Representation of Borel Probability Measures and Characterization of computability of Brownian motion

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Table

Measure Space and Realization

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Example

- Consider $X = [0, 1]$ equipped with Lebesgue measure λ.
- Consider Cantor space $\mathcal{C} = \{0, 1\}^{\mathbb{N}}$ equipped with canonical **fair** measure γ .
- Binary representation $\rho_b : \mathcal{C} \to X$ realizes λ on γ: $\lambda \preccurlyeq \gamma$.

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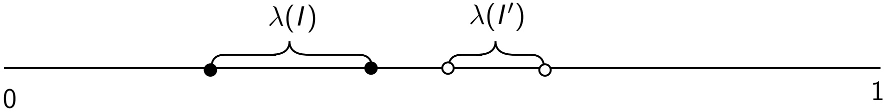

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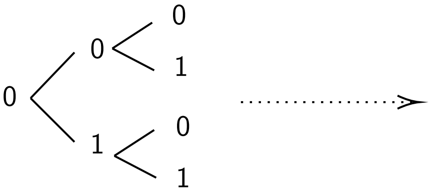

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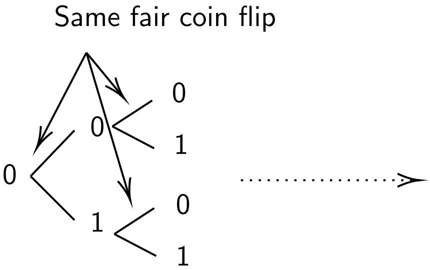
Canonical measure and Realizer

Fact (Schröder,Simpson 2006)

Let X be second countable T_0 space with Borel probability measure μ. Then there exists Borel probability measure $\bar{\gamma}$ on $\mathcal{C}$ s.t. μ has continuous partial realizer over $\bar{\gamma}$.



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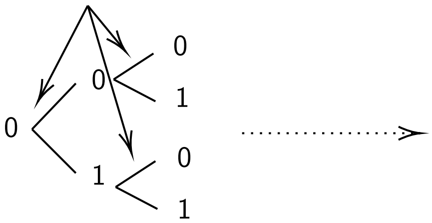
Using different *unfair* coin every time!



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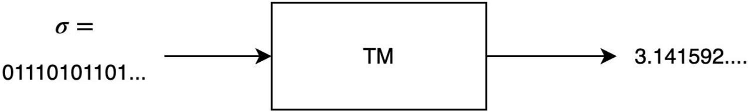

First main result

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Theorem (First main result)

For every Borel probability measure $\bar{\gamma}$ on $\mathcal{C}$, there exists an partial continuous realizer F which realizes $\bar{\gamma}$ on **fair** *measure γ.*

$$(\mathcal{C}, \gamma) \xrightarrow{\quad F \quad} (\mathcal{C}, \overline{\gamma}) \xrightarrow{\quad G \quad} (X, \mu)$$

$$\gamma\left(F^{-1}\left(G^{-1}(V)\right)\right) = \overline{\gamma}(G^{-1}(V)) \qquad \mu(V) = \overline{\gamma}(G^{-1}(V))$$

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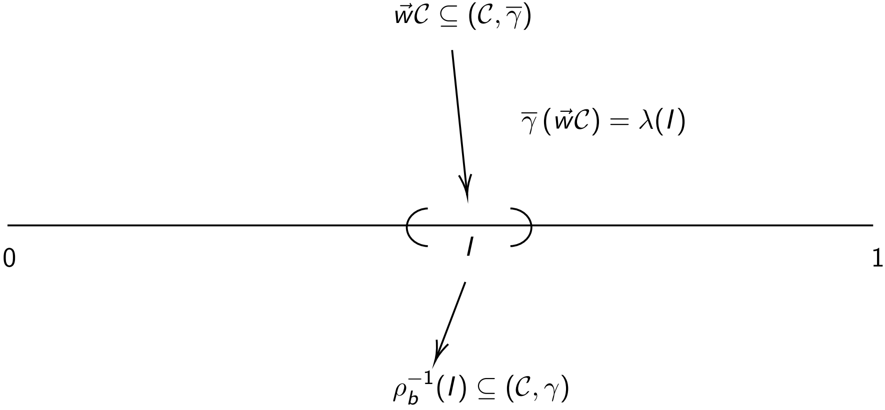
Corollary

Let X be second countable T_0 space with Borel probability measure μ. Then there exists an partial continuous realizer F which realizes μ on γ.

Proof sketch of theorem

Theorem

For every Borel probability measure $\bar{\gamma}$ on $\mathcal{C}$, there exists an almost surely continuous realizer $F : \mathcal{C} \to \mathcal{C}$ which realizes $\bar{\gamma}$ on fair measure γ.



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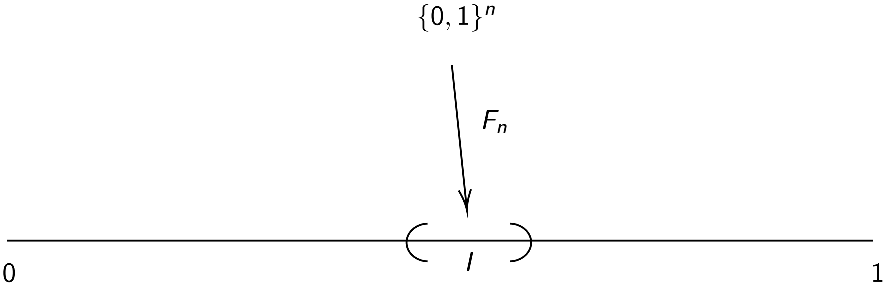

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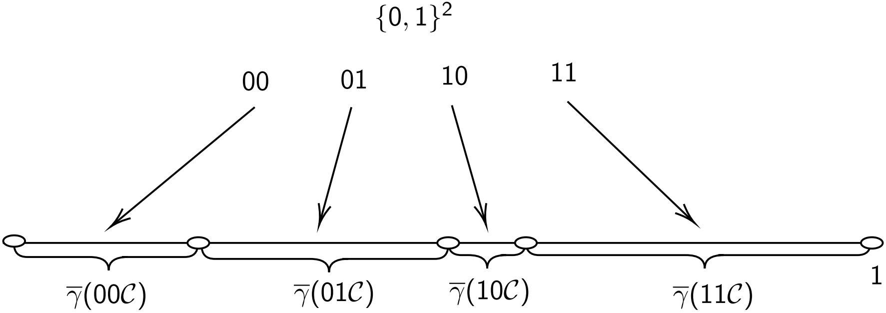

$$\lambda\left(F_n\left(\overline{w}\mathcal{C}\right)\right) = \overline{\gamma}\left(\overline{w}\mathcal{C}\right)$$

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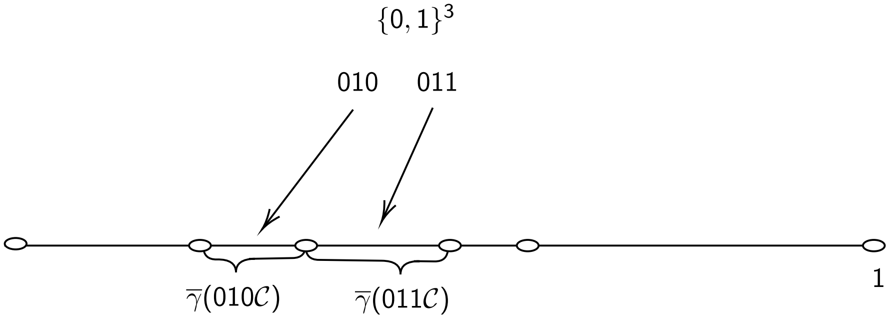

$$F_{n+1}(\vec{w}0) \cup F_{n+1}(\vec{w}1) \subseteq F_n(\vec{w})$$

$$=> \lim_n F_n \text{ is well defined!}$$

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Theorem

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Proof (Cont.)

So, $\gamma(\rho_b^{-1}(F(\vec{w}\mathcal{C}))) = \bar{\gamma}(\vec{w}\mathcal{C})$ for every $\vec{w} \in \{0,1\}^*$. And we can extend this result to hold for every Borel subset of $\mathcal{C}$. It means $(\rho_b^{-1} \circ F)^{-1}$ realizes $\bar{\gamma}$ on γ. □

Computability of measure

Definition

Let (X, μ) be measure space and ξ be representation of X.
A mapping $G :\subseteq \mathcal{C} \to dom(\xi)$ is said to be $\xi -$ *realizer* of μ if $\xi \circ G :\subseteq \mathcal{C} \to X$ realizes μ on γ, which is canonical fair measure of $\mathcal{C}$.
If G is computable, then we'll call μ is ξ-computable measure.

$$
\begin{tikzcd}
(\mathcal{C}, \gamma) \arrow[d, "G"'] \arrow[dr, "\xi \circ G"] & \\
dom(\xi) \arrow[r, "\xi"'] & (X, \mu)
\end{tikzcd}
$$

Example of Realizer

Example

1. The identity on $\mathrm{dom}(\rho_b) \subseteq \mathcal{C}$ is ρ_b-realizer of the Lebesgues measure μ on $[0, 1]$.
2. Dirac distribution δ_r is ρ-computable iff r is ρ-computable.
3. Let F be Gaussian CDF. Its inverse F^{-1} is realizer of Gaussian measure μ on Lebesgue measure λ. Then the mapping G below is $F^{-1} \circ \xi$-realizer of Gaussian measure μ.

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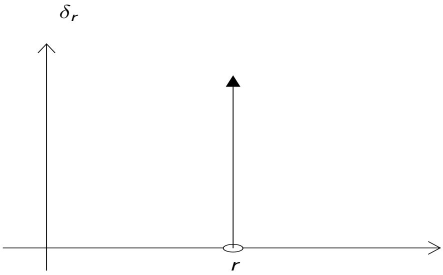

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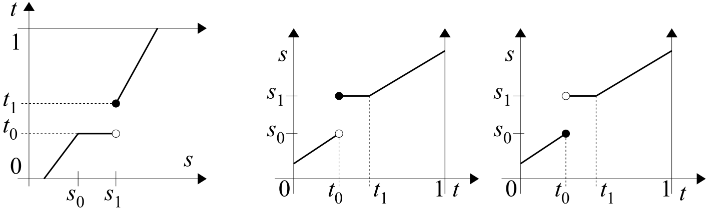
$$\begin{tikzcd}
(\mathcal{C}, \gamma) \arrow[d, "G"'] \arrow[rrd, "F^{-1} \circ \xi \circ G"] & & \\
dom(\xi) \arrow[r, "\xi"'] & ([0,1], \lambda) \arrow[r, "F^{-1}"'] & (\mathbb{R}, \mu)
\end{tikzcd}$$

Semi-inverse of CDF

Definition

Let $(\mathbb{R}, \mathcal{A}, \mu)$ be measure space. Recall cumulative distribution function of μ is $\mathbb{R} \ni s \mapsto \mu\big((-\infty, s]\big) \in [0, 1]$. The upper and lower semi-inverse of cumulative distribution function are

$$\begin{aligned} F^{\mu}_{>} : (0,1) \ &\ni\ t \ \mapsto \inf\left\{s \in \mathbb{R} \,\middle|\, \mu\big((-\infty, s)\big) > t\right\} \\ F^{\mu}_{<} : (0,1) \ &\ni\ t \ \mapsto \sup\left\{s \in \mathbb{R} \,\middle|\, \mu\big((-\infty, s]\big) < t\right\} \end{aligned}$$



Semi-inverse of CDF

Lemma

$F^{\mu}_{>}$ is upper-semicontinuous and $F^{\mu}_{<}$ is lower-semicontinuous. Both of them realize $(\mathbb{R}, \mathcal{A}, \mu)$ on $\big([0,1], \mathcal{B}, \lambda\big)$.

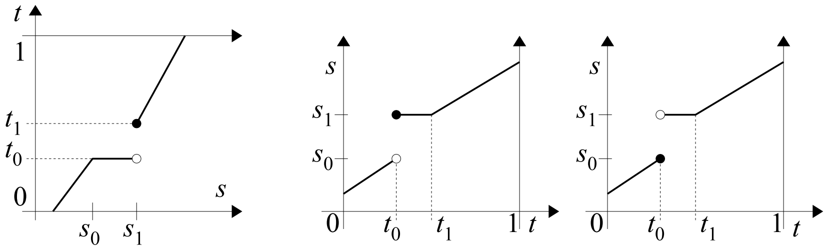


Figure: Cumulative distribution function with upper/lower semi-inverse

Characterization of computable measure on Reals

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Theorem (Our result)

Let μ be the Borel probability measure on $\mathbb{R}$ and $F^{\mu}_{<}$,$F^{\mu}_{>}$ be lower and upper semi inverse of its cumulative distribution function. μ is ρ-computable iff $F^{\mu}_{<}$ is $(\rho|^{[0,1]}, \rho_{<})$-computable and $F^{\mu}_{>}$ is $(\rho|^{[0,1]}, \rho_{>})$-computable

Brownian motion

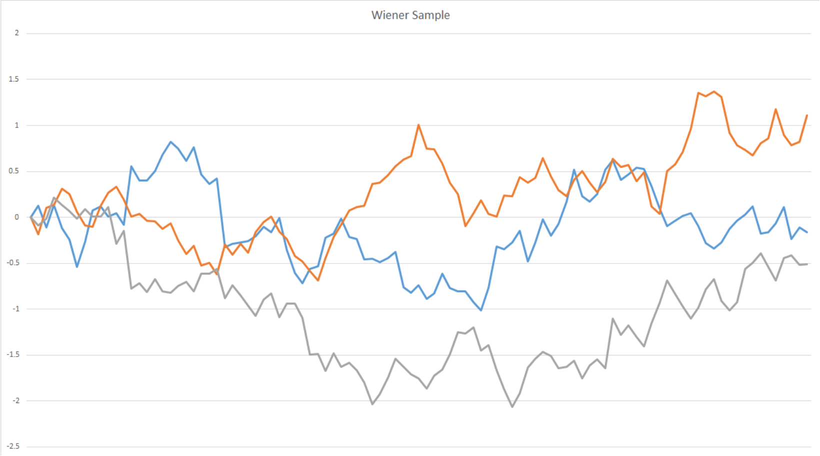
Definition

1D *Brownian motion*, or *Wiener process*, or *Wiener measure* is Borel probability measure on the space $\mathcal{C}[0, 1]$ which satisfies following conditions.

1. $W(0) = 0$ with probability 1.
2. For every $0 \leq r < s < t$, $W(t) - W(s)$ is independent of $W(r)$.
3. $W(t) - W(s)$ is normally distributed with mean 0 and variance $|t - s|$.

Main question : Is this probability measure computable?

Sample path of Brownian motion



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Algorithm to compute Brownian motion

Canonical representation of $\mathcal{C}[0,1]$(Wei00,§6.1) contains two information.

- Value of function $f(a/2^n)$ for every dyadic rationals.
- A binary modulus of continuity *moc*.

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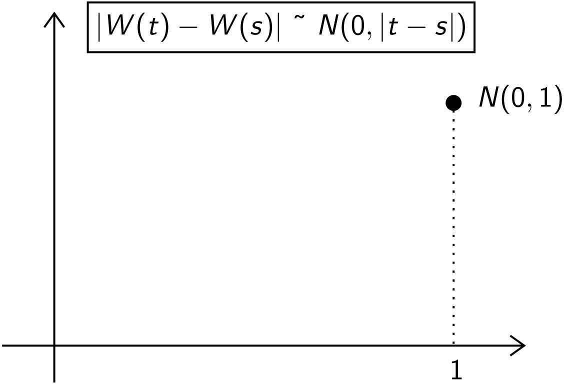

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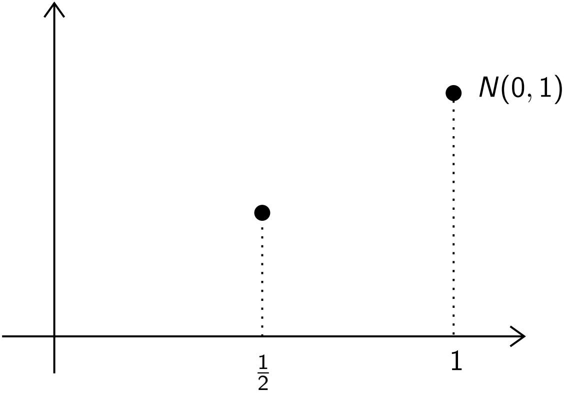

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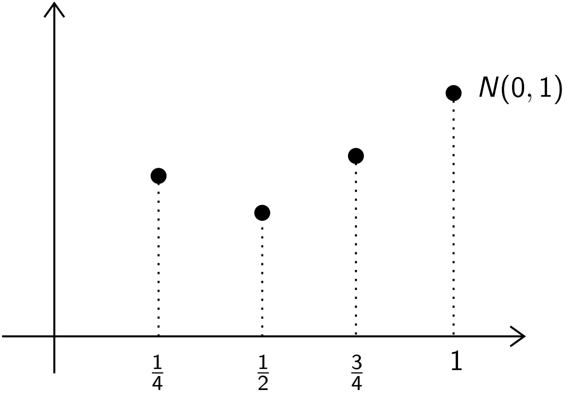

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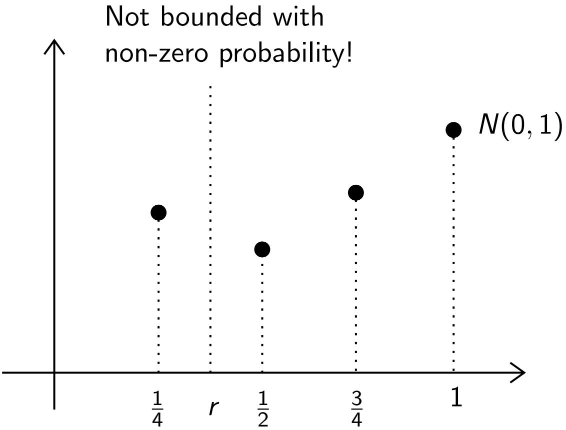

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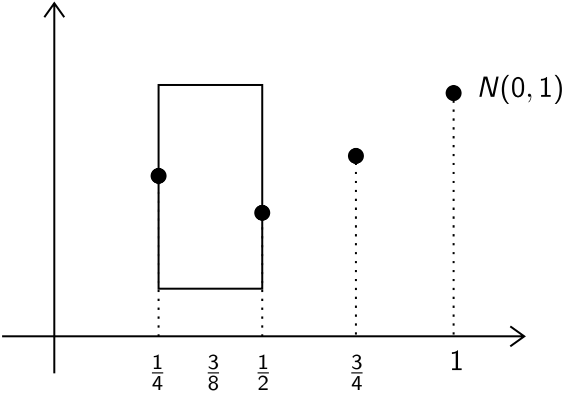

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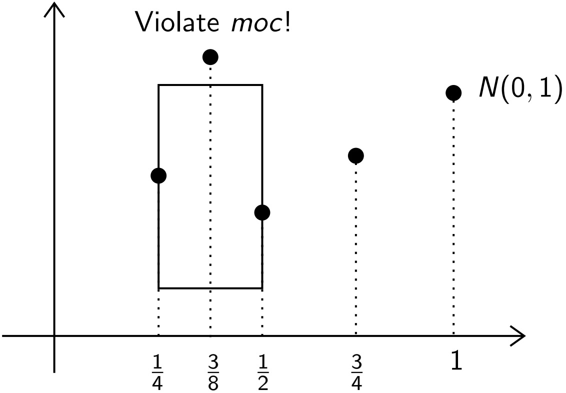

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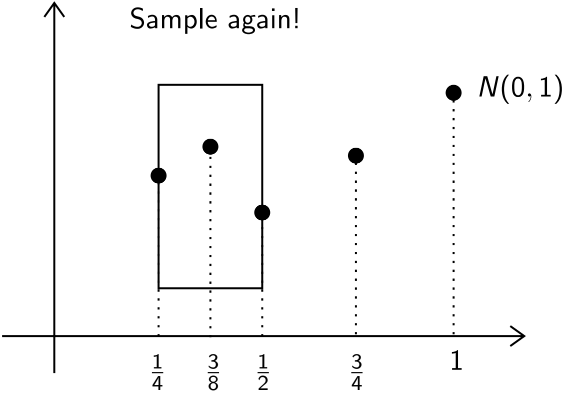

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Fact (Lèvy's modulus of continuity theorem)

$$\lim_{h \to 0} \sup_{|s-t| \leq h} \frac{|W(s) - W(t)|}{\sqrt{2h \ln 1/h}} = 1$$

with probability 1.

Parameterized modulus of continuity

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Lemma

Let $y_c = \sqrt{2\ln(ec)/c}$*. For every* $W \in (\mathcal{C}[0,1],\mu)$*,* W *has parameterized modulus of continuity* ω *with the smallest parameter* $c = c(W) \geq 1$ *s.t.*

$$\omega(h,c) = \begin{cases} \sqrt{2ch\ln(1/h)} & : \ h \leq 1/ec \\ y_c + (h - 1/ec)\cdot c \cdot \ln(c)/y_c & : \ h \geq 1/ec \end{cases}$$

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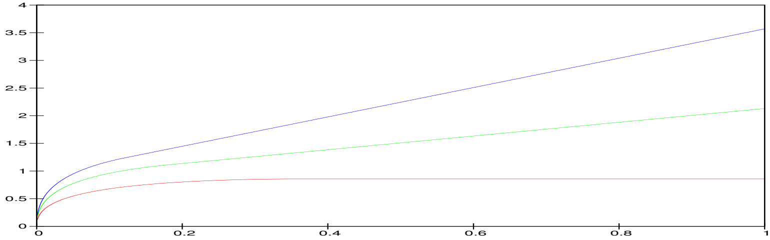

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Theorem (Second main result)

The Wiener measure is computable iff the random variable c *has a* **computable probability distribution**.

Characterization of Computability

Lemma

For every $W \in (\mathcal{C}[0,1], \mu)$, W has parameterized modulus of continuity $\omega(\bullet, c)$ with smallest parameter $c = c(W) \geq 1$.

Theorem (Second main theorem)

The Wiener measure is computable iff the random variable c has a **computable probability distribution**.

Theorem

Let $\tilde{\omega} : [0,1] \times [1, \infty) \to [0, \infty)$ denote **any** *strictly increasing computable which works as parameterized modulus of continuity of Wiener process. The Wiener measure is computable if and only if there exists a random variable $\tilde{c}$ with* **computable probability distribution**.

Thank you!