Duality (Part 1)

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Lagrangian dual function

Standard Form Optimization Problem:

$$
\begin{aligned}
p^* := \min \quad & f_0(x) \\
\text{s.t.} \quad & f_i(x) \leq 0, \;\; i = 1, \ldots, m \\
& h_i(x) = 0, \;\; i = 1, \ldots, p
\end{aligned}
$$

- $\forall x \in \mathcal{D} \subseteq \mathbb{R}^n$
- optimal value p^*

Lagrangian dual function

Standard Form Optimization Problem:

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \leq 0, \;\; i = 1, \dots, m \\ & h_i(x) = 0, \;\; i = 1, \dots, p \end{aligned}$$

Lagrangian:

$$\begin{aligned} L(x, \lambda, \nu) \quad & = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{p} \nu_j h_j(x) \\ & = \langle f, (1, \lambda, \nu) \rangle \end{aligned}$$

Lagrangian dual function

Standard Form Optimization Problem:

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \leq 0, \ \ i = 1, \dots, m \\ & h_i(x) = 0, \ \ i = 1, \dots, p \end{aligned}$$

Lagrange Dual Function:

$$\begin{aligned} g(\lambda, \nu) \quad & = \inf_{x \in \mathcal{D}} (L(x, \lambda, \nu)) \\ & = \inf_{x \in \mathcal{D}} (f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{p} \nu_j h_j(x)) \end{aligned}$$

Properties

- $g(\lambda, \nu)$ is concave.
- if $\lambda \geq 0$, $g(\lambda, \nu) \leq p^*$, $\forall \nu$

if $\hat{x}$ is a feasible point, and $\lambda \geq 0$ then:

$$L(\hat{x}, \lambda, \nu) = f_0(\hat{x}) + \sum_{i=1}^{m} \lambda_i f_i(\hat{x}) + \sum_{j=1}^{p} \nu_j h_j(\hat{x}) \leq f_0(\hat{x})$$

$$\begin{aligned} g(\lambda, \nu) &= \inf_{x \in \mathcal{D}} (f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{p} \nu_j h_j(x)) \\ &\leq L(\hat{x}, \lambda, \nu) \leq f_0(\hat{x}), \quad \forall \hat{x} \ \text{ feasible} \end{aligned}$$

Interpretations

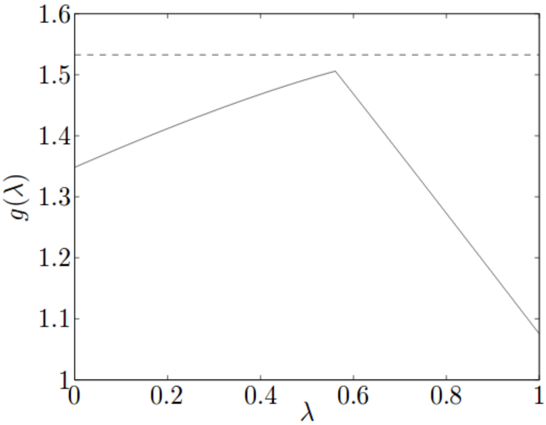


Figure 5.2 The dual function g for the problem in figure 5.1. Neither f_0 nor f_1 is convex, but the dual function is concave. The horizontal dashed line shows $p^\star$, the optimal value of the problem.

Dual function & Conjugate

Reminder:
We call f^* the conjugate of $f : \mathbb{R}^n \to \mathbb{R}$, the function:

$$f^*(y) = \sup_{x \in \mathcal{D}} (y^T x - f(x))$$

Dual function & Conjugate

$$f^*(y) = \sup_{x \in \mathcal{D}} (y^T x - f(x))$$

For the Linear Constraint Optimization problem:

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{s.t.} \quad & Ax \leq b \\ & Cx = d \end{aligned}$$

$$\begin{aligned} g(\lambda, \nu) \quad & = \inf_x (f_0(x) + \lambda^T (Ax - b) + \nu^T (Cx - d)) \\ & = -b^T \lambda - d^T \nu - \sup_x ((-A^T \lambda - C^T \nu)^T x - f_0(x)) \\ & = -b^T \lambda - d^T \nu - f_0^* ((-A^T \lambda - C^T \nu)) \end{aligned}$$

Least-norm solution of linear equations

$$\begin{array}{ll} \text{minimize} & x^T x \\ \text{subject to} & Ax = b \end{array}$$

dual function

- Lagrangian is $L(x, \nu) = x^T x + \nu^T (Ax - b)$
- to minimize L over x, set gradient equal to zero:

$$\nabla_x L(x, \nu) = 2x + A^T \nu = 0 \quad \Longrightarrow \quad x = -(1/2) A^T \nu$$

- plug in in L to obtain g:

$$g(\nu) = L((-1/2)A^T \nu, \nu) = -\frac{1}{4} \nu^T A A^T \nu - b^T \nu$$

  a concave function of ν

lower bound property: $p^\star \geq -(1/4)\nu^T A A^T \nu - b^T \nu$ for all ν

Standard form LP

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b, \quad x \succeq 0
\end{array}
$$

dual function

- Lagrangian is

$$
\begin{aligned}
L(x, \lambda, \nu) &= c^T x + \nu^T (Ax - b) - \lambda^T x \\
&= -b^T \nu + (c + A^T \nu - \lambda)^T x
\end{aligned}
$$

- L is affine in x, hence

$$
g(\lambda, \nu) = \inf_x L(x, \lambda, \nu) = \begin{cases} -b^T \nu & A^T \nu - \lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases}
$$

g is linear on affine domain $\{(\lambda, \nu) \mid A^T \nu - \lambda + c = 0\}$, hence concave

lower bound property: $p^\star \geq -b^T \nu$ if $A^T \nu + c \succeq 0$

Lagrange Dual Optimization Problem

Dual optimization problem:

$$\begin{aligned} d^* := \max \quad & g(\lambda, \nu) \\ \text{s.t.} \quad & \lambda \geq 0 \end{aligned}$$

- **dom** $\mathrm{g} = (\lambda, \nu) | g(\lambda, \nu) > -\infty$
- $g(\lambda, \nu)$ is concave, so this problem is a convex optimization problem

Definitions

Weak Duality:

$$d^* \leq p^*$$

Strong Duality:

$$d^* = p^*$$

Duality Gap:

$$\text{gap} = p^* - d^*$$

Slater's Constraint Qualification

If the primal problem is **convex** with the following form, we have conditions to ensure strong duality:

$$
\begin{aligned}
p^* := \text{minimize} \quad & f_0(x) \\
\text{s.t.} \quad & f_i(x) \leq 0, \quad i = 1 \ldots, m \\
& Ax = b
\end{aligned}
$$

Slater: If there exist $\hat{x}$ in the *relative interior of* $\mathcal{D}$ satisfying: $f_i(\hat{x}) < 0$ and $A\hat{x} = b$, then strong duality holds.
This can be reduced to the constraints f_i which are not affine.

Inequality form LP

primal problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \end{array}$$

dual function

$$g(\lambda) = \inf_x \left((c + A^T\lambda)^T x - b^T\lambda \right) = \begin{cases} -b^T\lambda & A^T\lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases}$$

dual problem

$$\begin{array}{ll} \text{maximize} & -b^T\lambda \\ \text{subject to} & A^T\lambda + c = 0, \quad \lambda \succeq 0 \end{array}$$

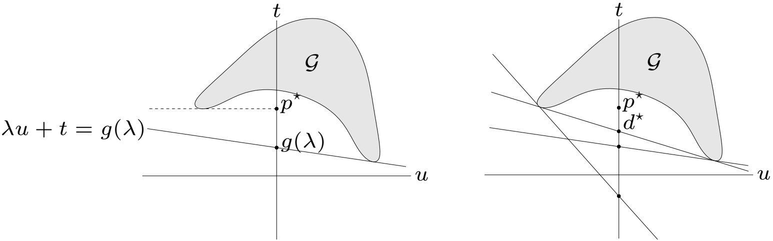
- from Slater's condition: $p^\star = d^\star$ if $A\tilde{x} \prec b$ for some $\tilde{x}$
- in fact, $p^\star = d^\star$ except when primal and dual are infeasible

Geometric interpretation

for simplicity, consider problem with one constraint $f_1(x) \leq 0$

interpretation of dual function:

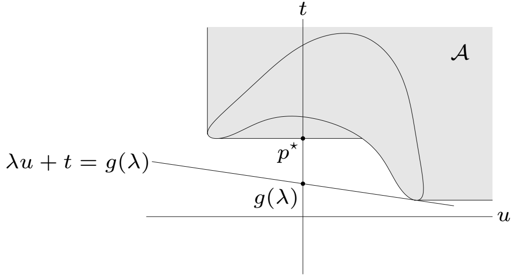
$$g(\lambda) = \inf_{(u,t)\in\mathcal{G}} (t + \lambda u), \qquad \text{where} \quad \mathcal{G} = \{(f_1(x), f_0(x)) \mid x \in \mathcal{D}\}$$



- $\lambda u + t = g(\lambda)$ is (non-vertical) supporting hyperplane to $\mathcal{G}$
- hyperplane intersects t-axis at $t = g(\lambda)$

epigraph variation: same interpretation if $\mathcal{G}$ is replaced with

$$\mathcal{A} = \{(u,t) \mid f_1(x) \leq u, f_0(x) \leq t \text{ for some } x \in \mathcal{D}\}$$



strong duality

- holds if there is a non-vertical supporting hyperplane to $\mathcal{A}$ at $(0, p^\star)$
- for convex problem, $\mathcal{A}$ is convex, hence has supp. hyperplane at $(0, p^\star)$
- Slater's condition: if there exist $(\tilde{u}, \tilde{t}) \in \mathcal{A}$ with $\tilde{u} < 0$, then supporting hyperplanes at $(0, p^\star)$ must be non-vertical

Proof of Slater's Constraint Qualification

[On Board]

Sub-optimality

If x is a primal feasible point, and (λ, ν) a dual feasible point:

$$p^* \geq g(\lambda, \nu)$$

$$0 \leq f_0(x) - p^* \leq f_0(x) - g(\lambda, \nu)$$

This leads to an absolute ϵ_{abs} and a relative ϵ_{rel} optimality criteria:

$$f_0(x) - g(\lambda, \nu) \leq \epsilon_{abs}$$

$$\frac{f_0(x) - g(\lambda, \nu)}{-f_0(x)} \leq \epsilon_{rel}, \quad f_0(x) < 0$$

$$\frac{f_0(x) - g(\lambda, \nu)}{g(\lambda, \nu)} \leq \epsilon_{rel}, \quad g(\lambda, \nu) > 0$$

KKT conditions - Non Convex

If $f_0, f_1, \ldots, f_m, h_0, \ldots, h_p$ are differentiable.
In case of strong duality, given the primal and dual optimal points, x^*, λ^*, ν^*.
The Optimality KKT conditions are:

$$
\begin{aligned}
f_i(x^*) \leq & \quad 0, \quad i = 1, \ldots, m \\
h_i(x^*) = & \quad 0, \quad i = 1, \ldots, p \\
\lambda_i^* \geq & \quad 0, \quad i = 1, \ldots, m \\
\lambda_i^* f_i(x^*) = & \quad 0, \quad i = 1, \ldots, m \\
\nabla f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(x^*) + \sum_{i=1}^{p} \nu_i^* \nabla h_i(x^*) = & \quad 0
\end{aligned}
$$

KKT conditions - Convex

$$\begin{aligned} f_i(\hat{x}) \leq & \quad 0, \quad i = 1, \dots, m \\ h_i(\hat{x}) = & \quad 0, \quad i = 1, \dots, p \\ \hat{\lambda}_i \geq & \quad 0, \quad i = 1, \dots, m \\ \hat{\lambda}_i f_i(\hat{x}) = & \quad 0, \quad i = 1, \dots, m \\ \nabla f_0(\hat{x}) + \sum_{i=1}^{m} \hat{\lambda}_i \nabla f_i(\hat{x}) + \sum_{i=1}^{p} \hat{\nu}_i \nabla h_i(\hat{x}) = & \quad 0 \end{aligned}$$

For a **convex problem**, any point satisfying the above conditions is optimal with a *0 duality gap*.
If Slater's conditions are true, then the KKT conditions are **necessary and sufficient conditions** for optimality.

Sumary

$$L(x, \lambda, \nu) = f_0(x) + \sum_i \lambda_i f_i(x) + \sum_j \nu_j h_j(x)$$

Original Problem

$$\begin{aligned} p^* = \min\ & f_0(x) \\ \text{s.t.}\ & f_i(x) \leq 0 \\ & h_j(x) = 0 \end{aligned}$$

Dual Problem

$$\begin{aligned} d^* &= \max\ \inf_x L(x, \lambda, \nu) \\ &= \max\ g(\lambda, \nu) \\ &\quad \text{s.t.}\ \lambda \geq 0 \end{aligned}$$

- $g(\lambda, \nu)$ concave
- $d^* \leq p^*$ :: weak duality
- $d^* = p^*$ :: strong duality

Sumary

Original Problem

$$\begin{aligned} p^* = \min\ & f_0(x) \\ \text{s.t. }\ & f_i(x) \leq 0 \\ & h_j(x) = 0 \end{aligned}$$

Dual Problem

$$\begin{aligned} d^* = \max\ & g(\lambda, \nu) \\ \text{s.t. }\ & \lambda \geq 0 \end{aligned}$$

Complementary Slackness:
If strong duality holds, with (x^*, λ^*, ν^*) the optimal solutions.

- $\lambda_i^* f_i(x^*) = 0, \ \forall i$
  - $\lambda_i^* > 0 \Rightarrow f_i(x^*) = 0$
  - $f_i(x^*) < 0 \Rightarrow \lambda_i^* = 0$

Perturbed Problem

Original Primal

$$\begin{aligned} p^* = \min \ & f_0(x) \\ \text{s.t.} \ & f_i(x) \leq 0 \\ & h_j(x) = 0 \end{aligned}$$

Perturbed Primal

$$\begin{aligned} p^*(u,v) = \min \ & f_0(x) \\ \text{s.t.} \ & f_i(x) \leq u_i \\ & h_j(x) = v_j \end{aligned}$$

$p^*(u,v) = \inf\{f_0(x) |\ \exists x \in \mathcal{D},\ f_i(x) \leq u_i,\ h_j(x) = v_j\}$
$p^*(0,0) = p^*$
If the original problem is convex, then $p^*(u,v)$ is a convex function of (u,v).(Exercise 5.32)

Global Sensibility Analysis

Let's assume:

- strong duality
- dual optimum is attained in (λ^*, ν^*).

$$p^*(u, v) \geq p^*(0, 0) - \lambda^{*\mathsf{T}} u - \nu^{*\mathsf{T}} v$$

- $\lambda_i^* \gg 0$ & $u_i < 0 \rightsquigarrow p^*(u, v) \nearrow$
- $(\nu_j^* \gg 0$ & $v_j < 0)$or$(\nu_j^* \ll 0$ & $v_j > 0) \rightsquigarrow p^*(u, v) \nearrow$
- NO guaranty for $\lambda_i^* \gg 0$ and $u_i > 0$ :: only a LOWER BOUND

Local Sensibility Analysis

If $p^*(u, v)$ is differentiable:

- $\lambda_i^* = \frac{\partial p^*(0,0)}{\partial u_i}$
- $\nu_j^* = \frac{\partial p^*(0,0)}{\partial v_j}$

By complementary slackness:

- $f_i(x^*) < 0 \Rightarrow \lambda_i^* = 0$ :: small change around $u_i = 0$
- $f_i(x^*) = 0 \Rightarrow \lambda_i^* > 0(\gg 0)$ :: p^* greatly affected by u_i

Introducing New Variables

Original Problem

$$
\begin{aligned}
p^* &= \min \; f_0(Ax+b) \\
d^* &= \max \; \inf_x (f_0(Ax+b)) \\
&= \max \; p^* = p^*
\end{aligned}
$$

Transformed Problem

$$
\begin{aligned}
p'^* &= \min \; f_0(y) \\
&\quad \text{s.t.} \;\; Ax+b=y \\
d'^* &= \max \; b^\mathsf{T}\nu - f_0^*(\nu) \\
&\quad \text{s.t.} \;\; A^\mathsf{T}\nu = 0
\end{aligned}
$$

Reminder:

$$
\begin{aligned}
g'(\nu) = &\quad \inf_{x,y} (f_0(y) - \nu^\mathsf{T} y + \nu^\mathsf{T} Ax + b^\mathsf{T}\nu) \\
= &\quad \begin{cases} -f_0^*(\nu) + b^\mathsf{T}\nu, & A^\mathsf{T}\nu = 0 \\ -\infty, & \text{otherwise} \end{cases}
\end{aligned}
$$

Introducing New Equality Constraints

Original Problem

$$\begin{aligned} p^* = \min\ & f_0(A_0 x + b_0) \\ \text{s.t.}\ & f_i(A_i x + b_i) \leq 0 \end{aligned}$$

Transformed Problem

$$\begin{aligned} p'^* = \min\ & f_0(y_0) \\ \text{s.t.}\ & f_i(y_i) \leq 0 \\ & A_i x + b_i = y_i \end{aligned}$$

Leads to a new dual:

$$\begin{aligned} d'^* = \max\ & \sum_i \nu_i^\intercal b_i - f_0^*(\nu_0) - \sum_i \lambda_i f_i^*(\nu_i/\lambda_i) \\ \text{s.t.}\ & \lambda \geq 0 \\ & \sum_i A_i^\intercal \nu_i = 0 \end{aligned}$$

Weak Alternatives

Either: (P1) has a solution OR (P2) has a solution OR none (Only implication of the infeasibility).
Example:

$$f_i(x) \leq 0, \ \forall i, \quad h_j(x) = 0, \ \forall j, \ \text{is feasible} \tag{1}$$

or

$$0 < g(\lambda, \nu), \ \lambda \geq 0, \ \text{is feasible} \tag{2}$$

If $\hat{x}$ a feasible point for (1), and $(\hat{\lambda}, \hat{\nu})$ feasible for (2) then:

$$0 < g(\hat{\lambda}, \hat{\nu}) \leq \underbrace{\sum_i \hat{\lambda}_i f_i(\hat{x})}_{\leq 0} + \underbrace{\sum_j \hat{\nu}_j h_j(\hat{x})}_{=0} \leq 0$$

Strong Alternatives

Either: (P1) has a solution OR (P2) has a solution (equivalence between feasibility of one and infeasibility of the other).
Example, with f_i convex $\forall i$:

$$f_i(x) < 0,\ \forall i, \quad \exists x \in \mathrm{relint}(\mathcal{D}),\ Ax = b,\ \text{is feasible} \tag{3}$$

or

$$0 \leq g(\lambda, \nu),\ \lambda \geq 0, \lambda \neq 0,\ \text{is feasible} \tag{4}$$

Farkas Lemma

The Farkas Lemma is a strong alternative:
Either

$$Ax = b, \quad x \geq 0$$

has a solution
or

$$A^\intercal y \geq 0, \quad b^\intercal y < 0$$

has a solution

Hilbert's Nullstellensatz

Some generalization of alternatives theorems:
Let's have $P_i \in \mathbb{R}[\mathbb{C}^n], \quad i = 1 \dots m$.
Either

$$P_1(x) = 0, \dots, P_m(x) = 0$$

has a solution $x \in \mathbb{C}^n$.
OR $\exists Q_1, \dots, Q_m \in \mathbb{R}[\mathbb{C}^n]$ such that

$$P_1(x)Q_1(x) + \cdots + P_m(x)Q_m(x) = -1$$

has a solution.

5.23.1

By contradiction, assume

$$-c \notin \left\{ \sum_{i \in I} z_i a_i \middle| z_i \geq 0 \right\} = K$$

K is a convex closed cone, thus by the *strict separating hyperplane theorem*, there exists $\zeta \in \mathbb{R}^m$ and $\gamma \in \mathbb{R}$ such that :

$$-c^T \zeta < \gamma, \ \left(\sum_{i \in I} z_i a_i^T \right) \zeta > \gamma, \ \forall z_i \geq 0$$

5.23.1

Lemma : Let $K \subset \mathbb{R}^q$ be a cone. Suppose there exist $p \in \mathbb{R}^q$ and $M \in \mathbb{R}$ such that

$$\forall x \in K, p^\intercal x \geq M$$

Then $M \geq 0$
Proof : We will prove this by contradiction. Suppose there exists $x' \in K$ such that $p^\intercal x' < 0$ (and thus, $M < 0$). Consider

$$\alpha = \frac{2M}{p^\intercal x'} > 0 \text{ and } y = \alpha x'$$

$y \in K$ since $\alpha > 0$ and $x' \in K$. But :

$$p^\intercal y = 2M \frac{p^\intercal x'}{p^\intercal x'} = 2M < M$$

since $M < 0$. That contradicts the fact that $p^\intercal x \geq M$ for all $x \in K$. $\square$

5.23.1

From the previous lemma, we see that $\gamma \geq 0$, so $\left(\sum_{i \in I} z_i a_i^\intercal\right) \zeta > 0,\ \forall z_i \geq 0$. Thus, by taking $z_i = 1$, $z_j = 0$ $\forall i \neq j$, we have :

$$a_i^\intercal \zeta > 0,\ \forall i \in I$$

Furthermore, since $-c^\intercal \zeta < \left(\sum_{i \in I} z_i a_i^\intercal\right) \zeta,\ \forall z_i \geq 0$, by taking $z_i = b_i - a_i^\intercal x^*$, which is nonnegative and equal to 0 for $i \in I$, we have :

$$\begin{aligned} -c^\intercal \zeta &< \left(\sum_{i \in I} (b_i - a_i^\intercal x^*) a_i^\intercal\right) \zeta \\ &= 0 \end{aligned}$$

5.23.1

To sum this up, we have :

$$-c^\mathsf{T}\zeta < 0,\ a_i^\mathsf{T}\zeta > 0,\ \forall i \in I$$

For an $\varepsilon > 0$ small enough, let us consider $d = x^* - \varepsilon\zeta$. This d is primal-feasible :

- For $i \notin I$, $a_i^\mathsf{T} d = \underbrace{a_i^\mathsf{T} x^*}_{<b_i} - \varepsilon a_i^\mathsf{T}\zeta < b_i$ for ε small enough (the strict inequality forms an open neighbourhood of x^*).
- For $i \in I$, $a_i^\mathsf{T} d = a_i^\mathsf{T} x^* - \varepsilon a_i^\mathsf{T}\zeta = b_i - \underbrace{\varepsilon a_i^\mathsf{T}\zeta}_{>0} \leq b_i$.

5.23.1

On top of that :

$$c^\intercal d = c^\intercal x^* - \varepsilon c^\intercal \zeta = p^* \underbrace{-\varepsilon c^\intercal \zeta}_{<0} < p^*$$

This contradicts the fact that p^* is the optimal value. So there exists $z \geq 0$ such that $z_i = 0 \; \forall i \notin I$ and $A^\intercal z + c = 0$. This z is dual feasible, thus $-b^\intercal z \leq p^*$ (weak duality theorem). But :

$$-b^\intercal z = -\sum_{i \in I} b_i z_i = -x^{*\intercal} \underbrace{\left(\sum_{i \in I} z_i a_i \right)}_{=-c} = c^\intercal x^* = p^*$$

So z is dual optimal.

5.23.2

We denote $(\mathcal{P})$ the primal problem (that we suppose infeasible), $(\mathcal{D})$ the dual one. Consider $(\mathcal{P}')$ and its dual $(\mathcal{D}')$:

$$\begin{aligned} p' = \min\ & 0 \\ \text{s.t.}\ & Ax \leq b \end{aligned} \qquad (\mathcal{P}')$$

$$\begin{aligned} d' = \max\ & -b^\intercal z \\ \text{s.t.}\ & A^\intercal z = 0 \\ & z \geq 0 \end{aligned} \qquad (\mathcal{D}')$$

5.23.2

Since $(\mathcal{P})$ is infeasible, so is $(\mathcal{P}')$, so $p'^* = \infty$. $(\mathcal{D}')$ is feasible (since $z = 0$ is feasible), so $d' \geq 0$. Suppose $d' = 0$. Since :

- $(\mathcal{D}')$ is feasible
- d' is finite
- $(\mathcal{P}')$ is the dual of $(\mathcal{D}')$

we have, through question 1, $d' = p'$. So $(\mathcal{P}')$ should be feasible, which is a contradiction. So $d' > 0$.

5.23.2

So there exists $\tilde{v} \geq 0$ such that $A^\intercal \tilde{v} = 0$ and $b^\intercal \tilde{v} < 0$. Let us denote v^* a feasible solution to $(\mathcal{D})$. Pose, for $\varepsilon > 0$:

$$z = v^* + \varepsilon \tilde{v} \geq 0$$

z is admissible for $(\mathcal{D})$:

$$A^\intercal z + c = \underbrace{A^\intercal v^* + c}_{=0} + \varepsilon \underbrace{A^\intercal \tilde{v}}_{=0} = 0$$

Furthermore,

$$-b^\intercal z = -b^\intercal v^* + \varepsilon \underbrace{\left(-b^\intercal \tilde{v}\right)}_{>0} \xrightarrow[\varepsilon \to +\infty]{} +\infty$$

So $(\mathcal{D})$ is unbounded above, and $d^* = +\infty$.

5.23.3

We clearly see that the problem in the example is infeasible, so $p^* = +\infty$. Let us find its dual problem :

$$\begin{aligned} \text{maximize} \ & -z_1 + z_2 \\ \text{subject to} \ & z_2 + 1 = 0 \\ & z_1, z_2 \geq 0 \end{aligned}$$

As we can see, $z_2 = -1$, which is not non-negative. So the problem is infeasible, and $d^* = -\infty$.