W. C. POWELL / 1977

We confine ourselves to algebras

$$\mathfrak{A} = \langle A, \bigwedge, \bigvee, \Longrightarrow, \wedge, \vee, f_i \rangle_{i \in I}$$

with distinguished binary operations $\Longrightarrow, \wedge, \vee$, and [second?]order operations $\bigwedge, \bigvee$ defined on subsets of A, in addition to possibly other ν_i-ary operations f_i on A for i in some index set I. Strictly speaking, $\bigwedge$ and $\bigvee$ [? ? ?] a family of first order operations on A, one for each cardinality $\leq 2^{|A|}$.

Assume the signature of the algebras are fixed throughout the discussion.

Terms of the algebras are built up from variables a,b,... that range over A [! I think, what he means is that we have a precise set of variables: the variables are the same as the elements of A; each variable does not "range" at all] as follows:

If X is a set of terms, $\bigwedge X$ and $\bigvee X$ are terms.
If σ and τ are terms, $\sigma \Longrightarrow \tau$, $\sigma \wedge \tau$, $\sigma \vee \tau$ are terms.
If σ_ξ is a term for each $\xi < \nu_i$, $f_i(\langle \sigma_\xi : \xi < \nu_i \rangle)$ is a term.

Formulas of the algebra are built up from equations $\sigma = \tau$ between terms of the algebra as follows:

If φ and ψ are formulas, then $\forall a\varphi$, $\exists a\varphi$, $\varphi \rightarrow \psi$, $\varphi \wedge \psi$, $\varphi \vee \psi$ are formulas.
If φ_ξ is a formula for each $\xi < \nu_i$, then $\overline{f}_i(\langle \varphi_\xi : \xi < \nu_i \rangle$ is a formula.

Def's: $1 \overset{\text{def}}{=} \bigvee A$ $\qquad \bot \overset{\text{def}}{=} 0=1$

[1 is a term; $\bot$ is a formula. Are we to consider each $a \in A$ a constant denoting itself? Not yet; so far, the grammar is OK].

$0 \overset{\text{def}}{=} \bigwedge A$ [term] $\qquad \neg\varphi \overset{\text{def}}{=} \varphi \rightarrow \bot$ [formula]

$a \Longleftrightarrow b \overset{\text{def}}{=} (a \Longrightarrow b) \wedge (b \Longrightarrow a)$ [term]

$$\sim a \overset{\text{def}}{=} a \Rightarrow 0 .$$ [term]

Define an interpretation $[\![\ ]\!]$ of formulas of the algebra as terms of the algebra as follows:

$$[\![\sigma=\tau]\!] \overset{\text{def}}{=} \sigma \Leftrightarrow \tau$$

$$[\![\forall a \varphi]\!] \overset{\text{def}}{=} \bigwedge \{ [\![\varphi(a)]\!] : a \in A \}$$

$$[\![\exists a \varphi]\!] \overset{\text{def}}{=} \bigvee \{ [\![\varphi(a)]\!] : a \in A \}$$

$$[\![\varphi \to \psi]\!] \overset{\text{def}}{=} [\![\varphi]\!] \Rightarrow [\![\psi]\!]$$

$$[\![\varphi \wedge \psi]\!] \overset{\text{def}}{=} [\![\varphi]\!] \wedge \psi]\!]$$

$$[\![\varphi \vee \psi]\!] \overset{\text{def}}{=} [\![\varphi]\!] \vee \psi]\!]$$

$$[\![f_i(\langle \varphi_\xi : \xi < v_i \rangle)]\!] \overset{\text{def}}{=} \underline{f_i}(\langle [\![\varphi_\xi]\!] : \xi < v_i \rangle .$$

We say $\mathcal{F}$ *is a (truth) filter* on $\mathfrak{A}$ if [!?: $\mathcal{F}$ is a set of terms and] $1 \in \mathcal{F}$, $0 \notin \mathcal{F}$ and

$$a \in \mathcal{F} \wedge a \Rightarrow b \in \mathcal{F} \longrightarrow b \in \mathcal{F} .$$

[Here I start having my doubts. Are we, after all identifying terms with their values in A, whereby each $a \in A$ denotes itself (has value itself)? Why did he not write $\sigma \in \mathcal{F} \wedge \sigma \Rightarrow \tau \in \mathcal{F} \longrightarrow \tau \in \mathcal{F}$?]

We say a formula φ of the algebra is $\mathcal{F}$-valid if $[\![\varphi]\!] \in \mathcal{F}$.

The following terms are called terms of a complete Heyting algebra (cHa terms):

$$\bigwedge_i (a_i \Rightarrow (b_i \Rightarrow a_i)$$
$$\bigwedge_i (a_i \Rightarrow (b_i \Rightarrow c_i)) \Rightarrow ((a_i \Rightarrow b_i) \Rightarrow (a_i \Rightarrow c_i))$$
$$\bigwedge_i ((a_i \wedge b_i) \Rightarrow a_i)$$
$$\bigwedge_i ((a_i \wedge b_i) \Rightarrow b_i)$$
$$\bigwedge_i (a_i \Rightarrow (b_i \Rightarrow (a_i \wedge b_i)))$$
$$\bigwedge_i (a_i \Rightarrow (a_i \vee b_i))$$
$$\bigwedge_i (b_i \Rightarrow (a_i \vee b_i))$$
$$\bigwedge_i ((a_i \Rightarrow c_i) \Rightarrow ((b_i \Rightarrow c_i) \Rightarrow (a_i \vee b_i \Rightarrow c_i))$$
$$\bigwedge_i \bigwedge_j (\bigwedge_j a_{ij} \Rightarrow a_{ij})$$
$$\bigwedge_i (\bigwedge_j (a_i \Rightarrow b_{ij}) \Rightarrow (a_i \Rightarrow \bigwedge_j b_{ij}))$$
$$\bigwedge_i \bigwedge_j (a_{ij} \Rightarrow \bigvee_j a_{ij})$$
$$\bigwedge_i (\bigwedge_j (a_{ij} \Rightarrow b_i) \Rightarrow (\bigvee_j a_{ij} \Rightarrow b_i))$$
$$\bigwedge_i (\bigwedge_j (a_{ij} \Rightarrow b_{ij}) \Rightarrow (\bigwedge_j a_{ij} \Rightarrow \bigwedge_j b_{ij}))$$

[Same questions as before: why not σ and τ for a and b ?]

We say $\mathfrak{A}$ (of more correctly) $\langle \mathfrak{A}, \mathcal{F} \rangle$) is a *complete Heyting filtered algebra* (cHfa) if $\mathcal{F}$ is a filter on $\mathfrak{A}$ and all cHa terms are in $\mathcal{F}$. WE say a formula φ of the algebra $\mathfrak{A}$ is satisfied by the filtered algebra ($\mathfrak{A} \Vdash \varphi$) if φ is $\mathcal{F}$-valid.

Although any cHa can be regarded as a cHa with the trivial filter $\mathcal{F}=\{1\}$, these are by no means the only cHfa's. For example, $\bigwedge \{a,b\} = a \wedge b$ need *not* be satisfied by a cHfa.

Thm For any sentences φ_i and ψ of the algebra,

$$\{\tau = 1 : \tau \text{ is a cHa term}\} \cup \{\varphi_i : i \in I\} \vdash \psi \quad \text{iff} \quad \forall \mathfrak{A}, \mathcal{F} (\forall i \in I. \mathfrak{A} \Vdash \varphi_i \rightarrow \mathfrak{A} \Vdash \psi) .$$

[...]

Pf Spose $\ldots \nvdash \psi$.

[There is something underneath $\nVdash$, and I think it is HPC: Heyting Predicate Calculus. Note that, if each ν_i is finite, all formulas are finitary: $\overline{f_i}$ is a ν_i-ary connective: it applies to formulas. Note also that $\bigwedge$ and $\bigvee$ are *not* connectives; when they want to be, they become quantifiers.]

By Kripke's completeness theorem, there is a model of ... but not of ψ . [Grammar clearly wrong; but intention clear] The first [?:] moment t_0 of the model is an algebra $\mathfrak{A}$. Let $\mathcal{F} \stackrel{\text{def}}{=} \{ [\![\varphi(\vec{a})]\!] : t_0 \Vdash \varphi(\vec{a}) \}$.

[At first reading, I did not understand this; but I am not saying anything is wrong or even difficult in this.]

We now generalize the cumulative hierarchy $\langle V_\alpha : \alpha \in ON \rangle$ by generalizing the powerset operation. Instead of viewing sets as (corresponding to) their characteristic functions which take either the values 0 or 1 , we allow generalized sets to be characteristic functions which take values in a cHfa.

Thus [?]. we define the generalized powerset operation $\mathcal{P}_{\mathfrak{A}}$ applied to any collection X to be the collection $\mathcal{P}_{\mathfrak{A}}(X)$ of all partial functions from X into A . (The partial functions can be viewed as total by stipulating that the extension is always 0 for new arguments.) Thus, we define the generalized cumulative hierarchy $\langle V^{\mathfrak{A}}_\alpha : \alpha \in ON \rangle$ by

$$V^{\mathfrak{A}}_\alpha \stackrel{\text{def}}{=} \bigcup \{ \mathcal{P}_{\mathfrak{A}}(V^{\mathfrak{A}}_\beta) : \beta < \alpha \} .$$

Formulas of the (internal) intensional [!] language of generalized models $V^{\mathfrak{A}} = \bigcup_\alpha V^{\mathfrak{A}}_\alpha$ of set theory are the $\eta,\equiv$ - formulas, that is formulas built up from $x \eta y$ and $x \equiv y$ (where η and $\equiv$ are the intensional membership and identity symbols) by the propositional operators induced by operation [sic!] of the algebra $\mathfrak{A}$.

[Does this mean that the intensional formulas have no quantifiers? The answer is "no"; see below.]

For every $\eta,\equiv$ - formula φ

[he does not use a different notation for $\eta,\equiv$ - formulas from the previous "algebra" formulas, although he might have, e.g., using capital Greeks now.]

we define by recursion its truth-value $[\![\varphi]\!]$ [:]

$$[\![x\eta y]\!] \overset{\Delta}{=} \bigvee \{x(y) : y \in x\}$$

$$[\![x \equiv y]\!] \overset{\Delta}{=} \bigvee \{1 : x = y\}$$

$$[\![f_i(\langle \varphi_\xi : \xi < v_i \rangle]\!] \overset{\Delta}{=} f_i(\langle [\![\varphi_\xi]\!] : \xi < v_i \rangle$$

$$[\![\forall x \varphi]\!] \overset{\Delta}{=} \bigwedge \{ [\![\varphi(x)]\!] : x \in V^{\mathfrak{A}}\}$$

$$[\![\exists x \varphi]\!] \overset{\Delta}{=} \bigvee \{ [\![\varphi(x)]\!] : x \in V^{\mathfrak{A}}\}$$

[There must be a mistake: $\varphi \wedge \psi$, $\varphi \vee \psi$, etc are not treated.]

We say $V^{\mathfrak{A}}$ satisfies $\varphi(x)$ ($V^{\mathfrak{A}} \models \varphi(x)$) if $[\![\varphi(x)]\!] \in \mathcal{F}$.

Prop $\quad Th(V^{\mathfrak{A}}) \vdash_{\overline{HPC}} \varphi \longrightarrow \varphi \in Th(V^{\mathfrak{A}})$.

Coro $\quad Th(V^{\mathfrak{A}})$ is a consistent theory.

Next we will show that $Th(V^{\mathfrak{A}})$ contains all the axioms of Zermelo-Fraenkel set theory other than η-induction and extensionality.

$$\eta\text{-pairs} \overset{\Delta}{=} \forall u,v \exists y (u \in y \wedge v \in y)$$ [sic; $\in$ should be η]

$$\eta\text{-union} \overset{\Delta}{=} \forall x \exists y \forall u (\exists v \eta x \ u \eta v \rightarrow u \eta y)$$

[$((\exists v (v \eta x \wedge u \eta v)) \rightarrow u \eta y)$ I think]

$$\eta\text{-infinity} \overset{\Delta}{=} \exists y (\exists u \ u \eta y \wedge \forall u \eta y \ \exists v \eta y \ u \eta v)$$

$$\text{weak } \eta\text{-powerset} \overset{\Delta}{=} \forall x \exists y \forall z (\forall u \eta y\, u \eta x \rightarrow {}^{*} z \eta y \text{ crossed out and replaced by } \exists z' \eta y \forall u (u \eta z \longleftrightarrow u \eta z')^{*}$$

$$\eta\text{-separation} \overset{\Delta}{=} \forall x \exists y \forall u (u \eta y \longleftrightarrow u \eta x \wedge \varphi(u))$$

$$\eta\text{-collection} \overset{\Delta}{=} \forall x \exists y \forall u \eta x (\exists v \varphi(u,v) \longrightarrow \exists v \eta y \varphi(u,v))$$

$$\eta\text{-induction} \overset{\Delta}{=} \forall p (\forall x (\forall u \eta x\, \varphi(u,p) \rightarrow \varphi(x,p)) \longrightarrow \forall x \varphi(x,p))$$

[Yes; η-ind is included. Parameter p added later.]

Thm $V^{\mathfrak{A}} \models \eta$-pairs, union, infinity, powerset and coll'n.

["separation" and "induction" are not listed]

Pf It is straight forward to check that the following terms yield all but η-coll'n.

$$\mathrm{dom}(\{u,v\}^{\mathfrak{A}}) = \{u,v\}, \qquad \{u,v\}(z) = [\![z \equiv u \vee z \equiv v]\!]$$

$$\mathrm{dom}(\cup^{\mathfrak{A}} x) = \cup\{\mathrm{dom}(y) : y \in \mathrm{dom}(x)\} \qquad \cup^{\mathfrak{A}} x(z) = [\![\exists y \eta x\, z \eta y]\!]$$

For $x \in V$, we define $x^{\mathfrak{A}} \in V^{\mathfrak{A}}$ by recursion

$$\mathrm{dom}(x^{\mathfrak{A}}) = \{y^{\mathfrak{A}} : y \in x\} \qquad x^{\mathfrak{A}}(y^{\mathfrak{A}}) = 1.$$

$$\mathrm{dom}(\mathcal{P}^{\mathfrak{A}}(x)) = \{z : \mathrm{dom}(z) \subseteq \mathrm{dom}(x)\} \qquad \mathcal{P}^{\mathfrak{A}}(x)(z) = [\![\forall u \eta z\, u \eta x]\!]$$

$$\mathrm{dom}(\{u \eta x : \varphi(u)\}^{\mathfrak{A}}) = \mathrm{dom}(x) \qquad \{u \eta x : \varphi(u)\}^{\mathfrak{A}}(z) = [\![u \eta x \wedge \varphi(u)]\!]$$

[Thus: separation *is* included]

For coll'n we need the following

Lemma $$[\![\exists u \eta x\, \varphi]\!] = \bigvee_{u \in \mathrm{dom}(x)} (x(u) \wedge [\![\varphi(u)]\!])$$

$$[\![\forall u \eta x\ \varphi]\!] = \bigwedge_{u \in dom(x)} (x(u) \Longrightarrow [\![\varphi(u)]\!])$$

Pf $[\![\exists u \eta x\ \varphi]\!] \Longleftrightarrow \bigvee_u ([\![u \eta x]\!] \wedge [\![\varphi(u)]\!]) \Longleftrightarrow \bigvee_u (\bigvee \{x(u) : u \in dom(x)\} \wedge [\![\varphi(u)]\!])$

$\Longleftrightarrow$

$$\bigvee_u (\bigvee_{\substack{u \in dom(x) \\ u' \equiv u}} \{x(u') \wedge [\![\varphi(u)]\!])$$

[$\uparrow$ it reads $=$]

?

$$\Longleftrightarrow \bigvee_{u \in dom(x)} (x(u) \wedge [\![\varphi(u)]\!])$$

η-coll'n: Let $X \overset{\Delta}{=} \{ [\![\varphi(u,v)]\!] : [\text{unreadable}] \wedge v \in V^{\mathfrak{A}} \}$. Then

$u \in dom(x)$

$$\forall a \in X \exists v (\exists u \in dom(x)\ a = [\![\varphi(u,v)]\!]).$$

$\uparrow$

[missing]

Let $dom(y) = Y$

[?; what is Y? I think, we use collection in the real world, to conclude that there exists Y such that

$$\forall a \in X \exists v \in Y (\exists u \in dom(x)\ a = [\![\varphi(u,v)]\!]\ ]$$

and $y(z)=1$. Hence,

$$\bigwedge_{u \in dom(x)} (x(u) \Longrightarrow \bigwedge_v ([\![\varphi(u,v)]\!]) \Longrightarrow \bigvee_{v \in dom(y)} (1 \wedge [\![\varphi(u,v)]\!]))) \in \mathcal{F}.$$

By the lemma,

$$[\![\forall u \eta x (\exists v \varphi(u,v) \longrightarrow \exists v \eta y \varphi(u,v))]\!] \in \mathcal{F}.$$

in order to conclude that η-induction is satisfied by $V^{\mathfrak{A}}$, we must assume that $\mathfrak{A}$ satisfies another condition. We call the terms of the algebra:

$$\bigwedge_i \Big(\bigwedge_\alpha \big(\bigwedge_{\beta<\alpha} a_{i\beta} \Longrightarrow a_{i\alpha} \big) \Longrightarrow \bigwedge_\alpha a_{i\alpha} \Big)$$

the inductive terms. We say that $\mathfrak{A}$ is *inductive* if all inductive terms are in $\mathcal{F}$.

Thm $\mathfrak{A}$ is inductive iff $V^{\mathfrak{A}} \models \eta$-induction.

Pf. "$\rightarrow$". Let $R^{\mathfrak{A}}_\alpha$ be defined by $\mathrm{dom}(R^{\mathfrak{A}}_\alpha)=V^{\mathfrak{A}}_\alpha$ and $R^{\mathfrak{A}}_\alpha(u)=1$. Since $[\![R^{\mathfrak{A}}_{\alpha+1}=\mathcal{P}(R^{\mathfrak{A}}_\alpha)]\!] \in \mathcal{F}$, if we let $a_{\alpha,\beta} \overset{\Delta}{=} [\![\forall u \eta R^{\mathfrak{A}}_{\alpha+1} \varphi(u,p)]\!]$, then

$$\bigwedge_p [\ [\![\forall x(\forall u \eta x\ \varphi(u,p) \rightarrow \varphi(x,p))]\!] \Longleftrightarrow \bigwedge_x [\ [\![\forall u \eta x\ \varphi(u,p) \rightarrow \varphi(x,p))]\!]$$

$$\Longrightarrow \bigwedge_\alpha [\ [\![\forall u \eta R^{\mathfrak{A}}_\alpha \quad \varphi(u,p) \rightarrow \forall u \eta R^{\mathfrak{A}}_{\alpha+1} \varphi(x,p))]\!]$$

$\uparrow$

my addition

$$\Longrightarrow \bigwedge_\alpha \big(\bigwedge_{\beta<\alpha} a_{\beta p} \Longrightarrow a_{\alpha p} \big)$$

$$\Longrightarrow \bigwedge_\alpha a_{\alpha,p}$$

$$\Longrightarrow \bigwedge_\alpha \ \bigwedge_{u \in V^{\mathfrak{A}}_{\alpha+1}} (1 \Longrightarrow [\![\varphi(u,p)]\!])$$

$$\Longleftrightarrow \bigwedge_u (1 \wedge [\![\varphi(u,p)]\!])$$

$$\Longleftrightarrow [\![\forall u \varphi(u,p)]\!]\ ] \in \mathcal{F}.$$

[For the converse:]

Let $TC(x) = \cup\cup x$ where [unreadable].

$\uparrow$

unreadable subscript

Let $y \in V^{\mathfrak{A}}$ be defined by $\mathrm{dom}(y)=\{ \langle \alpha^{\mathfrak{A}}, \tilde{a} \rangle^{\mathfrak{A}} : a=a_\alpha \}$ and $y(\langle \alpha^{\mathfrak{A}}, \tilde{a} \rangle^{\mathfrak{A}})=1$ where $\mathrm{dom}(\tilde{a})=\{0\}$ and $\tilde{a}(0)=a$. Let

$$\varphi(u,y) \overset{\Delta}{=} \forall u \eta TC(\{u\}) \forall a(\langle u,a\rangle \eta y \longrightarrow 0 \eta a)$$

Since $[\![\varphi(x,y)]\!]=1$ if $x \neq \alpha^{\mathfrak{A}}$ for some α,

$$\begin{aligned}
\bigwedge_{\alpha}(\bigwedge_{\beta<\alpha} a_{\beta} \Longrightarrow a_{\alpha}) & \Longleftrightarrow \bigwedge_{\alpha} [\![\forall \beta \eta \alpha^{\mathfrak{A}}\ \varphi(\beta,y) \longrightarrow \varphi(\alpha^{\mathfrak{A}},y)]\!] \\
& \Longrightarrow [\![\forall x(\forall \beta \eta x\ \varphi(x,y) \longrightarrow \varphi(x,y)]\!] \\
& \Longrightarrow [\![\forall x\ \varphi(x,y)]\!] \\
& \Longrightarrow \bigwedge_{\alpha} [\![\varphi(\alpha^{\mathfrak{A}},y)]\!] \\
& \Longleftrightarrow \bigwedge_{\alpha} a_{\alpha} \in \mathcal{F}.
\end{aligned}$$

Since we are interested only in models of transfinite induction, we confine ourselves to compete Heyting filtered algebras that are inductive.

Next we derive the non-logical axioms of ZF from the principles that we have shown to be true in $V^{\mathfrak{A}}$.

Since the usual proof of the recursion theorem depends neither on the law of excluded middle nor on the principle of extensionality, we may define $\in$ as follows:

$$x \in y \overset{\Delta}{\equiv} \exists x' \eta y(\forall z \eta x\ z \in x' \wedge \forall z \eta x'\ z \in x).$$

↑↑
guess

[η-recursion]

AS usual we also define

$$x \subseteq y \overset{\Delta}{\equiv} \forall z \in x\ z \in y$$

$$x = y \overset{\Delta}{\equiv} x \subseteq y \wedge y \subseteq x$$

The axiom of extensionality is formulated as follows:

(∈-ext) $\forall xy(x=y \longrightarrow \forall z(x\in z \longleftrightarrow y\in z))$

[sic!]

By induction on the complexity of φ, the full substitution principle follows from extensionality.

Prop ∈-extensionality $\longrightarrow$ $\forall xy(x=y \rightarrow (\varphi(x) \longleftrightarrow \varphi(y))$ for any ∈-formula.

For a binary class relation R we define

$$x \underset{\eta R}{\subseteq} y \overset{\Delta}{\equiv} \forall u \eta x \; uRy$$

$$x \underset{R}{\subseteq} y \overset{\Delta}{\equiv} \forall u R x \; uRy$$

$$x \underset{\eta R}{=} y \overset{\Delta}{\equiv} x \underset{\eta R}{\subseteq} y \wedge y \underset{\eta R}{\subseteq} x$$

$$x \underset{R}{=} y \overset{\Delta}{\equiv} x \underset{R}{\subseteq} y \wedge y \underset{R}{\subseteq} x$$

WE call a relation R extensional if it satisfies the principle of extensionality for R :

(R-ext) $\forall xy(x \underset{R}{=} y \longrightarrow \forall z(xRz \longleftrightarrow yRz))$.

Thm If $\mathfrak{A}$ is an extensional complete Heyting filtered algebra, $V^{\mathfrak{A}}$ satisfies

(1) $x\eta y \rightarrow x\in y$

(2) ∈-extensionality

(3) If R is an extensional relation containing η and [should last word be "then", I think]

$$\forall xy(x \underset{\eta R}{\subseteq} y \longrightarrow x \underset{R}{\subseteq} y) \qquad (*) .$$

(4) $\forall x\eta y\varphi \longleftrightarrow \forall x\in y\varphi$, where φ is an ε-formula

(5) $\exists x\eta y\varphi \longleftrightarrow \exists x\in y\varphi$ [same kind of φ, I think]

Pf By induction on $\langle rk(x), rk(y)\rangle$ we show

(i) $x \underset{\eta\in}{\equiv} x$ [$\in$ for R above!]

(ii) $x \underset{\eta\in}{\equiv} y \longrightarrow \forall z(y \underset{\eta\in}{\equiv} z \longrightarrow x \underset{\eta\in}{\equiv} z)$

(iii) $x \underset{\eta\in}{\subseteq} y \longrightarrow x \subseteq y$

(iv)=(2) $x=y \longrightarrow \forall z(x\in z \longleftrightarrow y\in z)$

(v)=(3)

(i): It suffices to show $x \underset{\eta\in}{\subseteq} x$. So suppose $u\eta x$. By IH(i), $u \underset{\eta\in}{\equiv} u$. Hence, $u\in x$.

[**Note:** $x\in y \overset{\Delta}{\equiv} \exists x'\eta y(\forall z\eta x\ z\in x' \wedge \forall z\eta x'\ z\in x) \equiv \exists x'\eta y\ x \underset{\eta\in}{\equiv} x'$]

↑↑

guess

(ii): Spose $x \underset{\eta\in}{\subseteq} y \underset{\eta\in}{\subseteq} z$. We will show $x \underset{\eta\in}{\subseteq} z$. So suppose $u\eta x$. Then $u\in y$. i.e. for some $u'\eta y$, $u \underset{\eta\in}{\equiv} u'$. Hence, $u'\in z$. Consequently, $\exists u''\eta z\ u'' \underset{\eta\in}{\equiv} u' \underset{\eta\in}{\equiv} u$. By IH(ii), $u'' \underset{\eta\in}{\equiv} u$. Hence, $u\in z$. Since by an (*almost*) similar argument $z \underset{\eta\in}{\subseteq} y \underset{\eta\in}{\subseteq} x \rightarrow z \underset{\eta\in}{\subseteq} x$, we are done.

(iii) Spose $x \underset{\eta\in}{\subseteq} y$ and $u\in x$. Hence, $\exists u'\in y\ u' \underset{\eta\in}{\equiv} u$. Hence, $\exists u''\eta y\ u'' \underset{\eta\in}{\equiv} u' \underset{\eta\in}{\equiv} u$. Hence, by IH(ii), $u\in y$.

(iv) Suppose $x=y$ and $x\in z$. THen $\exists x'\eta z\ x' \underset{\eta\in}{\equiv} x=y$. By IH(ii), $y\in z$.

(v) Suppose R is an extensional relation containing η s.t. (*) holds and $x\in y$. Since $x\in y$ and R contains η, for some $x'Ry$, $x' \underset{\eta\in}{\equiv} x$. By IH(v), $x \underset{\in R}{=} x \underset{\eta\in}{\equiv} x' \underset{\in R}{=} x'$. Hence,

↑

guess

$x' \underset{\in R}{=} x$. By (*), $x' \underset{R}{=} x$. So since R is extensional, xRy.

Now we have shown (2) and (3). (1) is an immediate consequence of (i). WE omit the proof of (5) since it is similar to the proof of (4). By (1), $\forall u\in x\varphi \rightarrow \forall u\eta x\varphi$. Towards showing the other half of (4), spose $\forall u\eta x\ \varphi$ and $u\in x$. Then for some $u'\eta x$, $u \underset{\eta\in}{\equiv} u'$. Hence, by (iii), we have $u=u'$ and $\varphi(u')$. Hence, by the previous proposition, $\varphi(u)$.

↑

corrected from u'

Note that (i), (ii) and (iii) are immediate consequences of (4). Note also that we have shown that $\in$ is the smallest extensional relation R containing η s.t. (*) holds.

Thm If $\mathfrak{A}$ is an inductive, compete Heyting filtered algebra, then $V^{\mathfrak{A}}$ satisfies $\in$-extensionality, induction, pairs, union, infinity, powerset, separation and coll'n.

Pf We have already $\in$-extensionality. Since the remaining $\in$-principles follow from their corresponding η-principles, we only check $\in$-separation.

Since any $\in$-formula φ is an η-formula, $\{u\eta x:\varphi(u)\}$ exists by η-separation. By (5) of the previous theorem,

$$\exists u'\eta x(\varphi(u')\wedge u=u') \longleftrightarrow \exists u'\in x(\varphi(u')\wedge u=u') \longleftrightarrow u\in x\wedge\varphi(u).$$

Hence, $\forall u(u\in\{u\eta x:\varphi(u)\} \longleftrightarrow u\in x\wedge\varphi(u))$.

Finally, we establish a connection between the theory of the algebra and the theory of the model of set theory induced by the algebra. In $V^{\mathfrak{A}}$, the powerset $\mathcal{P}(I)$ of a set I is an algebra with the same signature as $\mathfrak{A}$. For $I\in V^{\mathfrak{A}}$ and for each operation $f_i:A^{v_i}\longrightarrow A$ of the algebra, let f_i^I be the function defined in $V^{\mathfrak{A}}$ by

$$\mathrm{dom}(f_i^I) = {}^{v_i^{\mathfrak{A}}}\mathcal{P}(I) \text{ and } f_i^I(x)=\{i\in I:\bar{f}(\langle i\in x_{\xi^{\mathfrak{A}}}:\xi\in v_i\rangle)\}.$$

Then in $V^{\mathfrak{A}}$, $\mathfrak{A}^I \stackrel{\Delta}{=} \langle\mathcal{P}(I), \bigwedge^I, \bigvee^I, \Rightarrow^I, \wedge^I, \vee^I, f_i^I\rangle_{i\in I}$ [double use of I] $\in V^{\mathfrak{A}}$ is a Heyting algebra with signature γ^I. Furthermore, the theory of the algebra $\mathfrak{A}$ is just the theory of the

↑

[?]

powerset of a singleton in the model of set theory induced by the algebra.

Thm For any formula of the algebra $\mathfrak{A}$,

$$\forall \vec{a}\in A(\mathfrak{A}\models\varphi(\vec{a}) \text{ iff } V^{\mathfrak{A}}\models\mathfrak{A}^1\models\varphi(\vec{a}))$$

Pf By induction on the complexity of φ, we have $[\![\varphi(\vec{a})]\!] = [\![\mathfrak{A}^1\models\varphi(\tilde{\vec{a}})]\!]$.

Thm Complete Heyting algebras are inductive.

Pf It suffices to show by induction on α , that

$$\bigwedge_{\alpha}(\bigwedge_{\beta<\alpha}\Rightarrow a_\alpha)\le a_\alpha .$$

By the IH, $\bigwedge_{\alpha}(\bigwedge_{\beta<\alpha}\Rightarrow a_\alpha)\le \bigwedge_{\beta<\alpha} a_\beta$. Hence,

$$\bigwedge_{\alpha}(\bigwedge_{\beta<\alpha}\Rightarrow a_\alpha)\le \bigwedge_{\beta<\alpha} a_\beta \wedge \bigwedge_{\beta<\alpha}\Rightarrow a_\alpha \le a_\alpha . \square$$

In addition to cHa, there are cHfa's that are induced by combinatorial structures.

Consider a structure $\mathcal{T}=\langle T,\cdot,K,S\rangle$ where the elements of T are *constructions*, $\cdot : T^2 \xrightarrow[p]{} T$ is a binary partial operation called application, and K and S are distinguished constructions.

Terms built up from constructions by application are defined just in case (i) the term is a construction or (ii) the term is the application of two defined terms [??]. Thus all subterms of a defined term are defined. $t\simeq t'$ is short for t is defined iff t' is defined and if t and t' are defined they are equal (in value). Any construction t determines a partial function $\{t\}:T\xrightarrow[p]{}T$ defined by $\{t\}(t')=t\cdot t'$. t is said to be *total* if $\mathrm{dom}(\{t\})=T$.

We assume that for all $r,s,t\in T$, Srs and Krs are defined and $Krs=r$ and $Srst=rt(st)$.

By recursion on the complexity of a term t , we define $\langle\lambda x.t\rangle$ as follows.

If t is the variable x , then let $\langle\lambda x.t\rangle=SKK$.
If t is a constant or a variable other than x , let $\langle\lambda x.t\rangle=K\cdot t$.
If $t=t_1\cdot t_2$, let $\langle\lambda x.t\rangle=S\langle\lambda x.t_1\rangle\langle\lambda x.t_2\rangle$.

Let $\langle \lambda x_1,\ldots,x_N.t\rangle \overset{\Delta}{=} \langle \lambda x_1.\langle \lambda x_2.\ldots\langle \lambda x_N.t\rangle\ldots\rangle\rangle$.

Prop (λ-abstraction) $\langle \lambda x_1,\ldots,x_N.t\rangle$ is defined and

$$\langle \lambda x_1,\ldots,x_N.t\rangle \cdot x_1 \cdot \ldots \cdot x_N \simeq t .$$

This is not what we want: $\langle \lambda x_1.t\rangle \cdot x_1 \simeq t$: OK; we also want $\langle \lambda x_1.t\rangle \cdot y \simeq t[y/x_1]$??

WE define a pairing function $P \overset{\Delta}{=} \langle \lambda xyz.zxy\rangle$. The pair Pxy is always defined and

$$PxyK = Kxy = x$$
$$Pxy(K\langle \lambda z.z\rangle) = K\langle \lambda z.z\rangle xy = \langle \lambda z.z\rangle y = y .$$

Hence, $\langle \lambda z.zK\rangle$ and $\langle \lambda z.zK\langle \lambda z.z\rangle\rangle$ can be regarded as the 1st and 2nd projections. For $a,b\in \mathcal{P}(T)$, let

$$a\times b \overset{\Delta}{=} \{Pxy: x\in a \wedge y\in b\} .$$

(?) It is an open question as to whether there is a term t such that txy is always defined and txyK=x and txyS=y .

Doesn't this contradict Church's thesis

Let $w \overset{\Delta}{=} \langle \lambda xyz.x(yy)z\rangle$ and $\varphi=\langle \lambda x.wx(wx)\rangle$.

Fixed point theorem φ is total and $\varphi x=x(\varphi x)$.

still, should be written $\simeq$

nonsense! What if $x'y$ is not defined for any y?

Pf $\varphi xy \simeq wx(wx)y \simeq x(wx(wx))y \simeq x(\varphi x)y$.

correct to: $\varphi xy \simeq x(\varphi x)y$

We call $\mathcal{T}=\langle T,\cdot,K,S,D\rangle$ a *combinatorial* [sic!] structure if $D:\mathcal{P}(T)\to\mathcal{P}(T)$ satisfies

(1)	$a\subseteq Da$	
(2)	$D(a\times b) \subseteq Da \times Db$	for $a,b\in \mathcal{P}(T)$
(3)	$D(\cup X) \subseteq \cup D''X$	for $X \subseteq \mathcal{P}(T)$

(4) $\quad D(\cap X) \underset{\substack{\uparrow \\ [\,?\,]}}{\subseteq} \cup D''X$

Note that D defined by $Da=a$ or by $D(a)=T$ satisfies all these conditions.

Let $\mathfrak{A}_{\mathcal{T}}$ be the induced filtered algebra defined by $A=\mathcal{P}(T)$

$$a \Rightarrow b \stackrel{\Delta}{=} \{t \in T : Da \subseteq \mathrm{dom}(\{t\}) \wedge \{t\}''a \subseteq b$$

$$a \wedge b \stackrel{\Delta}{=} a \times b$$

$$a \vee b \stackrel{\Delta}{=} (\{K\} \times a \times T) \cup (\{S\} \times T \times b)$$

$$\bigwedge X \stackrel{\Delta}{=} \cap X$$

$$\bigvee X \stackrel{\Delta}{=} \cup X$$

$$\mathcal{F} \stackrel{\Delta}{=} \{a \in A : a \neq 0\} .$$

Thm $\mathfrak{A}_{\mathcal{T}}$ is an inductive complete Heyting filtered algebra for any combinatorial structure $\mathcal{T}$.

Pf Since $K \in T$, $\bigvee \mathcal{P}(T) = T \in \mathcal{F}$. By def'n, $0 \notin \mathcal{F}$. By (1),

$$x \in a \wedge e \in (a \Rightarrow b) \longrightarrow ex \in b .$$

Hence, $\mathcal{F}$ is a filter on $\mathfrak{A}_{\mathcal{T}}$.

$K \in a \Rightarrow (b \Rightarrow a)$

$S \in (a \Rightarrow (b \Rightarrow c)) \Rightarrow ((a \Rightarrow b) \Rightarrow (a \Rightarrow c))$.

$\langle \lambda z, zK \rangle \in (a \wedge b) \Rightarrow a$ by (2)

$\langle \lambda z, zK \langle \lambda z, z \rangle \rangle \in (a \wedge b) \Rightarrow b$ by (2) .

$P \in (a \Rightarrow (b \Rightarrow (a \wedge b))$

$\langle \lambda x. \langle \lambda y. \langle \lambda z. P(x(zK(K(SKK))))(y(z(K(SKKK))))(zKK) \rangle \rangle \rangle \in$
$(a \Rightarrow c) \Rightarrow ((b \Rightarrow c) \Rightarrow ((a \vee b) \Rightarrow c)$ by (2)

$\langle \lambda x.P(PKx)K\rangle \in a \Rightarrow (a \vee b)$

$\langle \lambda x.P(PSK)x\rangle \in b \Rightarrow (a \vee b)$

$\langle \lambda x.x\rangle \in \bigwedge_i a_i \Rightarrow a_i$

$\langle \lambda x.x\rangle \in \bigwedge_i (a \Rightarrow b_i) \Rightarrow (a \Rightarrow \bigwedge_i b_i)$ since we may assume the index set I is inhabited

$\langle \lambda x.x\rangle \in a_i \Rightarrow \bigvee_i a_i$

$\langle \lambda x.x\rangle \in \bigwedge_i (a_i \Rightarrow b) \Rightarrow (\bigvee_i a_i \Rightarrow b)$ by (3)

$\langle \lambda x.x\rangle \in \bigwedge_i (a_i \Rightarrow b_i) \Rightarrow (\bigwedge_i a_i \Rightarrow \bigwedge_i b_i)$ by (4).

Hence, $\mathfrak{A}_{\mathcal{T}}$ is a complete Heyting filtered algebra.

$$\varphi \in \bigwedge_{\alpha} (\bigwedge_{\beta<\alpha} a_\beta \Rightarrow a_\alpha) \Rightarrow \bigwedge_{\alpha} a_\alpha .$$

Pf Since φ is total it suffices to show that if $e \in \bigwedge_{\alpha} (\bigwedge_{\beta<\alpha} a_\beta \Rightarrow a_\alpha)$ then $\varphi e \in \bigwedge_{\alpha} a_\alpha$. So suppose $\forall\alpha\forall e' \in \bigwedge_{\beta<\alpha} a_\beta$, $ee' \in a_\alpha$. WE will show by induction on α that $\varphi e \in a_\alpha$. By IH, $\varphi e \in \bigcap_{\beta<\alpha} a_\beta$. Hence, by our supposition $\varphi e = e(\varphi e) \in a_\alpha$. □

Thus, $\mathfrak{A}_{\mathcal{T}}$ is inductive also. □

Thus, the models $V^{\mathfrak{A}_{\mathcal{T}}}$ satisfy all the axioms of ZF other than the law of excluded middle. Moreover, any model of set theory induced by a combinatorial structure satisfies the axiom of double complements -- at least in a classical metatheory.

(Ax of double complements) $\forall x \exists y \forall z(\neg\neg z \in x \longrightarrow z \in y)$

Very few Heyting-valued models of set theory satisfy this axiom.

Actually we can get the ax. of double complements in $V^{\mathfrak{A}}$ by only assuming the axiom of double complements [in V] (which trivially follows from the law of excluded middle).

First we consider the normalization of an element of $V^{\mathfrak{A}}$.

Def'n $\mathrm{dom}(N(x)) = \{N(y) : y \in \mathrm{dom}(x) \wedge x(y) \in \mathcal{F}\}$

$N(x)(N(y)) = x(y)$

Prop (i) $N(x) \subseteq R_x$

(ii) $N(x) = N(N(x))$

(iii) $[\![\forall x\ x = N(x)]\!] \in \mathcal{F}$

(iv) $[\![x=y]\!] \in \mathcal{F} \longrightarrow \mathrm{rk}(N(x)) = \mathrm{rk}(N(y))$, where $\mathrm{rk}(x) \overset{\Delta}{=} \sup\{\mathrm{rk}(y) : y \in \mathrm{dom}(x)\}$

Pf (i) and (ii) are by $\in$-induction.

(iii) $\langle I, I\rangle \in [\![x = N(x)]\!]$

(iv) By induction on $\langle \mathrm{rk}(x), \mathrm{rk}(y)\rangle$. By (ii)&(iii), wlog we may assume $x = N(x)$ and $y = N(y)$, Suppose

$$\bigwedge_{u \in \mathrm{dom}(x)} (x(u) \Rightarrow [\![u \in y]\!]) \wedge \bigwedge_{u \in \mathrm{dom}(y)} (y(u) \Rightarrow [\![u \in y]\!]) = [\![x=y]\!] \in \mathcal{F}.$$

↑

corrected from y

Then $\forall u \in \mathrm{dom}(N(x))\ [\![u \in y]\!] \in \mathcal{F}$ and $\forall u \in \mathrm{dom}(N(y))\ [\![u \in x]\!] \in \mathcal{F}$.

By definition of $[\![u \in v]\!]$,

$\forall u \in \mathrm{dom}(N(x))\ \exists u' \in \mathrm{dom}(y)\ [\![u \in u']\!] \in \mathcal{F}$ and

$\forall u \in \mathrm{dom}(N(y))\ \exists u' \in \mathrm{dom}(x)\ [\![u \in u']\!] \in \mathcal{F}$.

By IH, $\forall u \in \mathrm{dom}(N(x))\ \mathrm{rk}(u) < \mathrm{rk}(y)$ and $\forall u \in \mathrm{dom}(N(y))\ \mathrm{rk}(u) < \mathrm{rk}(x)$.

Hence, $\mathrm{rk}(x) = \mathrm{rk}(y)$.

Thm (Ax. of double complements) $\forall \mathcal{T}\ V^{\mathfrak{A}_{\mathcal{T}}} \models$ Ax of double complements

Pf Let $(--)^{\mathfrak{A}} : V^{\mathfrak{A}} \to V^{\mathfrak{A}}$ be defined by $\mathrm{dom}((--)^{\mathfrak{A}}x) = --R_x \cap V^{\mathfrak{A}}$

$$((--)^{\mathfrak{A}}x)y = [\![\neg\neg y \in x]\!].$$

Since $[\![\forall xy(x=y \rightarrow (--)^{\mathfrak{A}}x=(--)^{\mathfrak{A}}y]\!] \in \mathcal{F}$, wlog we may assume $x=N(x)$. By the previous proposition,

$$[\![z\in x]\!] \in \mathcal{F} \quad \longrightarrow \quad \exists z'\in \mathrm{dom}(x)\ x(z') \wedge [\![z=z']\!] \in \mathcal{F}$$

$$\longrightarrow \quad z=N(z) \subseteq R_{N(z)}=R_{N(z')} \subseteq R_{z'} \subseteq R_x .$$

Hence, $[\![z\in x]\!] \neq \emptyset \rightarrow \langle I,I\rangle \in [\![\neg\neg z\in x]\!] \wedge [\![z=z]\!] \subseteq \bigcup_{u\in --R_x\cap V^{\mathfrak{A}}} [\![\neg\neg u\in x]\!] \wedge [\![u=z]\!] =$

$[\![z\in (--)^{\mathfrak{A}}x]\!]$. I.e.,

$$(\lambda t.\langle I,I\rangle) \in [\![\neg\neg z\in x]\!] \Rightarrow [\![z\in (--)^{\mathfrak{A}}x]\!] = [\![\neg\neg z\in x \rightarrow z\in (--)^{\mathfrak{A}}x]\!] .$$

Thus, $[\![\forall z(\neg\neg z\in x \rightarrow z\in (--)^{\mathfrak{A}}x)]\!] \in \mathcal{F}$.