Nonstandard realizability with and without truth

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Overview

Interpretations without truth

- Herbrand realizability
- Bounded realizability
- Bounded functional interpretation

Interpretations with truth

- Bounded realizability with truth
- Bounded functional interpretations with truth

A parametrised approach

Some questions

Outline

Let E-HA$^\omega$ be Heyting arithmetic in all finite types with extensionality (as in Benno's talk).

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External quantifiers axioms:

- $\forall^{\mathrm{st}} x \, \Phi(x) \leftrightarrow \forall x (\mathrm{st}(x) \to \Phi(x))$.
- $\exists^{\mathrm{st}} x \, \Phi(x) \leftrightarrow \exists x (\mathrm{st}(x) \wedge \Phi(x))$

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Standardness axioms:

  - $x =_\sigma y \wedge \mathrm{st}^\sigma(x) \to \mathrm{st}^\sigma(y)$;
  - $\mathrm{st}^\sigma(t)$ for each closed term t;
  - $\mathrm{st}^{\sigma\to\tau}(f) \wedge \mathrm{st}^\sigma(x) \to \mathrm{st}^\tau(fx)$;

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External induction:

$(\Phi(0) \wedge \forall^{\mathrm{st}} n^0\, (\Phi(n) \to \Phi(n+1))) \to \forall^{\mathrm{st}} n^0\, \Phi(n)$

Herbrand realizability

Assign to each formula Φ of E-HA$^{\omega^*}_{\mathrm{st}}$ the formulas Φ^{hr} and $\Phi_{\mathrm{hr}}(a)$ of E-HA$^{\omega^*}_{\mathrm{st}}$ such that $\Phi^{\mathrm{hr}} \equiv \tilde{\exists}^{\mathrm{st}} a \, \Phi_{\mathrm{hr}}(a)$ according to the following clauses :

1. $\Phi^{\mathrm{hr}} :\equiv [\Phi]$ for internal atomic formulas Φ;
2. $\mathrm{st}(t)^{\mathrm{hr}} :\equiv \exists^{\mathrm{st}} a \, [t \in a]$;

If $\Phi^{\mathrm{hr}} \equiv \exists^{\mathrm{st}} a \, \Phi_{\mathrm{hr}}(a)$ and $\Psi^{\mathrm{hr}} \equiv \exists^{\mathrm{st}} b \, \Psi_{\mathrm{hr}}(b)$, then:

3. $(\Phi \wedge \Psi)^{\mathrm{hr}} :\equiv \exists^{\mathrm{st}} a, b \, [\Phi_{\mathrm{hr}}(a) \wedge \Psi_{\mathrm{hr}}(b)]$;
4. $(\Phi \vee \Psi)^{\mathrm{hr}} :\equiv \exists^{\mathrm{st}} a, b \, [\Phi_{\mathrm{hr}}(a) \vee \Psi_{\mathrm{hr}}(b)]$;
5. $(\Phi \to \Psi)^{\mathrm{hr}} :\equiv \exists^{\mathrm{st}} s \, [\forall^{\mathrm{st}} t \, (\Phi_{\mathrm{hr}}(t) \to \Psi_{\mathrm{hr}}(s[t]))]$;
6. $(\forall x \, \Phi)^{\mathrm{hr}} :\equiv \exists^{\mathrm{st}} a \, [\forall x \, \Phi_{\mathrm{hr}}(a)]$;
7. $(\exists x \, \Phi)^{\mathrm{hr}} :\equiv \exists^{\mathrm{st}} a \, [\exists x \, \Phi_{\mathrm{hr}}(a)]$.

$\exists^{\mathrm{st}}$-free formulas

Definition

We say that a formula of E-$\mathrm{HA}^{\omega}_{\mathrm{st}}$ is $\exists^{\mathrm{st}}$-*free* if and only if it is built:

1. from atomic internal formulas $s =_0 t$;
2. by conjunctions $\wedge$;
3. by disjunctions $\vee$;
4. by implications $\rightarrow$;
5. by quantifications $\forall$ and $\exists$;
6. by standard universal quantifications $\forall^{\mathrm{st}}$ (but not $\exists^{\mathrm{st}}$).

Lemma

- *For all* $\exists^{\mathrm{st}}$*-free formulas* $\Phi_{\nexists^{\mathrm{st}}}$ *of* $\text{E-HA}^{\omega^*}_{\mathrm{st}}$, *we have*
  - $(\Phi_{\nexists^{\mathrm{st}}})^{\mathrm{hr}} \equiv (\Phi_{\nexists^{\mathrm{st}}})_{\mathrm{hr}}(a)$;
  - $\text{E-HA}^{\omega^*}_{\mathrm{st}} \vdash (\Phi_{\nexists^{\mathrm{st}}})_{\mathrm{hr}} \leftrightarrow \Phi_{\nexists^{\mathrm{st}}}$.

Lemma

- *For all* $\exists^{\mathrm{st}}$*-free formulas* $\Phi_{\nexists^{\mathrm{st}}}$ *of* $\text{E-HA}^{\omega^*}_{\mathrm{st}}$*, we have*
  - $(\Phi_{\nexists^{\mathrm{st}}})^{\mathrm{hr}} \equiv (\Phi_{\nexists^{\mathrm{st}}})_{\mathrm{hr}}(a)$;
  - $\text{E-HA}^{\omega^*}_{\mathrm{st}} \vdash (\Phi_{\nexists^{\mathrm{st}}})_{\mathrm{hr}} \leftrightarrow \Phi_{\nexists^{\mathrm{st}}}$.
- *For all formulas* Φ *of* $\text{E-HA}^{\omega^*}_{\mathrm{st}}$*, the formula* $\Phi_{\mathrm{hr}}(a)$ *is* $\exists^{\mathrm{st}}$*-free.*

Caracteristic Principles

Definition

$$(\mathrm{HAC}) \equiv \forall^{\mathrm{st}} x\, \exists^{\mathrm{st}} y\, \Phi(x,y) \to \exists^{\mathrm{st}} F\, \forall^{\mathrm{st}} x\, \exists y \in Fx\ \Phi(x,y).$$

$$(\mathrm{HIP}_{\nexists^{\mathrm{st}}}) \equiv (\Phi_{\nexists^{\mathrm{st}}} \to \exists^{\mathrm{st}} x\, \Psi(x)) \to \exists^{\mathrm{st}} y\, (\Phi_{\nexists^{\mathrm{st}}} \to \exists x \in y\, \Psi(x)).$$

$$(\mathrm{NCR}) \equiv \forall x\, \exists^{\mathrm{st}} y\, \Phi(x,y) \to \exists^{\mathrm{st}} z\, \forall y\, \exists x\ \in z\, \Phi(x,y).$$

Soundness

Theorem (soundness theorem of hr)

For all formulas Φ *of* $\mathsf{E\text{-}HA}^{\omega^*}_{\mathrm{st}}$, *if*

$$\mathsf{E\text{-}HA}^{\omega^*}_{\mathrm{st}} + \mathsf{P} \vdash \Phi,$$

then there are closed terms t *of appropriate types such that*

$$\mathsf{E\text{-}HA}^{\omega^*}_{\mathrm{st}} \vdash \Phi_{\mathrm{hr}}(t).$$

Abbreviation

$\mathsf{P} := \mathsf{HAC} + \mathsf{NCR} + \mathsf{HIP}_{\nexists^{\mathrm{st}}}$.

Characterization

Theorem (Characterization theorem of hr)

For all formulas Φ *of* $\mathsf{E\text{-}HA}^{\omega^*}_{\mathrm{st}}$*, we have*

$$\mathsf{E\text{-}HA}^{\omega^*}_{\mathrm{st}} + \mathsf{P} \vdash \Phi \leftrightarrow \Phi^{\mathrm{hr}}.$$

Abbreviation

$\mathsf{P} := \mathsf{HAC} + \mathsf{NCR} + \mathsf{HIP}_{\nexists^{\mathrm{st}}}$.

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Let E-HA$^\omega$ be Heyting arithmetic in all finite types with full extensionality and with primitive equality only at type 0.

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  - $s \leq^*_{\rho\to\sigma} t :\equiv \forall v\, \forall u \leq^*_\rho v\, (su \leq^*_\sigma tv \wedge tu \leq^*_\sigma tv)$.

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- The Howard-Bezem strong majorizability $\leq^*_\sigma$ is defined by:
  - $s \leq^*_0 t :\equiv s \leq_0 t$;
  - $s \leq^*_{\rho\to\sigma} t :\equiv \forall v\, \forall u \leq^*_\rho v\, (su \leq^*_\sigma tv \wedge tu \leq^*_\sigma tv)$.
- $\leq^*_\sigma$ is not reflexive! We say that x^σ is monotone if and only if $x \leq^*_\sigma x$.

Majorizability

Proposition

1. $\text{E-HA}^\omega \vdash x \leq^*_\sigma y \to y \leq^*_\sigma y$;
2. $\text{E-HA}^\omega \vdash x \leq^*_\sigma y \wedge y \leq^*_\sigma z \to x \leq^*_\sigma z$.

Majorizability

Proposition

1. $\text{E-HA}^\omega \vdash x \leq^*_\sigma y \rightarrow y \leq^*_\sigma y$;
2. $\text{E-HA}^\omega \vdash x \leq^*_\sigma y \wedge y \leq^*_\sigma z \rightarrow x \leq^*_\sigma z$.

Theorem (Howard's majorizability theorem)

For all closed terms t^σ of E-HA^ω, *there is a closed term s^σ of* E-HA^ω *such that* $\text{E-HA}^\omega \vdash t \leq^*_\sigma s$.

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Enrich the language and the axioms of E-HA$^\omega$ as follows.

- $\mathrm{st}^\sigma(t^\sigma)$ (for each finite type σ).

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- External induction:
  $(\Phi(0) \wedge \forall^{\mathrm{st}} n^0\, (\Phi(n) \to \Phi(n+1))) \to \forall^{\mathrm{st}} n^0\, \Phi(n)$

Some abbreviations

- $\tilde{\forall} x\, \Phi(x)$ abbreviates $\forall x(x \leq^* x \to \Phi(x))$.
- $\tilde{\exists} x\, \Phi(x)$ abbreviates $\exists x(x \leq^* x \land \Phi(x))$.
- $\forall^{\mathrm{st}} x\, \Phi(x)$ abbreviates $\forall x(\mathrm{st}(x) \to \Phi(x))$.
- $\exists^{\mathrm{st}} x\, \Phi(x)$ abbreviates $\exists x(\mathrm{st}(x) \land \Phi(x))$.
- $\ldots$

Intuitionistic nonstandard bounded modified realizability

Assign to each formula Φ of E-HA$^{\omega}_{\mathrm{st}}$ the formulas Φ^{b} and $\Phi_{\mathrm{b}}(a)$ of E-HA$^{\omega}_{\mathrm{st}}$ such that $\Phi^{\mathrm{b}} \equiv \tilde{\exists}^{\mathrm{st}} a\, \Phi_{\mathrm{b}}(a)$ according to the following clauses :

1. $\Phi^{\mathrm{b}} :\equiv [\Phi]$ for internal atomic formulas Φ;
2. $\mathrm{st}(t)^{\mathrm{b}} :\equiv \tilde{\exists}^{\mathrm{st}} a\, [t \leq^* a]$;

If $\Phi^{\mathrm{b}} \equiv \tilde{\exists}^{\mathrm{st}} a\, \Phi_{\mathrm{b}}(a)$ and $\Psi^{\mathrm{b}} \equiv \tilde{\exists}^{\mathrm{st}} b\, \Psi_{\mathrm{b}}(b)$, then:

3. $(\Phi \wedge \Psi)^{\mathrm{b}} :\equiv \tilde{\exists}^{\mathrm{st}} a, b\, [\Phi_{\mathrm{b}}(a) \wedge \Psi_{\mathrm{b}}(b)]$;
4. $(\Phi \vee \Psi)^{\mathrm{b}} :\equiv \tilde{\exists}^{\mathrm{st}} a, b\, [\Phi_{\mathrm{b}}(a) \vee \Psi_{\mathrm{b}}(b)]$;
5. $(\Phi \to \Psi)^{\mathrm{b}} :\equiv \tilde{\exists}^{\mathrm{st}} B\, [\tilde{\forall}^{\mathrm{st}} a\, (\Phi_{\mathrm{b}}(a) \to \Psi_{\mathrm{b}}(Ba))]$;
6. $(\forall x\, \Phi)^{\mathrm{b}} :\equiv \tilde{\exists}^{\mathrm{st}} a\, [\forall x\, \Phi_{\mathrm{b}}(a)]$;
7. $(\exists x\, \Phi)^{\mathrm{b}} :\equiv \tilde{\exists}^{\mathrm{st}} a\, [\exists x\, \Phi_{\mathrm{b}}(a)]$.

Monotonicity

Lemma (monotonicity of b)

For all formulas Φ *of* $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$, *we have*

$$\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \vdash \Phi_{\mathrm{b}}(a) \wedge a \leq^{*} c \to \Phi_{\mathrm{b}}(c).$$

$\tilde{\exists}^{\mathrm{st}}$-free formulas

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2. by conjunctions $\wedge$;
3. by disjunctions $\vee$;
4. by implications $\to$;
5. by quantifications $\forall$ and $\exists$ (so also $\tilde{\forall}$ and $\tilde{\exists}$);
6. by monotone standard universal quantifications $\tilde{\forall}^{\mathrm{st}}$ (but not $\tilde{\exists}^{\mathrm{st}}$).

$\tilde{\exists}^{\mathrm{st}}$-free formulas

Lemma

- *For all $\tilde{\exists}^{\mathrm{st}}$-free formulas $\Phi_{\tilde{\nexists}^{\mathrm{st}}}$ of* E-HA$^{\omega}_{\mathrm{st}}$*, we have*
  - $(\Phi_{\tilde{\nexists}^{\mathrm{st}}})^{\mathrm{b}} \equiv (\Phi_{\tilde{\nexists}^{\mathrm{st}}})_{\mathrm{b}}(a)$;
  - E-HA$^{\omega}_{\mathrm{st}} \vdash (\Phi_{\tilde{\nexists}^{\mathrm{st}}})_{\mathrm{b}} \leftrightarrow \Phi_{\tilde{\nexists}^{\mathrm{st}}}$.

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  - E-HA$^{\omega}_{\mathrm{st}} \vdash (\Phi_{\tilde{\nexists}^{\mathrm{st}}})_{\mathrm{b}} \leftrightarrow \Phi_{\tilde{\nexists}^{\mathrm{st}}}$.
- *For all formulas Φ of* E-HA$^{\omega}_{\mathrm{st}}$, *the formula $\Phi_{\mathrm{b}}(a)$ is $\tilde{\exists}^{\mathrm{st}}$-free.*

Caracteristic Principles

Definition

- $\mathsf{mAC}^\omega \equiv \tilde{\forall}^{\mathrm{st}} x\, \tilde{\exists}^{\mathrm{st}} y\, \Phi \to \tilde{\exists}^{\mathrm{st}} Y\, \tilde{\forall}^{\mathrm{st}} x\, \tilde{\exists} y \leq^* Yx\, \Phi$;
- $\mathsf{R}^\omega \equiv \forall x\, \exists^{\mathrm{st}} y\, \Phi \to \tilde{\exists}^{\mathrm{st}} z\, \forall x\, \exists y \leq^* z\, \Phi$;
- $\mathsf{IP}^\omega_{\tilde{\nexists}^{\mathrm{st}}} \equiv (\Phi_{\tilde{\nexists}^{\mathrm{st}}} \to \tilde{\exists}^{\mathrm{st}} x\, \Psi) \to \tilde{\exists}^{\mathrm{st}} y\, (\Phi_{\tilde{\nexists}^{\mathrm{st}}} \to \tilde{\exists} x \leq^* y\, \Psi)$;
- $\mathsf{MAJ}^\omega \equiv \forall^{\mathrm{st}} x\, \exists^{\mathrm{st}} y\, (x \leq^* y)$.

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- $\mathrm{R}^\omega \equiv \forall x\, \exists^{\mathrm{st}} y\, \Phi \to \tilde{\exists}^{\mathrm{st}} z\, \forall x\, \exists y \leq^* z\, \Phi$;
- $\mathrm{IP}^\omega_{\tilde{\nexists}^{\mathrm{st}}} \equiv (\Phi_{\tilde{\nexists}^{\mathrm{st}}} \to \tilde{\exists}^{\mathrm{st}} x\, \Psi) \to \tilde{\exists}^{\mathrm{st}} y\, (\Phi_{\tilde{\nexists}^{\mathrm{st}}} \to \tilde{\exists} x \leq^* y\, \Psi)$;
- $\mathrm{MAJ}^\omega \equiv \forall^{\mathrm{st}} x\, \exists^{\mathrm{st}} y\, (x \leq^* y)$.

Proposition

The principle R^ω *implies the principle* MAJ^ω, *that is* $\text{E-HA}^\omega_{\mathrm{st}} + \mathrm{R}^\omega$ *proves all instances of* MAJ^ω

Soundness

Theorem (soundness theorem of b)

For all formulas Φ *of* $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$, *if*

$$\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} + \mathsf{P} \vdash \Phi,$$

then there are closed monotone terms t *of appropriate types such that*

$$\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \vdash \Phi_{\mathrm{b}}(t).$$

Abbreviation

$\mathsf{P} := \mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} + \mathsf{mAC}^{\omega} + \mathsf{R}^{\omega} + \mathsf{IP}^{\omega}_{\nexists\mathrm{st}} + \mathsf{MAJ}^{\omega}.$

Characterization

Theorem (Characterization theorem of b)

For all formulas Φ of $\text{E-HA}^{\omega}_{\mathrm{st}}$, we have

$$\text{E-HA}^{\omega}_{\mathrm{st}} + \mathsf{P} \vdash \Phi \leftrightarrow \Phi^{\mathrm{b}}.$$

Abbreviation

$\mathsf{P} := \text{E-HA}^{\omega}_{\mathrm{st}} + \mathsf{mAC}^{\omega} + \mathsf{R}^{\omega} + \mathsf{IP}^{\omega}_{\nexists\mathrm{st}} + \mathsf{MAJ}^{\omega}.$

Outline

Intuitionistic nonstandard bounded functional interpretation

Assign to each formula Φ of E-$\mathrm{HA}^{\omega}_{\mathrm{st}}$ the formulas Φ^{B} and $\Phi_{\mathrm{B}}(a; b)$ of E-$\mathrm{HA}^{\omega}_{\mathrm{st}}$ such that $\Phi^{\mathrm{B}} \equiv \tilde{\exists}^{\mathrm{st}} a \, \tilde{\forall}^{\mathrm{st}} b \, \Phi_{\mathrm{B}}(a; b)$ according to the following clauses.

$\Phi^{\mathrm{B}} :\equiv [\Phi]$ for internal atomic formulas Φ;

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Intuitionistic nonstandard bounded functional interpretation

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$$\Phi^{\mathrm{B}} :\equiv [\Phi] \text{ for internal atomic formulas } \Phi;$$

$$\mathrm{st}(t)^{\mathrm{B}} :\equiv \tilde{\exists}^{\mathrm{st}} a\, [t \leq^* a].$$

If $\Phi^{\mathrm{B}} \equiv \tilde{\exists}^{\mathrm{st}} a\, \tilde{\forall}^{\mathrm{st}} b\, \Phi_{\mathrm{B}}(a;b)$ and $\Psi^{\mathrm{B}} \equiv \tilde{\exists}^{\mathrm{st}} c\, \tilde{\forall}^{\mathrm{st}} d\, \Psi_{\mathrm{B}}(c;d)$ then:

$$\begin{aligned}
(\Phi \wedge \Psi)^{\mathrm{B}} &:\equiv \tilde{\exists}^{\mathrm{st}} a, c\, \tilde{\forall}^{\mathrm{st}} b, d\, [\Phi_{\mathrm{B}}(a;b) \wedge \Psi_{\mathrm{B}}(c;d)];\\
(\Phi \vee \Psi)^{\mathrm{B}} &:\equiv \tilde{\exists}^{\mathrm{st}} a, c\, \tilde{\forall}^{\mathrm{st}} e, f\, [\tilde{\forall} b \leq^* e\, \Phi_{\mathrm{B}}(a;b) \vee \tilde{\forall} d \leq^* f\, \Psi_{\mathrm{B}}(c;d)];\\
(\Phi \to \Psi)^{\mathrm{B}} &:\equiv \tilde{\exists}^{\mathrm{st}} C, B\, \tilde{\forall}^{\mathrm{st}} a, d\, [\tilde{\forall} b \leq^* Bad\, \Phi_{\mathrm{B}}(a;b) \to \Psi_{\mathrm{B}}(Ca;d)];\\
(\forall x\, \Phi)^{\mathrm{B}} &:\equiv \tilde{\exists}^{\mathrm{st}} a\, \tilde{\forall}^{\mathrm{st}} b\, [\forall x\, \Phi_{\mathrm{B}}(a;b)];\\
(\exists x\, \Phi)^{\mathrm{B}} &:\equiv \tilde{\exists}^{\mathrm{st}} a\, \tilde{\forall}^{\mathrm{st}} c\, [\exists x\, \tilde{\forall} b \leq^* c\, \Phi_{\mathrm{B}}(a;b)].
\end{aligned}$$

Monotonicity

Lemma (monotonicity of B)

For all formulas Φ *of* $\text{E-HA}^{\omega}_{\mathrm{st}}$, *we have*

$$\text{E-HA}^{\omega}_{\mathrm{st}} \vdash \Phi_{\mathrm{B}}(a; b) \wedge a \leq^{*} c \to \Phi_{\mathrm{B}}(c; b).$$

Characteristic principles

Definition

- $\mathrm{mAC}^\omega \equiv \tilde{\forall}^{\mathrm{st}} x\, \tilde{\exists}^{\mathrm{st}} y\, \Phi \to \tilde{\exists}^{\mathrm{st}} Y\, \tilde{\forall}^{\mathrm{st}} x\, \tilde{\exists} y \leq^* Yx\, \Phi$;
- $\mathrm{R}^\omega \equiv \forall x\, \exists^{\mathrm{st}} y\, \Phi \to \tilde{\exists}^{\mathrm{st}} z\, \forall x\, \exists y \leq^* z\, \Phi$;
- $\mathrm{I}^\omega \equiv \tilde{\forall}^{\mathrm{st}} z\, \exists x\, \forall y \leq^* z\, \phi \to \exists x\, \forall^{\mathrm{st}} y\, \phi$;
- $\mathrm{IP}^\omega_{\tilde{\forall}^{\mathrm{st}}} \equiv (\tilde{\forall}^{\mathrm{st}} x\, \phi \to \tilde{\exists}^{\mathrm{st}} y\, \Psi) \to \tilde{\exists}^{\mathrm{st}} z\, (\tilde{\forall}^{\mathrm{st}} x\, \phi \to \tilde{\exists} y \leq^* z\, \Psi)$;
- $\mathrm{M}^\omega \equiv (\tilde{\forall}^{\mathrm{st}} x\, \phi \to \psi) \to \tilde{\exists}^{\mathrm{st}} y\, (\tilde{\forall} x \leq^* y\, \phi \to \psi)$;
- $\mathrm{BUD}^\omega \equiv \tilde{\forall}^{\mathrm{st}} u, v\, (\forall x \leq^* u\, \phi \vee \forall y \leq^* v\, \psi) \to \forall^{\mathrm{st}} x\, \phi \vee \forall^{\mathrm{st}} y\, \psi$;
- $\mathrm{MAJ}^\omega \equiv \forall^{\mathrm{st}} x\, \exists^{\mathrm{st}} y\, (x \leq^* y)$.

Proposition

- $\text{E-HA}^{\omega}_{\mathrm{st}} + \mathsf{I}^{\omega} \vdash \mathsf{BUD}^{\omega}$.
- $\text{E-HA}^{\omega}_{\mathrm{st}} + \mathsf{R}^{\omega} \vdash \mathsf{MAJ}^{\omega}$.

Soundness

Theorem (soundness theorem of B)

For all formulas Φ *of* $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$, *if*

$$\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} + \mathsf{P} \vdash \Phi,$$

then there are closed monotone terms t *of appropriate types such that*

$$\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \vdash \tilde{\forall}^{\mathrm{st}} b\, \Phi_{\mathrm{B}}(t; b).$$

Abbreviation

$\mathsf{P} := \mathsf{mAC}^{\omega} + \mathsf{R}^{\omega} + \mathsf{I}^{\omega} + \mathsf{IP}^{\omega}_{\tilde{\forall}^{\mathrm{st}}} + \mathsf{M}^{\omega} + \mathsf{BUD}^{\omega} + \mathsf{MAJ}^{\omega}$.

Characterization

Theorem (characterization theorem of B)

For all formulas Φ *of* $\text{E-HA}^{\omega}_{\mathrm{st}}$, *we have*

$$\text{E-HA}^{\omega}_{\mathrm{st}} + \mathsf{P} \vdash \Phi \leftrightarrow \Phi^{\mathrm{B}}.$$

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Transfer Principles

Definition

1. $(\mathsf{T}_\forall) \equiv \forall^{\mathrm{st}} f\, (\forall^{\mathrm{st}} x\, \phi \rightarrow \forall x\, \phi)$;
2. $(\mathsf{T}_\exists) \equiv \forall^{\mathrm{st}} f\, (\exists x\, \phi \rightarrow \exists^{\mathrm{st}} x\, \phi)$;

where f are all the free variables in the internal formula ϕ.

Adding Transfer

Theorem

1. *Adding* $\mathsf{T}_\forall$ *or* $\mathsf{T}_\exists$ *to* $\mathsf{E\text{-}HA}^{\omega^*}_{\mathrm{st}} + \mathsf{R} + \mathsf{HGMP}^{\mathrm{st}}$ *leads to nonconservativity over* HA*.*
2. *Adding* $\mathsf{T}_\forall$ *or* $\mathsf{T}_\exists$ *to* $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$ *leads to inconsistency.*

Recovering standard interpretations

If we restrict ourselves to the purely external fragment (only quantifiers of the form $\exists^{\mathrm{st}}$ and $\forall^{\mathrm{st}}$) then we recover the bounded functional interpretation.

Diamond translation

Definition

The *diamond translation* $\diamond$ assigns to each formula ϕ of $\mathrm{HA}^{\omega}_{\trianglelefteq}$ the formula $\phi^{\diamond}$ of $\text{E-HA}^{\omega}_{\mathrm{st}}$ accordingly to the following clauses. For atomic formulas, we define:

1. $(s =_0 t)^{\diamond} :\equiv s =_0 t$;
2. $(s \trianglelefteq_{\sigma} t)^{\diamond} :\equiv s \leq^{*}_{\sigma} t$.

For the remaining formulas, we define:

3. $(\phi \circ \psi)^{\diamond} :\equiv \phi^{\diamond} \circ \psi^{\diamond}$ for $\circ \in \{\vee, \wedge, \to\}$;
4. $(\daleth x \trianglelefteq t\, \phi)^{\diamond} :\equiv \daleth x \leq^{*} t\, \phi^{\diamond}$ for $\daleth \in \{\forall, \exists\}$;
5. $(\daleth x\, \phi)^{\diamond} :\equiv \daleth^{\mathrm{st}} x\, \phi^{\diamond}$ for $\daleth \in \{\forall, \exists\}$.

Recovering standard interpretations

Theorem

For all formulas ϕ of $\mathsf{HA}^{\omega}_{\trianglelefteq}$, we have:

1. $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \vdash \phi_{\mathrm{B}'}(a;b)^{\diamond} \leftrightarrow (\phi^{\diamond})_{\mathrm{B}}(a;b)$;
2. $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} \vdash (\phi^{\mathrm{B}'})^{\diamond} \leftrightarrow (\phi^{\diamond})^{\mathrm{B}}$.

Krivine's negative translation

$A^{\mathrm{K}} :\equiv \neg A_{\mathrm{K}}$ (Φ_{at} is an atomic formula)

- $(\Phi_{\mathrm{at}})_{\mathrm{K}} :\equiv \neg \Phi_{\mathrm{at}}$,
- $(\neg \Phi)_{\mathrm{K}} :\equiv \neg \Phi_{\mathrm{K}}$,
- $(\Phi \vee \Psi)_{\mathrm{K}} :\equiv \Phi_{\mathrm{K}} \wedge \Psi_{\mathrm{K}}$,
- $(\forall x\, \Phi)_{\mathrm{K}} :\equiv \exists x\, \Phi_{\mathrm{K}}$.

Theorem (Soundness and characterization of K)

For all formulas Φ of the language of $\text{E-PA}^{\omega}_{\mathrm{st}}$, we have:

1. $\text{E-PA}^{\omega}_{\mathrm{st}} \vdash \Phi \;\Rightarrow\; \text{E-HA}^{\omega}_{\mathrm{st}} + \text{I-LEM} \vdash \Phi^{\mathrm{K}}$;
2. $\text{E-PA}^{\omega}_{\mathrm{st}} \vdash \Phi \leftrightarrow \Phi^{\mathrm{K}}$.

Factorization $\mathrm{U} = \mathrm{K}\,\mathrm{B}$

Theorem (factorisation $\mathrm{U} = \mathrm{K}\,\mathrm{B}$)

For all formulas Φ of the language of $\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}}$*, we have:*

1. $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} + \mathsf{I\text{-}LEM} \vdash \tilde{\forall} a, b\, (\Phi_{\mathrm{U}}(a; b) \leftrightarrow \neg \tilde{\forall} c \leq^{*} b\, (\Phi_{\mathrm{K}})_{\mathrm{B}}(a; c))$;
2. $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} + \mathsf{I\text{-}LEM} \vdash \tilde{\forall} a, B\, (\Phi_{\mathrm{U}}(a; Ba) \leftrightarrow (\Phi^{\mathrm{K}})_{\mathrm{B}}(a; B))$;
3. $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} + \mathsf{I\text{-}LEM} + \mathsf{mAC}^{\omega}_{\mathrm{st}} \vdash \Phi^{\mathrm{U}} \leftrightarrow (\Phi^{\mathrm{K}})^{\mathrm{B}}$.

Application

- Using the factorization $\mathrm{U} = \mathrm{K}\,\mathrm{B}$ and the soundness theorem of B one gets new proofs of the soundness and characterization theorems of U.

Outline

Realizability with q-truth

Assigns to each formula Φ of E-HA$^{\omega}_{\mathrm{st}}$ the formula $\Phi^{\mathrm{bq}} :\equiv \tilde{\exists}^{\mathrm{st}} a\, \Phi_{\mathrm{bq}}(a)$ of E-HA$^{\omega}_{\mathrm{st}}$ according to the following clauses, $\Phi^{\mathrm{bq}} \equiv \tilde{\exists}^{\mathrm{st}} a\, \Phi_{\mathrm{bq}}(a)$ and $\Psi^{\mathrm{bq}} \equiv \tilde{\exists}^{\mathrm{st}} b\, \Psi_{\mathrm{bq}}(b)$):

$$\phi^{\mathrm{bq}} :\equiv [\phi],$$

$$\mathrm{st}(t)^{\mathrm{bq}} :\equiv \tilde{\exists}^{\mathrm{st}} a\, [t \leq^* a],$$

$$(\Phi \wedge \Psi)^{\mathrm{bq}} :\equiv \tilde{\exists}^{\mathrm{st}} a, b\, [\Phi_{\mathrm{bq}}(a) \wedge \Psi_{\mathrm{bq}}(b)],$$

$$(\Phi \vee \Psi)^{\mathrm{bq}} :\equiv \tilde{\exists}^{\mathrm{st}} a, b\, [(\Phi_{\mathrm{bq}}(a) \wedge \Phi) \vee (\Psi_{\mathrm{bq}}(b) \wedge \Psi)],$$

$$(\Phi \to \Psi)^{\mathrm{bq}} :\equiv \tilde{\exists}^{\mathrm{st}} B\, \tilde{\forall}^{\mathrm{st}} a\, [\Phi_{\mathrm{bq}}(a) \wedge \Phi \to \Psi_{\mathrm{bq}}(Ba)],$$

$$(\forall x\, \Phi)^{\mathrm{bq}} :\equiv \tilde{\exists}^{\mathrm{st}} a\, [\forall x\, \Phi_{\mathrm{bq}}(a)],$$

$$(\exists x\, \Phi)^{\mathrm{bq}} :\equiv \tilde{\exists}^{\mathrm{st}} a\, [\exists x\, (\Phi_{\mathrm{bq}}(a) \wedge \Phi)].$$

Realizability with $\mathfrak{t}$-truth

$$\phi^{\mathrm{bt}} :\equiv [\phi],$$

$$\mathrm{st}(t)^{\mathrm{bt}} :\equiv \tilde{\exists}^{\mathrm{st}} a \, [t \leq^* a],$$

$$(\Phi \wedge \Psi)^{\mathrm{bt}} :\equiv \tilde{\exists}^{\mathrm{st}} a, b \, [\Phi_{\mathrm{bt}}(a) \wedge \Psi_{\mathrm{bt}}(b)],$$

$$(\Phi \vee \Psi)^{\mathrm{bt}} :\equiv \tilde{\exists}^{\mathrm{st}} a, b \, [\Phi_{\mathrm{bt}}(a) \vee \Psi_{\mathrm{bt}}(b)],$$

$$(\Phi \rightarrow \Psi)^{\mathrm{bt}} :\equiv \tilde{\exists}^{\mathrm{st}} B \, \tilde{\forall}^{\mathrm{st}} a \, [(\Phi_{bt}(a) \rightarrow \Psi_{bt}(Ba)) \wedge (\Phi \rightarrow \Psi)],$$

$$(\forall x \, \Phi)^{\mathrm{bt}} :\equiv \tilde{\exists}^{\mathrm{st}} a \, [\forall x \, \Phi_{\mathrm{bt}}(a)],$$

$$(\exists x \, \Phi)^{\mathrm{bt}} :\equiv \tilde{\exists}^{\mathrm{st}} a \, [\exists x \, \Phi_{\mathrm{bt}}(a)].$$

$\mathrm{bt} = \mathrm{bq} \wedge \mathrm{id}$

Theorem

For all formulas Φ *of* $\text{E-HA}^{\omega}_{\mathrm{st}}$, *we have*

$$\text{E-HA}^{\omega}_{\mathrm{st}} \vdash \forall^{\mathrm{st}} a \, (\Phi_{\mathrm{bt}}(a) \leftrightarrow \Phi_{\mathrm{bq}}(a) \wedge \Phi).$$

Soundness of bq and bt

Theorem

For all formulas Φ *of* $\mathsf{E}\text{-}\mathsf{HA}^{\omega}_{\mathrm{st}}$*, if*

$$\mathsf{E}\text{-}\mathsf{HA}^{\omega}_{\mathrm{st}} \pm \mathsf{mAC}^{\omega} \pm \mathsf{R}^{\omega} \pm \mathsf{IP}^{\omega}_{\tilde{\nexists}\mathrm{st}} \pm \mathsf{MAJ}^{\omega} \vdash \Phi,$$

then there are closed monotone terms t *such that*

$$\mathsf{E}\text{-}\mathsf{HA}^{\omega}_{\mathrm{st}} \pm \mathsf{mAC}^{\omega} \pm \mathsf{R}^{\omega} \pm \mathsf{IP}^{\omega}_{\tilde{\nexists}\mathrm{st}} \pm \mathsf{MAJ}^{\omega} \vdash \Phi_{\mathrm{bq}}(t),$$
$$\mathsf{E}\text{-}\mathsf{HA}^{\omega}_{\mathrm{st}} \pm \mathsf{mAC}^{\omega} \pm \mathsf{R}^{\omega} \pm \mathsf{IP}^{\omega}_{\tilde{\nexists}\mathrm{st}} \pm \mathsf{MAJ}^{\omega} \vdash \Phi_{\mathrm{bt}}(t).$$

Characterization of bq and bt

Theorem

For all formulas Φ *of* $\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}}$, *we have*

$$\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} + \mathsf{mAC}^{\omega} + \mathsf{R}^{\omega} + \mathsf{IP}^{\omega}_{\tilde{\nexists}\mathrm{st}} + \mathsf{MAJ}^{\omega} \vdash \Phi^{\mathrm{bq}} \leftrightarrow \Phi,$$
$$\mathsf{E\text{-}HA}^{\omega}_{\mathrm{st}} + \mathsf{mAC}^{\omega} + \mathsf{R}^{\omega} + \mathsf{IP}^{\omega}_{\tilde{\nexists}\mathrm{st}} + \mathsf{MAJ}^{\omega} \vdash \Phi^{\mathrm{bt}} \leftrightarrow \Phi.$$

Outline

Intuitionistic nonstandard bounded functional interpretation with q-truth

Assign to each formula Φ of E-HA$^{\omega}_{\mathrm{st}}$ the formula $\Phi^{\mathrm{Bq}} :\equiv \tilde{\exists}^{\mathrm{st}} a\, \Phi_{\mathrm{Bq}}(a; b)$ of E-HA$^{\omega}_{\mathrm{st}}$ according to the following clauses (where $\Phi_{\mathrm{Bq}}(a; b)$ is the part inside square brackets, ϕ is an internal atomic formula, $\Phi^{\mathrm{Bq}} \equiv \tilde{\exists}^{\mathrm{st}} a\, \tilde{\forall}^{\mathrm{st}} b\, \Phi_{\mathrm{bq}}(a)$ and $\Psi^{\mathrm{Bq}} \equiv \tilde{\exists}^{\mathrm{st}} c\, \tilde{\forall}^{\mathrm{st}} d\, \Psi_{\mathrm{bq}}(b)$):

$$
\begin{array}{ll}
\phi^{\mathrm{Bq}} & :\equiv [\phi], \\
\mathrm{st}(t)^{\mathrm{Bq}} & :\equiv \tilde{\exists}^{\mathrm{st}} a\, [t \leq^* a], \\
(\Phi \wedge \Psi)^{\mathrm{Bq}} & :\equiv \tilde{\exists}^{\mathrm{st}} a, c\, \tilde{\forall}^{\mathrm{st}} b, d\, [\Phi_{\mathrm{Bq}}(a; b) \wedge \Psi_{\mathrm{Bq}}(c; d)], \\
(\Phi \vee \Psi)^{\mathrm{Bq}} & :\equiv \tilde{\exists}^{\mathrm{st}} a, c\, \tilde{\forall}^{\mathrm{st}} e, f\ \ [(\tilde{\forall} b \leq^* e\, \Phi_{\mathrm{Bq}}(a; b) \wedge \Phi) \vee (\tilde{\forall} d \leq^* f\, \Psi_{\mathrm{Bq}}(c; d) \wedge \Psi)], \\
(\Phi \to \Psi)^{\mathrm{Bq}} & :\equiv \tilde{\exists}^{\mathrm{st}} C, B\, \tilde{\forall}^{\mathrm{st}} a, d\, [\tilde{\forall} b \leq^* Bad\, \Phi_{\mathrm{Bq}}(a; b) \wedge \Phi \to \Psi_{\mathrm{Bq}}(Ca; d)], \\
(\forall x\, \Phi)^{\mathrm{Bq}} & :\equiv \tilde{\exists}^{\mathrm{st}} a\, \tilde{\forall}^{\mathrm{st}} b\, [\forall x\, \Phi_{\mathrm{Bq}}(a; b)], \\
(\exists x\, \Phi)^{\mathrm{Bq}} & :\equiv \tilde{\exists}^{\mathrm{st}} a\, \tilde{\forall}^{\mathrm{st}} c\, [\exists x\, (\tilde{\forall} b \leq^* c\, \Phi_{\mathrm{Bq}}(a; b) \wedge \Phi)].
\end{array}
$$

Intuitionistic nonstandard bounded functional interpretation with t-truth

$$
\begin{array}{ll}
\Phi^{\mathrm{Bt}} & :\equiv [\Phi], \\
\mathrm{st}(t)^{\mathrm{Bt}} & :\equiv \tilde{\exists}^{\mathrm{st}} a\, [t \leq^* a], \\
(\Phi \wedge \Psi)^{\mathrm{Bt}} & :\equiv \tilde{\exists}^{\mathrm{st}} a, c\, \tilde{\forall}^{\mathrm{st}} b, d\, [\Phi_{\mathrm{Bt}}(a; b) \wedge \Psi_{\mathrm{Bt}}(c; d)], \\
(\Phi \vee \Psi)^{\mathrm{Bt}} & :\equiv \tilde{\exists}^{\mathrm{st}} a, c\, \tilde{\forall}^{\mathrm{st}} e, f\, [\tilde{\forall} b \leq^* e\, \Phi_{\mathrm{Bt}}(a; b) \vee \tilde{\forall} d \leq^* f\, \Psi_{\mathrm{Bt}}(c; d)], \\
(\Phi \to \Psi)^{\mathrm{Bt}} & :\equiv \tilde{\exists}^{\mathrm{st}} C, B\, \tilde{\forall}^{\mathrm{st}} a, d\, \ [\tilde{\forall} b \leq^* Bad\, \Phi_{\mathrm{Bt}}(a; b) \to \Psi_{\mathrm{Bt}}(Ca; d) \wedge (\Phi \to \Psi)], \\
(\forall x\, \Phi)^{\mathrm{Bt}} & :\equiv \tilde{\exists}^{\mathrm{st}} a\, \tilde{\forall}^{\mathrm{st}} b\, [\forall x\, \Phi_{\mathrm{Bt}}(a; b)], \\
(\exists x\, \Phi)^{\mathrm{Bt}} & :\equiv \tilde{\exists}^{\mathrm{st}} a\, \tilde{\forall}^{\mathrm{st}} c\, [\exists x\, \tilde{\forall} b \leq^* c\, \Phi_{\mathrm{Bt}}(a; b)].
\end{array}
$$

Factorization $\mathrm{Bt} = \mathrm{Bq} \wedge \mathrm{id}$

Theorem

For all formulas Φ *of* $\text{E-HA}^{\omega}_{\mathrm{st}}$, *we have*

$$\text{E-HA}^{\omega}_{\mathrm{st}} \vdash \tilde{\forall}^{\mathrm{st}} a, b \, (\Phi_{\mathrm{Bt}}(a; b) \leftrightarrow \Phi_{\mathrm{Bq}}(a; b) \wedge \Phi).$$

Soundnesses of Bq and Bt

Theorem

For all formulas Φ *of* $\mathsf{E}\text{-}\mathsf{HA}^{\omega}_{\mathrm{st}}$, *if*

$$\mathsf{P} \vdash \Phi,$$

then there are closed monotone terms t *such that*

$$\mathsf{P} \vdash \tilde{\forall}^{\mathrm{st}} b\, \Phi_{\mathrm{Bq}}(t; b),$$
$$\mathsf{P} \vdash \tilde{\forall}^{\mathrm{st}} b\, \Phi_{\mathrm{Bt}}(t; b).$$

Abbreviation

$\mathsf{P} := \mathsf{E}\text{-}\mathsf{HA}^{\omega}_{\mathrm{st}} \pm \mathsf{mAC}^{\omega} \pm \mathsf{R}^{\omega} \pm \mathsf{I}^{\omega} \pm \mathsf{IP}^{\omega}_{\tilde{\forall}^{\mathrm{st}}} \pm \mathsf{M}^{\omega} \pm \mathsf{BUD}^{\omega} \pm \mathsf{MAJ}^{\omega}$.

Unfortunately, we are unable to present a theorem that characterizes the *least* theory containing E-$\mathsf{HA}^{\omega}_{\mathrm{st}}$ and proving $\Phi^{\mathrm{Bq}} \leftrightarrow \Phi$ for all formulas Φ of E-$\mathsf{HA}^{\omega}_{\mathrm{st}}$) and Bt.

A parametrised approach

Herbrandized functional interpretation (for IL).
Let $\mathcal{T}_{\mathbf{P}} = \mathcal{S}_{\mathbf{P}} = \mathsf{HA}^{\omega}_{\mathrm{st}}$. We instantiate the three parameters as follows:

$$
\begin{array}{lcl}
x \prec_{\mathrm{st}} a & \equiv & \exists^{\mathrm{st}} a \wedge (x \in a) \\
x \prec_{\tau} a & \equiv & \tau^*(a) \wedge (x \in a) \\
\forall x \sqsubset a\, A & \equiv & \forall x \in a\, A \\
\mathsf{W}(x) & \equiv & \tau^*(x)
\end{array}
$$

Proposition (Herbrandized functional interpretation)

With the parameters instantiated as above we have:

$$
\begin{array}{lcl}
\{\{\tau(x)\}\}^{a} & \Leftrightarrow & \exists^{\mathrm{st}} a \wedge (x \in a) \\
\{\{A \to B\}\}^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{w}} & \Leftrightarrow & \forall \boldsymbol{y} \in \boldsymbol{f}\boldsymbol{x}\boldsymbol{w}\{\{A\}\}^{\boldsymbol{x}}_{\boldsymbol{y}} \to \{\{B\}\}^{\boldsymbol{g}\boldsymbol{x}}_{\boldsymbol{w}} \\
\{\{A \wedge B\}\}^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} & \Leftrightarrow & \{\{A\}\}^{\boldsymbol{x}}_{\boldsymbol{y}} \wedge \{\{B\}\}^{\boldsymbol{v}}_{\boldsymbol{w}} \\
\{\{A \vee B\}\}^{b,\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} & \Leftrightarrow & \exists z \prec_{\mathrm{B}'} b(\forall^{\mathrm{st}} \boldsymbol{y}' \in \boldsymbol{y}\{\{A\}\}^{\boldsymbol{x}}_{\boldsymbol{y}'} \lozenge_z \forall^{\mathrm{st}} \boldsymbol{w}' \in \boldsymbol{w}\{\{B\}\}^{\boldsymbol{v}}_{\boldsymbol{w}'}) \\
\{\{\exists z^{\tau} A\}\}^{\boldsymbol{x}}_{\boldsymbol{y}} & \Leftrightarrow & \exists z^{\tau} \forall \boldsymbol{y}' \in \boldsymbol{y}\{\{A\}\}^{\boldsymbol{x}}_{\boldsymbol{y}'} \\
\{\{\forall z^{\tau} A\}\}^{\boldsymbol{x}}_{\boldsymbol{y}} & \Leftrightarrow & \forall z^{\tau}\, \{\{A\}\}^{\boldsymbol{x}}_{\boldsymbol{y}}
\end{array}
$$

so that $\{\{A\}\}^{\boldsymbol{x}}_{\boldsymbol{y}}$ *can be seen to correspond to* $A_{D_{\mathrm{st}}}(\boldsymbol{x}; \boldsymbol{y})$.

Some questions and work in progress

- What applications are there for the interpretations with truth? Can they give additional information about Transfer?

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- What applications are there for the interpretations with truth? Can they give additional information about Transfer?
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Thank you!