Representation of Borel Probability Measures and Characterization of computability of Brownian motion

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Measure Space and Realization

Definition

Let $(X, \mu)$ and $(Y, \nu)$ be measure space and $F$ be measurable partial mapping $F : \subseteq X \rightarrow Y$. $\nu$ is called push forward measure if $\mu(F^{-1}[V]) = \nu(V)$. We say $F$ realizes $\nu$ on $\mu$ and write $\nu \preceq \mu$.

Example

- Consider $X = [0, 1]$ equipped with Lebesgue measure $\lambda$.
- Consider Cantor space $C = \{0, 1\}^\mathbb{N}$ equipped with canonical fair measure $\gamma$.
- Binary representation $\rho_b : C \rightarrow X$ realizes $\lambda$ on $\gamma$: $\lambda \preceq \gamma$. 
Definition

Let \((X, \mu)\) and \((Y, \nu)\) be measure space and \(F\) be measurable partial mapping \(F : \subseteq X \rightarrow Y\). \(\nu\) is called **push forward measure** if \(\mu(F^{-1}[V]) = \nu(V)\). We say \(F\) **realizes** \(\nu\) on \(\mu\) and write \(\nu \preceq \mu\).
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- Binary representation \(\rho_b : C \rightarrow X\) realizes \(\lambda\) on \(\gamma\): \(\lambda \preceq \gamma\).
Fact (Schröder,Simpson 2006)

Let $X$ be second countable $T_0$ space with Borel probability measure $\mu$. Then there exists Borel probability measure $\bar{\gamma}$ on $C$ s.t. $\mu$ has continuous partial realizer over $\bar{\gamma}$. 

Same fair coin flip

```
0
/|
0 1
/|  \
0 0 1
/  \\
1 1
```
Canonical measure and Realizer

**Fact (Schröder, Simpson 2006)**

Let $X$ be second countable $T_0$ space with Borel probability measure $\mu$. Then there exists Borel probability measure $\bar{\gamma}$ on $C$ s.t. $\mu$ has continuous partial realizer over $\bar{\gamma}$.

Using different *unfair* coin every time!
Canonical measure and Realizer

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Let $X$ be second countable $T_0$ space with Borel probability measure $\mu$. Then there exists Borel probability measure $\tilde{\gamma}$ on $C$ s.t. $\mu$ has continuous partial realizer over $\tilde{\gamma}$.

We want *fair* probability measure instead of arbitrary one!
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We want fair probability measure instead of arbitrary one!

\[
\sigma = 01110101101\ldots \quad \rightarrow \quad \text{TM} \quad \rightarrow \quad 3.141592\ldots
\]
First main result

Fact (Schröder, Simpson 2006)
Let $X$ be second countable $T_0$ space with Borel probability measure $\mu$. Then there exists Borel probability measure $\bar{\gamma}$ on $C$ s.t. $\mu$ has continuous partial realizer over $\bar{\gamma}$.

Theorem (First main result)
For every Borel probability measure $\bar{\gamma}$ on $C$, there exists an partial continuous realizer $F$ which realizes $\bar{\gamma}$ on fair measure $\gamma$.

\[
(C, \gamma) \xrightarrow{F} (C, \bar{\gamma}) \xrightarrow{G} (X, \mu)
\]

\[
\gamma(F^{-1}(G^{-1}(V))) \quad \mu(V) = \bar{\gamma}(G^{-1}(V)) = \bar{\gamma}(G^{-1}(V))
\]
First main result

Fact (Schröder, Simpson 2006)
Let $X$ be second countable $T_0$ space with Borel probability measure $\mu$. Then there exists Borel probability measure $\bar{\gamma}$ on $\mathcal{C}$ s.t. $\mu$ has continuous partial realizer over $\bar{\gamma}$.

Theorem (First main result)
For every Borel probability measure $\bar{\gamma}$ on $\mathcal{C}$, there exists an partial continuous realizer $F$ which realizes $\bar{\gamma}$ on \textbf{fair} measure $\gamma$.

Corollary
Let $X$ be second countable $T_0$ space with Borel probability measure $\mu$. Then there exists an partial continuous realizer $F$ which realizes $\mu$ on $\gamma$. 
Proof sketch of theorem

**Theorem**

For every Borel probability measure $\bar{\gamma}$ on $C$, there exists an almost surely continuous realizer $F : C \to C$ which realizes $\bar{\gamma}$ on fair measure $\gamma$. 

\[
\bar{\omega}C \subseteq (C, \bar{\gamma}) \\
\bar{\gamma}(\bar{\omega}C) = \lambda(I) \\
\rho_b^{-1}(I) \subseteq (C, \gamma)
\]
Proof sketch of theorem

**Theorem**

For every Borel probability measure $\bar{\gamma}$ on $\mathcal{C}$, there exists an almost surely continuous realizer $F : \mathcal{C} \rightarrow \mathcal{C}$ which realizes $\bar{\gamma}$ on fair measure $\gamma$. 
Proof sketch of theorem

**Theorem**

For every Borel probability measure $\bar{\gamma}$ on $\mathbb{C}$, there exists an almost surely continuous realizer $F : \mathbb{C} \rightarrow \mathbb{C}$ which realizes $\bar{\gamma}$ on fair measure $\gamma$.

$$\lambda \left( F_n \left( \overline{wC} \right) \right) = \bar{\gamma} \left( \overline{wC} \right)$$
Proof sketch of theorem

**Theorem**

For every Borel probability measure $\bar{\gamma}$ on $C$, there exists an almost surely continuous realizer $F : C \to C$ which realizes $\bar{\gamma}$ on fair measure $\gamma$.

$\{0, 1\}^3$

010 011

$\bar{\gamma}(010C)$  $\bar{\gamma}(011C)$

$F_{n+1} (\vec{w}0) \cup F_{n+1} (\vec{w}1) \subseteq F_n (\vec{w})$

$\Rightarrow \lim_{n} F_n$ is well defined!
Proof sketch of theorem

Theorem

For every Borel probability measure $\tilde{\gamma}$ on $\mathcal{C}$, there exists an almost surely continuous realizer $F : \mathcal{C} \to \mathcal{C}$ which realizes $\tilde{\gamma}$ on fair measure $\gamma$.

Proof (Cont.)

So, $\gamma(\rho_b^{-1}(F(\tilde{w}C))) = \tilde{\gamma}(\tilde{w}C)$ for every $\tilde{w} \in \{0, 1\}^*$. And we can extend this result to hold for every Borel subset of $\mathcal{C}$. It means $(\rho_b^{-1} \circ F)^{-1}$ realizes $\tilde{\gamma}$ on $\gamma$. □
Computability of measure

Definition

Let \((X, \mu)\) be measure space and \(\xi\) be representation of \(X\). A mapping \(G : \subseteq C \rightarrow \text{dom}(\xi)\) is said to be \(\xi -\)realizer of \(\mu\) if \(\xi \circ G : \subseteq C \rightarrow X\) realizes \(\mu\) on \(\gamma\), which is canonical fair measure of \(C\). If \(G\) is computable, then we’ll call \(\mu\) is \(\xi\)-computable measure.
Example of Realizer

Example

1. The identity on \( \text{dom}(\rho_b) \subseteq C \) is \( \rho_b \)-realizer of the Lebesgues measure \( \mu \) on \([0, 1]\).

2. Dirac distribution \( \delta_r \) is \( \rho \)-computable iff \( r \) is \( \rho \)-computable.

3. Let \( F \) be Gaussian CDF. Its inverse \( F^{-1} \) is realizer of Gaussian measure \( \mu \) on Lebesgue measure \( \lambda \). Then the mapping \( G \) below is \( F^{-1} \circ \xi \)-realizer of Gaussian measure \( \mu \).
The identity on \(\text{dom}(\rho_b) \subseteq \mathcal{C}\) is \(\rho_b\)-realizer of the Lebesgue measure \(\mu\) on \([0, 1]\).

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Example of Realizer

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1. The identity on $\text{dom}(\rho_b) \subseteq \mathcal{C}$ is $\rho_b$-realizer of the Lebesgues measure $\mu$ on $[0, 1]$.

2. Dirac distribution $\delta_r$ is $\rho$-computable iff $r$ is $\rho$-computable.

3. Let $F$ be Gaussian CDF. Its inverse $F^{-1}$ is realizer of Gaussian measure $\mu$ on Lebesgue measure $\lambda$. Then the mapping $G$ below is $F^{-1} \circ \xi$-realizer of Gaussian measure $\mu$.

\[
\begin{array}{ccc}
(C, \gamma) & \xrightarrow{G} & \text{dom}(\xi) \\
 & F^{-1} \circ \xi \circ G & \xrightarrow{F^{-1}} (\mathbb{R}, \mu) \\
\end{array}
\]
Semi-inverse of CDF

**Definition**

Let \((\mathbb{R}, \mathcal{A}, \mu)\) be measure space. Recall cumulative distribution function of \(\mu\) is \(\mathbb{R} \ni s \mapsto \mu\left((-\infty, s]\right) \in [0, 1]\). The upper and lower semi-inverse of cumulative distribution function are

\[
F^\mu_\geq : (0, 1) \ni t \mapsto \inf \left\{ s \in \mathbb{R} \mid \mu\left((-\infty, s]\right) > t \right\}
\]

\[
F^\mu_\leq : (0, 1) \ni t \mapsto \sup \left\{ s \in \mathbb{R} \mid \mu\left((-\infty, s]\right) < t \right\}
\]
Semi-inverse of CDF

Lemma

$F_{\mu}^\mu >$ is upper-semicontinuous and $F_{\lambda}^\mu <$ is lower-semicontinuous. Both of them realize $(\mathbb{R}, \mathcal{A}, \mu)$ on $([0, 1], \mathcal{B}, \lambda)$.

Figure: Cumulative distribution function with upper/lower semi-inverse
Characterization of computable measure on Reals

**Lemma**

$F^\mu_>$ is upper-semicontinuous and $F^\mu_<$ is lower-semicontinuous. Both of them realize $(\mathbb{R}, \mathcal{A}, \mu)$ on $([0, 1], \mathcal{B}, \lambda)$.

**Theorem (Our result)**

Let $\mu$ be the Borel probability measure on $\mathbb{R}$ and $F^\mu_<$, $F^\mu_>$ be lower and upper semi inverse of its cumulative distribution function. $\mu$ is $\rho$-computable iff $F^\mu_<$ is $(\rho|^{[0,1]}, \rho_<)$-computable and $F^\mu_>$ is $(\rho|^{[0,1]}, \rho_>)$-computable.
Brownian motion

Definition

1D Brownian motion, or Wiener process, or Wiener measure is Borel probability measure on the space $C[0, 1]$ which satisfies following conditions.

1. $W(0) = 0$ with probability 1.
2. For every $0 \leq r < s < t$, $W(t) - W(s)$ is independent of $W(r)$.
3. $W(t) - W(s)$ is normally distributed with mean 0 and variance $|t - s|$.

Main question: Is this probability measure computable?
Sample path of Brownian motion
Computability of measure

Definition

Let \((X, \mu)\) be measure space and \(\xi\) be representation of \(X\). A mapping \(G : \subseteq C \rightarrow \text{dom}(\xi)\) is said to be \(\xi - \text{realizer}\) of \(\mu\) if \(\xi \circ G : \subseteq C \rightarrow X\) realizes \(\mu\) on \(\gamma\), which is canonical fair measure of \(C\). If \(G\) is computable, then we'll call \(\mu\) is \(\xi\)-computable measure.
Algorithm to compute Brownian motion

Canonical representation of $C[0, 1]$(Wei00, §6.1) contains two information.
- Value of function $f(a/2^n)$ for every dyadic rationals.
- A binary modulus of continuity $moc$. 
Algorithm to compute Brownian motion

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$$\left|W(t) - W(s)\right| \sim N(0, |t - s|)$$

$N(0, 1)$
Algorithm to compute Brownian motion

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\[
\begin{align*}
\mathbb{N}(0, 1) \\
\frac{1}{2} & \quad 1
\end{align*}
\]
Algorithm to compute Brownian motion

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![Diagram](image-url)
Algorithm to compute Brownian motion

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![Diagram](Not bounded with non-zero probability!)

$N(0, 1)$
Algorithm to compute Brownian motion

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![Diagram showing violation of moc](image-url)
Algorithm to compute Brownian motion

Canonical representation of $C[0, 1](\text{Wei00, §6.1})$ contains two information.

- Value of function $f(a/2^n)$ for every dyadic rationals.
- A binary modulus of continuity $moc$.

Sample again!
Computability of Brownian motion

Canonical representation of $C[0, 1]$ contains two information.

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- A binary modulus of continuity $moc$. 
Computability of Brownian motion

Canonical representation of $C[0, 1]$ contains two information.

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Fact (Lèvy’s modulus of continuity theorem)

$$\lim_{h \to 0} \sup_{|s-t| \leq h} \frac{|W(s) - W(t)|}{\sqrt{2h \ln 1/h}} = 1$$

with probability 1.
Parameterized modulus of continuity

Fact (Lévy’s modulus of continuity theorem)

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\lim_{h \to 0} \sup_{|s - t| \leq h} \frac{|W(s) - W(t)|}{\sqrt{2h \ln 1/h}} = 1
\]

with probability 1.

Lemma

Let \( y_c = \sqrt{2 \ln (ec)}/c \). For every \( W \in (C[0, 1], \mu) \), \( W \) has parameterized modulus of continuity \( \omega \) with the smallest parameter \( c = c(W) \geq 1 \) s.t.

\[
\omega(h, c) = \begin{cases} 
\sqrt{2ch \ln (1/h)} & : h \leq 1/ec \\
y_c + (h - 1/ec) \cdot c \cdot \ln (c)/y_c & : h \geq 1/ec 
\end{cases}
\]
Parameterized modulus of continuity

Lemma

Let \( y_c = \sqrt{2 \ln(ec)}/c \). For every \( W \in (C[0, 1], \mu) \), \( W \) has parameterized modulus of continuity \( \omega \) with the smallest parameter \( c = c(W) \geq 1 \) s.t.

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$$\omega(h, c) = \begin{cases} 
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y + (h - 1/ec) \cdot c \cdot \ln(c)/y & : h \geq 1/ec 
\end{cases}$$
Characterization of Computability

Lemma

Let $y_c = \sqrt{2 \ln (ec) / c}$. For every $W \in (C[0, 1], \mu)$, $W$ has parameterized modulus of continuity $\omega$ with smallest parameter $c = c(W) \geq 1$ s.t.

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Theorem (Second main result)

*The Wiener measure is computable iff the random variable $c$ has a computable probability distribution.*
**Lemma**

For every $W \in (C[0,1], \mu)$, $W$ has parameterized modulus of continuity $\omega(\bullet, c)$ with smallest parameter $c = c(W) \geq 1$.

**Theorem (Second main theorem)**

The Wiener measure is computable iff the random variable $c$ has a computable probability distribution.

**Theorem**

Let $\tilde{\omega} : [0, 1] \times [1, \infty) \to [0, \infty)$ denote any strictly increasing computable which works as parameterized modulus of continuity of Wiener process. The Wiener measure is computable if and only if there exists a random variable $\tilde{c}$ with computable probability distribution.
Thank you!