Toposes for modified realizability (jww Mees de Vries)

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Section 1

Motivation

Topos theory

Topos theory provides a framework subsuming all the main approaches to the semantics of (intuitionistic) higher-order logic. Indeed, it subsumes:

- Kripke models (presheaf models)
- Topological models, Beth models, Heyting-valued, forcing models (sheaf models)
- Realizability (realizability models)
- Dialectica (Dialectica toposes)

In addition, topos theory is a rich mathematical theory, with many abstract ideas applicable to all of these topics.

Example: glueing

(Artin) glueing was a construction on toposes motivated by topology, but can be applied to realizability toposes to obtain toposes for "with truth"-variants.

A topos, in this talk

- Topos: elementary topos with natural numbers object (category with finite limits, exponentials, subobject classifier, and natural numbers object).
- Model of higher-order arithmetic (**HAH**).
- So if φ is a sentence in higher-order arithmetic and $\mathcal{E}$ is a topos, the following is well-defined: $\mathcal{E} \models \varphi$.
- Model is *extensional*: extensionality for functions, propositions and subsets.

$$\begin{array}{l}
\forall f, g \in Y \to X \left(\forall y \in Y \, (fy = gy) \to f = g \right) \\
\forall p, q \in \Omega \left((p \leftrightarrow q) \to p = q \right) \\
\forall A, B \in \mathrm{Pow}(X) \left(\forall x \in X \, (x \in A \leftrightarrow x \in B) \to A = B \right)
\end{array}$$

Realizability toposes

Aim

One aim of the theory of realizability toposes: give a more semantic and conceptual account of the various forms of realizability that one finds in the proof-theoretic literature.

- First example of a realizability topos: Hyland's effective topos Eff ($\sim$1980). A topos for number realizability.
- Since then: many more examples!
- Standard reference: Jaap van Oosten's book: *Realizability: An Introduction to its Categorical Side*. Elsevier, 2008.

Section 2

Number realizability and the effective topos

Number realizability: the proof-theoretic story

Given any sentence φ in the language of **HA**, we define a new sentence $x \,\mathbf{rn}\, \varphi$, also in the language of **HA**, with one additional free variable x.

Soundness

From a derivation of $\varphi(x_1, \ldots, x_n)$ in **HA** one can effectively extract a numeral n and a derivation of $n \cdot (x_1, \ldots, x_n) \,\mathbf{rn}\, \varphi$, also in **HA**.

Characteric principles

Let ECT_0 be extended Church's Thesis:

$$\forall x\,(Ax \to \exists y\, Bxy) \to \exists z\, \forall x\, \big(Ax \to z \cdot x \downarrow \wedge B(x, z \cdot x)\big),$$

where A does not contain existential quantifiers or disjunctions.

1. $\mathbf{HA} + \mathrm{ECT}_0 \vdash \varphi \leftrightarrow \exists x\,(\, x \,\mathbf{rn}\, \varphi\,)$.
2. If φ is closed, then $\mathbf{HA} + \mathrm{ECT}_0 \vdash \varphi \Leftrightarrow \mathbf{HA} \vdash n \,\mathbf{nr}\, \varphi$ for some numeral n.

What to expect from a topos for number realizability?

$$\mathcal{E} \models \varphi \Longleftrightarrow \exists x \, (\, x \, \mathbf{rn} \, \varphi \,).$$

In particular, $\mathcal{E} \models \mathrm{ECT}_0$,

But wasn't the RHS a formal statement in the language of **HA**? So perhaps we should just say: the latter sentence holds in the standard model.

The effective topos has this property.

Question

Are there are other toposes with this property? (For instance, is there a subtopos of the effective topos in which the same **HA**-sentences are true as in Eff?)

Higher-order aspects of the effective topos

1. Uniformity Principle (Kreisel-Troelstra)

$$\forall X \in \mathrm{Pow}(\mathbb{N})\, \exists n \in \mathbb{N}\, \varphi(X, n) \to \exists n \in \mathbb{N}\, \forall X \in \mathrm{Pow}(\mathbb{N})\, \varphi(X, n).$$

2. Shanin's principle: every subset of the natural numbers is the surjective image of a $\neg\neg$-closed subset of the natural numbers.

Question

How certain are we that any extension of number realizability to higher-order arithmetic must satisfy these principles?

Section 3

Modified realizability

Arithmetic in finite types

Roughly, the system HA^ω is:

- Constructive logic.
- The language is sorted. Its sorts are the finite types:

$$\text{type} = \mathbb{N} \,|\, \text{type} \times \text{type} \,|\, \text{type} \to \text{type}.$$

- There is a constant 0 of type $\mathbb{N}$ and a constant S of type $\mathbb{N} \to \mathbb{N}$ for which we have the Peano axioms.
- The term language is a combinator version of the typed λ-calculus with recursor (Gödel's T), with **k** and **s**, and λ-abstraction defined.
- Induction for all formulas in this language.

(See, for example, Troelstra and Van Dalen, *Constructivism in mathematics*.)

HA$^\omega$

The axioms and rules of HA$^\omega$ are:

(i) All the axioms and rules of many-sorted intuitionistic logic.

(ii) Equality is an equivalence relation at all types:

$$x = x, \qquad x = y \to y = x, \qquad x = y \wedge y = z \to x = z$$

(iii) The congruence laws for equality at all types:

$$f = g \to fx = gx, \qquad x = y \to fx = fy$$

(v) The successor axioms:

$$\neg S(x) = 0, \qquad S(x) = S(y) \to x = y$$

(v) For any formula φ in the language of HA$^\omega$, the induction axiom:

$$\varphi(0) \to \big(\forall x^0 \, (\, \varphi(x) \to \varphi(Sx) \,) \to \forall x^0 \, \varphi(x) \,\big).$$

HA^ω and E-HA^ω

(vi) The axioms for the combinators:

$$\begin{array}{rcl} \mathbf{k}xy & = & x \\ \mathbf{s}xyz & = & xz(yz) \\ \mathbf{p}_0(\mathbf{p}xy) & = & x \\ \mathbf{p}_1(\mathbf{p}xy) & = & y \\ \mathbf{p}(\mathbf{p}_0x)(\mathbf{p}_1x) & = & x \end{array}$$

as well as for the recursor:

$$\begin{array}{rcl} \mathbf{R}xy0 & = & x \\ \mathbf{R}xy(Sn) & = & yn(\mathbf{R}xyn) \end{array}$$

The system E-HA^ω is obtained from HA^ω by adding the axiom of extensionality:

$$\begin{array}{lcl} \mathrm{EXT} & : & \forall f^{\sigma\to\tau}, g^{\sigma\to\tau}\big((\forall x^\sigma fx =_\tau gx) \to f =_{\sigma\to\tau} g\big) \\ & & \forall x^{\sigma\times\tau}, y^{\sigma\times\tau}\big(\mathbf{p}_0x =_\sigma \mathbf{p}_0y \wedge \mathbf{p}_1x =_\tau \mathbf{p}_1y \to x =_{\sigma\times\tau} y\big). \end{array}$$

Modified realizability: the proof-theoretic story

Type of a formula

We define the type $\mathrm{tp}(\varphi)$ of a formula φ as follows:

(i) $\mathrm{tp}(\varphi) = 0$ if φ is atomic.

(ii) If the type of φ is σ and that of ψ is τ, then the type of $\varphi \wedge \psi$ is $\sigma \times \tau$ and that of $\varphi \to \psi$ is $\sigma \to \tau$.

(iii) If the type of φ is τ then the type of $\exists x^\sigma \, \varphi$ is $\sigma \times \tau$ and the type of $\forall x^\sigma \, \varphi$ is $\sigma \to \tau$.

We can regard disjunction as a defined symbol:

$$\varphi \vee \psi :\equiv \exists n^0 \big((n = 0 \to \varphi) \wedge (n \neq 0 \to \psi) \big).$$

Intuition

One should think of the type of φ as the type of potential modified realizers of φ.

Modified realizability: the proof-theoretic story, continued

Modified realizability (Kreisel)

To any formula φ in the language of HA^ω we associate a new formula $x \text{ mr } \varphi$ as follows, where $x \text{ mr } \varphi$ is also a formula in the language of HA^ω whose free variables are those of φ plus possibly a variable x of type $\mathrm{tp}(\varphi)$:

$$
\begin{aligned}
x \text{ mr } \varphi &:= \varphi \qquad \text{if } \varphi \text{ is atomic.} \\
x \text{ mr } (\varphi \wedge \psi) &:= \mathbf{p}_0 x \text{ mr } \varphi \wedge \mathbf{p}_1 x \text{ mr } \psi \\
x \text{ mr } (\varphi \to \psi) &:= \forall y^{\mathrm{tp}(\varphi)} (\, y \text{ mr } \varphi \to x(y) \text{ mr } \psi \,) \\
x \text{ mr } \exists y^\sigma \varphi &:= \mathbf{p}_1 x \text{ mr } \varphi(\mathbf{p}_0 x) \\
x \text{ mr } \forall y^\sigma \varphi &:= \forall y^\sigma (\, x(y) \text{ mr } \varphi \,)
\end{aligned}
$$

Modified realizability: the proof-theoretic story, continued

Soundness

Let φ be a formula in the language of HA^ω. If φ is provable in HA^ω, then one can find effectively from this proof a term t of type $\mathrm{tp}(\varphi)$ in the language of HA^ω such that:

1. any variables occurring freely in t also occur freely in φ, and
2. $\mathsf{HA}^\omega \vdash t \text{ mr } \varphi$.

The same statement holds for $\mathsf{E\text{-}HA}^\omega$.

Characteric principles

$$\begin{array}{rcl} \mathsf{AC} & : & \forall x^\sigma \, \exists y^\tau \, \alpha(x,y) \to \exists f^{\sigma\to\tau} \, \forall x^\sigma \, \alpha(x, f(x)) \\ \mathsf{IP} & : & (\varphi \to \exists x^\sigma \, \psi) \to \exists x^\sigma (\varphi \to \psi) \end{array}$$

where x does not occur in φ and φ is existence-free.

1. $\mathsf{HA}^\omega + \mathsf{AC} + \mathsf{IP} \vdash \varphi \leftrightarrow \exists x \, (\, x \text{ mr } \varphi \,)$.
2. If φ is closed, then $\mathsf{HA}^\omega + \mathsf{AC} + \mathsf{IP} \vdash \varphi \Leftrightarrow \mathsf{HA}^\omega \vdash t \, \text{mr} \, \varphi$ for some term t in Gödel's T.

Towards a modified realizability topos

What do we want from a modified realizability topos?

$$\mathcal{E} \models \varphi \Longleftrightarrow \exists x\, x \,\mathsf{mr}\, \varphi.$$

Then we automatically get $\mathcal{E} \models \mathsf{AC}$ and $\mathcal{E} \models \mathsf{IP}$.

But wasn't the RHS a formal statement in the language of HA^ω? So perhaps we should just say: the latter sentence holds in the standard model.

But what is the standard model of Gödel's T?

One might say: the set-theoretic model.

Then Sets is a perfectly good modified realizability topos! Probably we need something more computational.

One could say: HRO. But this is impossible, because this refutes extensionality.

Criterion for a modified realizability topos?

So we need a different model. The natural choice (I think): HEO. (Also the interpretation of finite types in Eff.)

Criterion

For $\mathcal{E}$ to be a modified realizability topos, we need for any sentence φ in the language of HA^ω that $\mathcal{E} \models \varphi$ if and only if there is an element x in the HEO-model of Gödel's T such that $x \,\mathrm{mr}\, \varphi$. In particular, we want $\mathcal{E} \models \mathsf{AC}$ and $\mathcal{E} \models \mathsf{IP}$.

Grayson's topos

There does exist a modified realizability topos, due to Grayson (and discussed in Jaap's book), but it does not satisfy our criterion! Indeed, in this topos AC fails (because Church's thesis holds).

Today: a different topos which satisfies our criterion. Actually, I will discuss two!

Section 4

Triposes

Tripos (over set)

For what follows: see Jaap's book (or Mees' MSc thesis!).

Let us write **PreHey** for the category of *preHeyting algebras* (preorders whose poset reflections are Heyting algebras).

Tripos (Hyland, Johnstone, Pitts)

A tripos is a functor $P : \mathit{Sets} \to \mathbf{PreHey}$ such that:

- for each function $f : Y \to X$, the operation $Pf : PX \to PY$ has both adjoints satisfying the Beck-Chevally condition.
- There is a set Σ and an element $\top \in P(\Sigma)$ such that for any $A \in P(X)$ there is some map $a : X \to \Sigma$ (not necessarily unique) such that $P(a)(\top) \cong A$.

Examples

Slogan: choose a good Σ!

First example

Choose $\Sigma = H$, a complete Heyting (Boolean) algebra. Elements of $P(X)$ are functions $\varphi : X \to \Sigma$, ordered as follows: we have $\varphi \preceq \psi : X \to H$ if $\varphi(x) \leq \psi(x)$ for every $x \in X$.

Second example (effective tripos)

Choose $\Sigma = \mathrm{Pow}(\mathbb{N})$. Elements of $P(X)$ are functions $\varphi : X \to \Sigma$, ordered as follows: we have $\varphi \preceq \psi : X \to \Sigma$ if

> *there is a partial recursive function f such that for any $x \in X$ and $n \in \varphi(x)$, the function f is defined on n and $f(n) \in \psi(x)$.*

The latter definition reflects the idea of $\mathrm{Pow}(\mathbb{N})$ as a non-standard set of truth values (Dragalin, Powell, Scott).

Tripos to topos construction

Given a tripos P, we can construct a topos $\mathcal{C}[P]$ as follows:

- Objects are pairs (X, R) where X is a set and $R \in P(X \times X)$ is a symmetric and transitive relation (in the sense of the tripos).
- Morphisms $F : (X, R) \to (Y, S)$ are equivalence classes of elements $F \in P(X \times Y)$, which are functional relations (in the sense of the tripos) and with $F \sim G$, if F and G are extensionally equal (in the sense of the tripos).

For the previous triposes this yields ...

1. the topos of H-valued sets.
2. Hyland's effective topos.

Section 5

Modified realizability topos(es)

Grayson's modified realizability tripos

For his modified realizability tripos, Grayson chooses:

$$\Sigma = \{ (A_a, A_p) \, : \, A_a \subseteq A_p \subseteq \mathbb{N}, 0 \in A_p \}.$$

For $\varphi, \psi : X \to \Sigma$ we write $\varphi \preceq \psi$, if:

> *there is a partial recursive function f such that for any $x \in X$ and $n \in \varphi(x)_p$, we have that $f(n)$ is defined and belongs to $\psi(x)_p$; also, if $n \in \varphi(x)_a$, then $f(n) \in \psi(x)_a$.*

We see here:

- A distinction between potential and actual realizers.
- The idea that there should always be a canonical potential realizer.

However, this did not satisfy our criterion for a modified realizability topos (because AC fails in this topos).

One solution

Our solution is to add a notion of "extensional equality" to the set of potential realizers. That is, we choose:

$$\Sigma = \{ (A_a, A_p, \sim) \,:\, A_a \subseteq A_p \subseteq \mathbb{N}, 0 \in A_p, \sim \text{ equivalence relation on } A_p \}.$$

For $\varphi, \psi : X \to \Sigma$ we write $\varphi \preceq \psi$, if:

> *there is a partial recursive function f such that for any $x \in X$ and $n \in \varphi(x)_p$, we have that $f(n)$ is defined and belongs to $\psi(x)_p$; also, if $m, n \in \varphi(x)_p$ and $m \sim_{\varphi(x)} n$, then $f(m) \sim_{\psi(x)} f(n)$; and also, if $n \in \varphi(x)_a$, then $f(n) \in \psi(x)_a$.*

Theorem

This defines a tripos and in the resulting topos $\mathcal{E}$ we have:

$$\mathcal{E} \models \varphi \iff (\exists x)\, x \text{ mr } \varphi,$$

if we read the RHS wrt the HEO-model of Gödel's T.

So we also have: $\mathcal{E}$ models AC and IP.

Second solution

We can take Σ as before, but also demand that the elements of A_a are closed under $\sim$ (so if $m, n \in A_p$ with $m \sim n$, then $m \in A_a$ implies $n \in A_a$). The order relation is as before.

Theorem

This also defines a tripos and in the resulting topos we again have for any HA^ω-sentence φ:

$$\mathcal{E} \models \varphi \Longleftrightarrow (\exists x)\, x \text{ mr } \varphi,$$

if we read the RHS wrt the HEO-model of Gödel's T.

So we again have: $\mathcal{E}$ models AC and IP.

Indeed, this topos is a subtopos of the previous.

Many thanks to Eric Faber for putting me straight here!

Third topos

There is a different way of extensionalising Grayson's topos, where we only have a notion of extensional equality for the actual realizers:

$$\Sigma = \{ (A_a, A_p, \sim) \,:\, A_a \subseteq A_p \subseteq \mathbb{N}, 0 \in A_p, \sim \text{ equivalence relation on } A_a \}.$$

For $\varphi, \psi : X \to \Sigma$ we write $\varphi \preceq \psi$, if:

> *there is a partial recursive function f such that for any $x \in X$ and $n \in \varphi(x)_p$, we have that $f(n)$ is defined and belongs to $\psi(x)_p$; also, if $n \in \varphi(x)_a$, then $f(n) \in \psi(x)_a$; and, finally, if $m, n \in \varphi(x)_a$ and $m \sim_{\varphi(x)} n$, then $f(m) \sim_{\psi(x)} f(n)$.*

Claim

This also defines a tripos and in the resulting topos AC holds.

I believe/conjecture:

- IP fails in this topos.
- Jaap has a topos he calls Ext′ of which Eff is an open subtopos. This is its closed complement.

Question

So what is a modified realizability topos?

See: Mees de Vries, *An extensional modified realizability topos*. Master thesis, ILLC, University of Amsterdam, 2017. Available from: https://eprints.illc.uva.nl/1568/1/MoL-2017-27.text.pdf.

THANK YOU!