \in \mathcal{X}$ it holds that $(x, \sigma) \rightsquigarrow (z, \tau)$. It i **(completely) controllable in time** T if for each $x, z \in \mathcal{X}$, it holds that $x \underset{T}{\rightsquigarrow} z$. It is juste **(completely) controllable** is $x \rightsquigarrow z$ for all x, z.

Properties of reachability

Properties

Let Σ be a linear system and pick any $\sigma, \tau, T \in \mathcal{T}$.

- Σ is controllable on $[\sigma, \tau]$ iff $(0, \sigma) \rightsquigarrow (y, \tau)$ for all $y \in \mathcal{X}$
- If Σ is time-invariant, then Σ is controllable in time T iff $0 \underset{T}{\rightsquigarrow} y$ for all $y \in \mathcal{X}$
- If Σ is continuous-time, then it is controllable on $[\sigma, \tau]$ iff $(x, \sigma) \rightsquigarrow (0, \tau)$ for all $x \in \mathcal{X}$
- If Σ is time-invariant and continuous-time, then Σ is controllable in time T iff $x \underset{T}{\rightsquigarrow} 0$ for all $x \in \mathcal{X}$
- The conclusions of the two previous points still hold for discrete-time systems if $A(k)$ is invertible for all $k \in [\sigma, \tau]$.

List of exercices

Form 3 groups. Each group choose one of the following (list of) exercice(s), solve it, and one member write the solution on the blackboard.

1. Exercices 2.7.3 and 2.7.5 (p. 48) : rather theortical exercices on linear systems
2. Exercices 2.7.15 and 2.7.16 (p.52) : application on linearization
3. Exercice 2.10.2 (sampling of some examples) and Exercice 3.1.2/3.1.3