Reading group: Calculus of Variations and Optimal Control Theory by Daniel Liberzon

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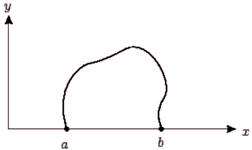
Dido's isoperimetric problem

How to build the biggest city with the hide of an ox?



This can be formulated as a maximization problem :

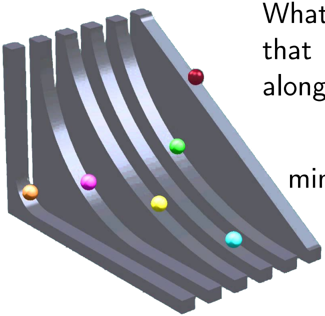
$$\text{maximize}_{y\in\mathscr{C}^1} \int_a^b y(x)dx$$
$$\text{such that } y(a) = y(b) = 0$$
$$\int_a^b \sqrt{1+(y'(x))^2}dx = C_0$$

Catenary problem

What equation describes a chain hanging between two fixed points ?

$$\begin{aligned}
\text{minimize}_{\, y \in \mathscr{C}^1} \quad & \int_a^b y(x)\sqrt{1+(y'(x))^2}\,dx \\
\text{such that} \quad & y(a) = y_a \\
& y(b) = y_b \\
& \int_a^b \sqrt{1+(y'(x))^2}\,dx = C_0
\end{aligned}$$

Brachistochrone problem



What path between two points is such that a particle sliding without friction along it takes the shortest time?

$$\text{minimize}_{y \in \mathscr{C}^1} \int_a^b \sqrt{\frac{1+(y'(x))^2}{y(x)}}dx$$

$$\text{such that } y(a) = y_a$$
$$y(b) = y_b$$

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Existence of minimum

We are usually interested not in local minima, but more on global minima. How could we be sure that they exist?

Theorem : Weierstrass Theorem

If f is a continuous functions and D is a compact set, then there exists a global minimum of f over D.

A compact set D is defined as a set for which every open cover has a finite subcover. Equivalently, D is compact if every sequence in D has a subsequence converging in D.
For a subset D of $\mathbb{R}^n$, compactness if easily defined: D must be closed and bounded.

First-order conditions

Consider $f : \mathbb{R}^n \to \mathbb{R}$ a $\mathscr{C}^1$-function, $D = dom\ f$ that we suppose open and non-empty.
Denote x^* a local minimum of f on D (which must be an interior point). For α small enough, $x^* + \alpha d \in D$.
Let us consider

$$g(\alpha) = f(x^* + \alpha d).$$

By construction, g is $\mathscr{C}^1$, so :

$$g(\alpha) = g(0) + \alpha g'(0) + o(\alpha)$$

First-order conditions

$$g(\alpha) = g(0) + \alpha g'(0) + o(\alpha)$$

We can easily prove that $g'(0) = 0$. For that, suppose that $g'(0) \neq 0$. Then, by the definition of $o(\alpha)$, there exists $\varepsilon > 0$ such that :

$$|\alpha| < \varepsilon,\ \alpha \neq 0 \implies |o(\alpha)| < |g'(0)\alpha|$$

And so :

$$g(\alpha) - g(0) < \alpha g'(0) + |\alpha g'(0)|$$

Restrict α to have the opposite sign to $g'(0)$, such that we have $g(\alpha) - g(0) < 0$ and so :

$$f(x^* + \alpha d) < f(x^*)$$

which contradicts the fact that x^* is a minimum. So $g'(0) = 0$.

First-order conditions

A simple application of the chain rule implies :

$$g'(0) = \nabla f(x^*) \cdot d$$

But since d is arbitrary, we conclude that :

$$\nabla f(x^*) = 0$$

Second-order conditions

Suppose now that f is $\mathscr{C}^2$. As before:

$$g(\alpha) = g(0) + \alpha g'(0) + \frac{1}{2}g''(0)\alpha^2 + o(\alpha^2)$$

We easily prove that $g''(0) \geq 0$. The same way we did for 1st order conditions, suppose $g''(0) < 0$. Then there exists $\varepsilon > 0$ such that :

$$|\alpha| < \varepsilon,\ \alpha \neq 0 \implies |o(\alpha^2)| < \frac{1}{2}|g''(0)\alpha^2|$$

And we prove exactely the same way that $g(\alpha) - g(0) < 0$, thus contradicting the fact that x^* is a local minimum.

Second-order conditions

Once again, using the chain rule:

$$g''(0) = d^\intercal \nabla^2 f(x^*) d \geq 0$$

and since d is arbitrary, $\nabla^2 f(x^*)$ must be positive semidefinite.

Unconstrained border conditions

We supposed so far that D is an open set. What if D is closed and $x^* \in \partial D$? There will be some d such that $x^* + \alpha d \notin D$ whatever α.

Definition: Feasible direction

We call a feasible direction at x^* a vector d such that $x^* + \alpha d \in D$ for small enough $\alpha > 0$.

Unconstrained border conditions

We can actually easily prove that we can replace the first and second-order conditions by: if x^* is a local minimum of f on D:

- if f is $\mathscr{C}^1$, then $\nabla f(x^*) \cdot d \geq 0$ for every feasible direction d
- if f is $\mathscr{C}^2$, then $d^\intercal \nabla^2 f(x^*) d \geq 0$ for all feasible directions satisfying $\nabla f(x^*) \cdot d = 0$.

Also, if D is convex, then every point of D is written $x^* + \alpha d$, so this kind of approach is clearly suitable for this kind of set.

Lagrange multipliers

We want now to minimize a function f on a set D defined by equalities:

$$h_1(x) = ... = h_m(x) = 0$$

Definition: regular point

x^* is called a regular point if $\nabla h_i(x^*)$, $i = 1, ..., m$ are linearly independant.

As we discussed before, the method used with feasible direction is not necessarily suitable for constrained cases. So here, we would rather use *feasible curves*.

Lagrange multipliers

We consider a curve $x(\alpha) \in D$, supposed $\mathscr{C}^1$, with $x(0) = x^*$. Given an aribtrary curve of this kind, we can consider

$$g(\alpha) = f(x(\alpha))$$

For $\alpha = 0$, we can prove:

$$g'(0) = \nabla f(x^*)x'(0) = 0$$

This vector $x'(0)$ is a tangent vector to D at x^*. It lives in the tangent space to D at x^*, which is denoted by $T_{x^*}D$.

From this, we can prove that if x^* is a minimum of f in D, then there exists $\lambda \in \mathbb{R}^m$ such that:

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla h_i(x^*) = 0$$

This is a really geometrical point of view. But we will prove that in a more algebraic way.

Lagrange multipliers

Suppose that $m = 1$. Given two vectors d_1, d_2 arbitrary, consider :

$$F : (\alpha_1, \alpha_2) \mapsto (f(x^* + \alpha_1 d_1 + \alpha_2 d_2), h(x^* + \alpha_1 d_1 + \alpha_2 d_2))$$

The Jacobian matrix of F at $(0, 0)$ is :

$$J_F(0,0) = \begin{pmatrix} \nabla f(x^*) d_1 & \nabla f(x^*) d_2 \\ \nabla h(x^*) d_1 & \nabla h(x^*) d_2 \end{pmatrix}$$

Theorem: Inverse Function Theorem

If the total derivative of a continuously differentiable function F defined from an open set of $\mathbb{R}^n$ into $\mathbb{R}^n$ is invertible at a point p (i.e., the Jacobian determinant of F at p is non-zero), then F is an invertible function near p. Moreover, the inverse function F^{-1} is also continuously differentiable.

Lagrange multipliers

If $J_F(0,0)$ is non singular, then by the Inverse Function Theorem, there are neighbourhoods of $(0,0)$ and $F(0,0) = (f(x^*), 0)$ on which F is a bijection.
That implies that there is a point $x = x^* + \alpha_1 d_1 + \alpha_2 d_2$, with (α_1, α_2) close to $(0,0)$, such that $h(x) = 0$ and $f(x) < f(x^*)$.
This cannot be true since x^* is a minimum.

So $J_F(0,0)$ must be singular.

Lagrange multipliers

$$J_F(0,0) = \begin{pmatrix} \nabla f(x^*)d_1 & \nabla f(x^*)d_2 \\ \nabla h(x^*)d_1 & \nabla h(x^*)d_2 \end{pmatrix}$$

We suppose that x^* is a regular point. In this case, it means that $\nabla h(x^*) \neq 0$. Choose d_1 such that $\nabla h(x^*)d_1 \neq 0$. Let

$$\lambda = -\frac{\nabla f(x^*)d_1}{\nabla h(x^*)d_1}$$

Since $J_F(0,0)$ is singular, its first row must be a constant multiple of its second row. Thus :

$$\nabla f(x^*)d_2 + \lambda \nabla h(x)d_2 = 0, \ \forall d_2 \in \mathbb{R}^n$$

Since d_2 is arbitrary, we have the well know equation:

$$\nabla f(x^*) + \lambda \nabla h(x) = 0$$

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Can we use this in an infinite-dimensional framework?

Consider now

$$J : V \to \mathbb{R}$$

V : infinite-dimensional vector space (usually, a function space).
J is called a functional.
Main difference with finite-dimensional case: no "generic" space.

There can be several V where the problem may be well defined, and choosing this V may be part of the problem.

Can we use this in an infinite-dimensional framework?

In finite-dimensional cases, all norms are equivalent. It is not the case for function spaces.

- On $\mathscr{C}^0([a,b],\mathbb{R}^n)$, we can use:

$$\|y\|_0 = \max_{a \leq x \leq b} |y(x)|$$

- On $\mathscr{C}^1([a,b],\mathbb{R}^n)$, we can use:

$$\|y\|_1 = \max_{a \leq x \leq b} |y(x)| + \max_{a \leq x \leq b} |y'(x)|$$

- There may be other norms (L^p,...)

Once this norm is chosen, we can clearly define a local optimum for the functional J over V.

First variation

Also, what do we call a derivative of a functional?

Definition: First Variation

For a function $y \in V$, we call the first variation of J at y, the linear functional $\delta J|_y : V \to \mathbb{R}$ satisfying, for all η and all α:

$$J(y + \alpha\eta) = J(y) + \delta J|_y(\eta)\alpha + o(\alpha)$$

First-order condition

Can we the same way, define a first-order condition? Suppose we want to find a local minimum of a functional J over a subset A of V.

Definition: Admissible perturbation

We call a perturbation $\eta \in V$ *admissible* if $y^* + \alpha\eta \in A$ for all α close enough to 0.

Reasonning the same way as before, we prove that if y^* is a local minimum, then for all admissible perturbations η, we must have

$$\delta J_{|y}(\eta) = 0$$

Problem : how do we compute $\delta J_{|y}$?

Second variation

Definition: Bilinear functional and quadratic form

$B : V \times V \to \mathbb{R}$ is bilinear if it is linear in each argument.
$Q(y) = B(y, y)$ is called a quadratic form on V.

Definition: Second variation

For a function $y \in V$, we call the second variation of J at y, the quadratic form $\delta^2 J|_y : V \to \mathbb{R}$ satisfying, for all η and all α:

$$J(y + \alpha\eta) = J(y) + \delta J|_y(\eta)\alpha + \delta^2 J|_y(\eta)\alpha^2 + o(\alpha^2)$$

Second-order condition

We still have that, if y^* is a local minimum of J over A, then for all admissible perturbations η, we have

$$\delta^2 J|_y(\eta) \geq 0$$

But the condition here is not sufficient! In the finite dimensional-case, sufficiency is proved by the fact that the unit ball is compact (and then using Weierstrass Theorem). This is not the case in infinite-dimensional case!

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Exercices

- page 6-7: prove the necessary (and sufficient) conditions in the case where $x^* \in \partial D$ (suppose D convex).
- page 15: Prove the first order condition in the constrained case with $m > 1$.
- Exercise 1.2 : have a look at the following constrained problem :

$$\begin{aligned} &\min x_1 + x_2 \\ \text{s.t. } &(x_1 + 2)^2 + x_2^2 = 2 \\ &(x_1 - 2)^2 + x_2^2 = 2 \end{aligned}$$

  Find the minimum solution (you can prove before that it exists!). Is it a regular point? Have a look also on the KKT conditions (Lagrange multipliers).
- Exercise 1.5 and 1.6 (first and second variations)