Hurewicz Fibrations in Toposes

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Overview

- Goal: realisability models of HoTT
- Inspiration: *A Notion of Homotopy for the Effective Topos*, J.v.Oosten 2013
- So far: structure of category with fibrant objects from a notion of interval (on a topos equipped with a such)
- AmSud Math project *Categories, Complexity and Logic*

The lore of toposes

Definition 1. *A topos is a finitely complete CCC with a subobject classfier Ω.*

Definition 2. *Let $\mathbb{T}$ be a topos and $X \in \mathbb{T}$. X's powerobject is the exponential Ω^X.*

Theorem 3. (Paré, 1974) *The contravariant powerobject functor $\Omega^{(-)}\colon \mathbb{T}^{\mathrm{op}} \longrightarrow \mathbb{T}$ is monadic.*

Remark 4. A monadic functor $\mathbb{C} \to \mathbb{D}$ creates limits in $\mathbb{D}$ (essentially: any limiting cone in $\mathbb{D}$ is the image of a limiting cone in $\mathbb{C}$). Hence a topos is finitely cocomplete.

Definition 5. *Let $\mathbb{T}$ be a topos. A natural number object $\mathbb{N}$, if it exists, is an initial object in the category of sequences $1 \to X \to X$.*

Remark 6. An NNO $1 \xrightarrow{\text{zero}} \mathbb{N} \xrightarrow{\text{succ}} \mathbb{N}$ implements Peano arithmetic internally.

Definition 7. *An arithmetic topos is a topos equipped with an NNO.*

Definition 8. *A countable family of subobjects of $X \in \mathbb{T}$ is a morphism $\phi\colon \mathbb{N} \to \Omega^X$.*

Notation. *We shall write* $(X_n \lhd X)_{n:\mathbb{N}}$ *for a countable family bounded by* X *when the indexing morphism* ϕ *is unambigous (or arbitrary).*

Remark 9. $(X_n \lhd X)_{n:\mathbb{N}}$ be a countable bounded family in $\mathbb{T}$.

1. $(X_n \lhd X)_{n:\mathbb{N}}$ admits a bounded product

$$\prod_{n:\mathbb{N}} X_n = \{\theta\colon X^{\mathbb{N}} \mid \forall n\colon \mathbb{N}.\theta(n) \in X_n\}$$

with n-th projection $\pi_n = \lambda\theta.\theta(n)$.

2. Essentially by Paré's argument, $(X_n)_{n:\mathbb{N}}$ admits a bounded coproduct $\coprod_{n:\mathbb{N}} X_n$ as well.

   [As sugested by Jaap, this needs to be made more precise]

Definition 10. *Let* $X \in \mathbb{T}$. *The object*

$$\mathrm{List}(X) \stackrel{\text{def.}}{=} \{x\colon (\star + X\ )^{\mathbb{N}} \mid \exists n\colon \mathbb{N}.\forall k\colon \mathbb{N}.(x_k \in X \iff k < n)\}$$

is called list object over X*.*

Remark 11. $\mathrm{List}(-)$ is a submonad of the *maybe monad* $(\star + (-))$.

Definition 12. *An* effective set $(X,\approx)$ *is a set equipped with an* effective equality, *that is a non-standard predicate* $|{-}\approx{-}|\colon X\times X\to\mathcal{P}\mathbb{N}$ *which is*

i. symmetric: $|x\approx x'|\vdash|x'\approx x|$

ii. transitive: $|x\approx x'|\wedge|x'\approx x''|\vdash|x\approx x''|$

Remark 13. Notice that we do not alway have reflexivity, that is $\mathcal{P}\mathbb{N}\vDash|x\approx x|$. In fact, the latter assert's x's *existence*. Accordingly, $E(x)\overset{\text{def.}}{=}|x\approx x|$ is called the existence predicate on $(X,\approx)$. Call $x\in X$ ghost if its existence is empty... In particular two equal inhabitants cannot be ghosts:

$$|x\approx x'|\vdash E(x)\wedge E(x')$$

Definition 14. *Let* $(X,\approx)$ *and* $(Y,\approx)$ *be effective sets. A functional relation*

$$\Phi\colon(X,\approx)\rightsquigarrow(Y,\approx)$$

is a predicate $\Phi\colon X\times Y\to\mathcal{P}\mathbb{N}$ *which is*

i. extensional: $\Phi(x,y)\wedge|x\approx x'|\wedge|y\approx y'|\vdash\Phi(x',y')$

ii. strict: $\Phi(x,y)\vdash E(x)\wedge E(y)$

iii. single-valued: $\Phi(x,y)\wedge\Phi(x,y')\vdash|y\approx y'|$

iv. total: $E(x)\vdash\bigcup_{y\in Y}E(y)\wedge\Phi(x,y)$

Two functional relations $\Phi,\Psi\colon(X,\approx)\rightsquigarrow(Y,\approx)$ are equivalent if $\Phi\dashv\vdash\Psi$.

Definition 15. *A* map *of effective sets is an equivalence class of functional relations.*

Notation 16. *We shall write $\mathfrak{R}_f$ for an arbitrary but fixed representant of the map f.*

We shall use this notation as a concise way to indicate that a property of a map formulated in terms of an arbitrary representant is well-defined.

Proposition 17. *Let $f\colon(X,\approx)\to(Y,\approx)$ and $g\colon(Y,\approx)\to(Z,\approx)$ be maps. Their composition is given by*

$$(\mathfrak{R}_g\circ\mathfrak{R}_f)(x,z):=\exists y.\mathfrak{R}_f(x,y)\wedge\mathfrak{R}_g(y,z)$$

in $\mathcal{E}ff$'s internal logic.

So in layman's terms this is

$$(\mathfrak{R}_g\circ\mathfrak{R}_f)(x,z)\ =\ \bigcup_{y\in Y}E(y)\wedge\mathfrak{R}_f(x,y)\wedge\mathfrak{R}_g(y,z)$$

Proposition 18. *Let* $\mathrm{id}_X\colon (X,\approx)\to(X,\approx)$ *be the map represented by*

$$\mathfrak{R}_{\mathrm{id}_X}(x,x') := |x\approx x'|$$

We have $f=\mathrm{id}_X\circ f$ *and* $f=\mathrm{id}_Y\circ f$ *for any map* $f\colon (X,\approx)\to(Y,\approx)$.

Theorem 19. (Hyland) *Effective sets and their maps aggregate to the category* $\mathcal{E}ff$*. This category is a topos.*

Definition 20. *An* assembly *or* ω-set *is an object* $(X,\approx)\in\mathcal{E}ff$ *such that* $|x\approx x'|=\varnothing$ *if* $x\neq x'$.

Example 21.

1. The assembly $(\{\star\},\mathbb{N})$ is terminal in $\mathcal{E}ff$.
2. The assembly $(\mathbb{N}, n\mapsto\{n\})$ is an NNO in $\mathcal{E}ff$.
3. Non-example: Ω.

Remark 22. Assume $(X,\approx)$. Let $\bar{X}\stackrel{\text{def.}}{=}\{x\in X\,|\,E(x)\neq\varnothing\}$. The relation $\approx$ is an equivalence relation on $\bar{X}$. Call equality class an element of the quotient. A global section $\star\to(X,\approx)$ selects an equality class in X.

Definition 23. *Let* $(X, \approx), (Y, \approx) \in \mathcal{E}ff$. *A function* $f : X \to Y$ *is* effective *is there is a* tracking code *or* tracker $t \in \mathbb{N}$ *such that , for all* $x, x' \in X$ *and* $n \in |x \approx x'|$ *we have* $t \cdot n \in |f(x) \approx f(x')|$.

Remark 24. A effective function induces a morphism $(X, \approx) \to (Y, \approx)$ represented by

$$\mathfrak{R}_f(x, y) = \bigcup_{x' \in X} \{\langle m, n \rangle \mid m \in |x \approx x'|, n \in |f(x') \approx y|\}$$

Proposition 25. *Any map to an assembly is induced by a unique effective function.*

Corollary 26. *A map* $f : (X, \approx) \to (Y, \approx)$ *among assemblies is induced by a* supereffective *function* $f : X \to Y$ *for which there exists a* tracking code *or* tracker $t \in \mathbb{N}$ *such that, for all* $x \in X$ *and* $n \in E(x)$, *we have* $t \cdot n \in E(f(x))$.

Definition 27. *An object* $X \in \mathbb{C}$ *in a category* $\mathbb{C}$ *is well-pointed if, given (arbitrary) morphisms* $f, g\colon X \to Y$, $f(x) = g(x)$ *for all global sections* $x\colon \star \to X$ *implies* $f = g$.

Remark 28. A boolean topos can be characterised as a topos where every object is well-pointed.

Definition 29. *Let* $\mathbb{T}$ *be a topos.* $I \in \mathbb{T}$ *is an elementary interval provided it*

i. is well-pointed;

ii. has precisely two global sections $\#0, \#1\colon \star \to I$.

Example 30. Consider the *effective topos* $\mathcal{E}ff$ and the assembly

$$I \stackrel{\text{def.}}{=} (\{i_0, i_1\}; E(i_0) = \{0,1\}, E(i_1) = \{1,2\})$$

Notice that $I \cong \mathbf{\Delta 2}$. Assume $(X, \approx) \in \mathcal{E}ff$. A morphism $s\colon I \to (X, \approx)$ determines and is determined by global sections $\lceil x \rceil, \lceil x' \rceil\colon \star \to (X, \approx)$ such that there are $x_0, x_0' \in X$ verifying

i. $\lceil x \rceil = \lceil x_0 \rceil$;

ii. $\lceil x' \rceil = \lceil x_0' \rceil$;

iii. $E(x_0) \cap E(x_0') \neq \varnothing$.

The latter item follows from the fact that $\mathfrak{R}_s$ is total. The morphism s is thus determined by the global sections $s \circ \lceil i_0 \rceil$ and $s \circ \lceil i_1 \rceil$, which entails in particular that I is well-pointed.

Definition 31. *Let $\mathbb{T}$ be a topos and I an elementary interval in $\mathbb{T}$. Elementary intervals of length n are obtained by gluing copies of I*

$$I_0 \stackrel{\text{def.}}{=} \star$$
$$I_{n+1} \stackrel{\text{def.}}{=} I_n +_1 I$$

by pushout

$$\begin{tikzcd}
\star \arrow[r, "\#0"] \arrow[d, "\#(n-1)"'] & I \arrow[d] \\
I_n \arrow[r] & I_{n+1} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Let $X \in \mathbb{T}$.We shall call a morphism $I_n \to X$ (rigid) path (of length n) in X and X^{I_n} the object of paths of length n.

Example 32. In $\mathcal{E}ff$ we can construct I_n as the assembly

$$\left(\{i_0^{(n)}, \cdots, i_{n-1}^{(n)}\}; E\left(i_k^{(n)}\right) = \{k, k+1\}\right)$$

A morphism $s: I_n \to (X, \approx)$ determines and is determined by a list $[\lceil x_0 \rceil; \cdots, \lceil x_{n-1} \rceil]$ of global sections of $(X, \approx)$ such that for all $0 \leqslant i < n-1$ there is x_i' verifying

i. $\lceil x_i \rceil = \lceil x_i' \rceil$;

ii. $E(x_i') \cap E(x_{i+1}') \neq \varnothing$.

Remark 33. Let $X \in \mathbb{H}$ and $n \in \mathbb{N}$. We have

$$X^{I_n} \;\cong\; \{l\colon \mathrm{List}(X) | \exists \omega\colon X^{I_n}. \forall i\colon [n]. l(i) = \omega_{\# i}\}$$

since I_n is well-pointed, so in particular $X^{I_n} \lhd \mathrm{List}(X)$ for all $n \in \mathbb{N}$. Hence the bounded coproduct $\coprod_{n:\mathbb{N}} X^{I_n}$ exists.

Notation. *When convenient, we shall use the list notation* $[\omega_{\#0}; \cdots, \omega_{\#(n-1)}]$ *for a path* ω.

Remark 34. Let I be an elementary interval. For any $n \geqslant 0$ and $0 \leqslant i \leqslant n$ there is the i*-th coface function*

$$\begin{aligned} \delta^{(i)} \colon \Gamma(I_n) &\longrightarrow \Gamma(I_{n+1}) \\ \#j &\mapsto \begin{cases} \#j & j < i \\ \#(j+1) & j \geqslant i \end{cases} \end{aligned}$$

Similarly, for any $n \geqslant 1$ and $0 \leqslant i \leqslant n-1$ there is the i*-th codegeneracy function*

$$\begin{aligned} \sigma^{(i)} \colon \Gamma(I_{n+1}) &\longrightarrow \Gamma(I_n) \\ \#j &\mapsto \begin{cases} \#j & j \leqslant i \\ \#(j-1) & j > i \end{cases} \end{aligned}$$

Definition 35. *An elementary interval I is* cosimplical *provided coface functions $\delta^{(i)}$ and codegeneracy functions $\sigma^{(i)}$ uniquely determine* coface operators $\delta_i \colon I_n \longrightarrow I_{n+1}$ *and* codegeneracy operators $\sigma_i \colon I_{n+1} \longrightarrow I_n$*, respectively.*

Example 36. The elementary interval $\Delta\mathbf{2}$ in $\mathcal{E}ff$ is cosimplicial. The I_n's are assemblies, so a global section $*\!\longrightarrow\! I_n$ is uniquely determined by an element of the underlying set while a morphism $f\colon I_m \longrightarrow I_n$ is uniquely determined by a tracked function on the underlying sets. The *i-th coface function*

$$\delta^{(i)}\colon \Gamma(I_n) \;\to\; \Gamma(I_{n+1})$$

admits the tracker

$$\Lambda j.\mathbf{if}\ j < i\ \mathbf{then}\ j\ \mathbf{else}\ j+1$$

when seen as a function $\{i_0, \cdots, i_{n-1}\} \to \{i_0, \cdots, i_n\}$. Similarly, the *$i$-th codegenacy function*

$$\sigma^{(i)}\colon \Gamma(I_{n+1}) \;\longrightarrow\; \Gamma(I_n)$$

admits the tracker

$$\Lambda j.\mathbf{if}\ j \leqslant i\ \mathbf{then}\ j\ \mathbf{else}\ j-1$$

when seen as a function $\{i_0, \cdots, i_n\} \to \{i_0, \cdots, i_{n-1}\}$.

Remark 37. Let I be a cosimplicial interval and $\mathbb{I} \subset \mathbb{T}$ be the subcategory with objects the I_n's and monotone morphisms (the latter are generated by coface and codegeneracy operators modulo cosimplicial identities). $\mathbb{I}$ is a monoidal category with tensor given by pushout

$$\begin{tikzcd} \star \arrow[r, "\#0"] \arrow[d, "\#(m-1)"'] & I_n \arrow[d] \\ I_m \arrow[r] \arrow[ur, phantom, "\ulcorner"] & I_m \otimes I_n = I_{m+n} \end{tikzcd}$$

Any morphism in $\mathbb{I}$ admits a normal form. In other words, $\mathbb{I}$ is equivalent to Δ^+ (the "algebraist's simplicial category").

Remark 38. Let $X \in \mathbb{H}$. The (bounded) family $\mathrm{Path}(X) \stackrel{\text{def.}}{=} (X^{I_n})_{n:\mathbb{N}}$ is a simplicial object with faces and degeneracies given by precomposition

$$\begin{aligned} d_i &= \lambda\omega\colon \mathrm{Path}(X)_n . \omega \circ \delta_i \\ s_i &= \lambda\omega\colon \mathrm{Path}(X)_n . \omega \circ \sigma_i \end{aligned}$$

we shall call the path complex of X. Notice that $\mathrm{Path}(X)_0 = X^{I_0} \cong X$ as $I_0 \cong \star$.

[Unilke the claim in the version projected during the talk, the Eilenberg-Zilber lemma does not hold in this setting. As recalled by Benno, it is not constructive, which is a fact this speaker was embarassingly not aware.]

Definition 39. *Let $X \in \mathbb{H}$, I a cosimplicial interval and $\sim_0$ be the relation on $\coprod_{n:\mathbb{N}} \mathrm{Path}(X)_n$ such that $u \sim_0 v$ if t is a degeneracy of s. The path object $X^{\langle I \rangle}$ is the quotient*

$$X^{\langle I \rangle} \stackrel{\text{def.}}{=} \coprod_{n:\mathbb{N}} \mathrm{Path}(X)_n / \sim$$

of $\coprod_{n:\mathbb{N}} \mathrm{Path}(X)_n$ by the equivalence relation generated by $\sim_0$.

Notation. *Given $\omega \colon X^{\langle I \rangle}$, we shall write $\omega\langle n \rangle$ for a representative of ω of length n and $\tilde{\omega}$ for the whole equivalence class.*

Definition 40. *A path $\omega \colon X^{\langle I \rangle}$* is constant *if its canonical representative is of length 0.*

Remark 41. $\star^{\langle I \rangle} \cong \star$ as there is only the trivial path up to degeneracy, so the quotient collapses.

Remark 42. Let $\omega\colon \mathrm{Path}(X)_m$ and $\varpi\colon \mathrm{Path}(X)_n$ be paths such that $\varpi = s(\omega)$ for some degeneracy $s\colon \mathrm{Path}(X)_m \to \mathrm{Path}(X)_n$. Assume $m, n\colon \mathbb{N}$. Then

$$\begin{aligned} \varpi\langle n\rangle_0 &= \omega\langle m\rangle_0 \\ \varpi\langle n\rangle_{n-1} &= \omega\langle m\rangle_{m-1} \end{aligned}$$

hence the source and target morphisms $\partial_X^-, \partial_X^+\colon X^{\langle I\rangle} \to X$ given by

$$\begin{aligned} \partial_X^- &\stackrel{\text{def.}}{=} \lambda\omega\colon X^{\langle I\rangle}.\omega\langle n\rangle_{\#0} \\ \partial_X^+ &\stackrel{\text{def.}}{=} \lambda\omega\colon X^{\langle I\rangle}.\omega\langle n\rangle_{\#(n-1)} \end{aligned}$$

are well-defined. Hence

- $\partial_X^-, \partial_X^+\colon X^{\langle I\rangle} \rightrightarrows X$ is an internal graph in $\mathbb{H}$;
- composition “by concatenation” $\otimes_X\colon X^{\langle I\rangle} \times X^{\langle I\rangle} \to X^{\langle I\rangle}$ is well-defined;
- the constant path morphism $\iota_X \stackrel{\text{def.}}{=} \lambda x\colon X.\,\widetilde{\mathrm{in}_0(x)}\colon X \to X^{\langle I\rangle}$ (with $\tilde{x}$ being x's equivalence class) is a section of both ∂_X^- and ∂_X^+.

Theorem 43.

1. $X^{\langle I \rangle}$ *is an internal category with composition* $\otimes_X$ *and unit* ι_X.
2. *There is an involution* $\mathbf{rev}\colon (-)\colon X^{\langle I \rangle} \to X^{\langle I \rangle}$ *given by list reversal.*
3. *The assignment* $(-)^{\langle I \rangle}\colon \mathbb{H} \to \mathbb{H}$ *is an endofunctor acting by postcomposition*

$$f^{\langle I \rangle}(u) = f \circ u$$

4. *The morphisms* ι_X, ∂_X^- *and* ∂_X^+ *are natural in* X.

Definition 44. *A finite face filtration (of length n) is a sequence $d = (d^{(i)})_{0 \leqslant i < n}$ of face operators*

$$\mathrm{Path}(X)_{m(0)} \xleftarrow{d^{(0)}} \mathrm{Path}(X)_{m(1)} \xleftarrow{d^{(1)}} \cdots \xleftarrow{d^{(n-1)}} \mathrm{Path}(X)_{m(n)}$$

Remark 45. Assume $n\colon \mathbb{N}$ and $\omega\colon \mathrm{Path}(X)_n$. If we see ω as a list, the $\mathrm{first\,face}$ of ω is the tail

$$d_0(\omega) \;=\; [\omega_{\#1}; \omega_{\#2}; \cdots; \omega_{\#(n-1)}]$$

while it's last face is the maximal prefix

$$d_{n-1}(\omega) \;=\; [\omega_{\#0}; \omega_{\#2}; \cdots; \omega_{\#(n-2)}]$$

We have in particular the tail filtration $t = \left(t_n^{(i)}\right)$ given by

$$\mathrm{Path}(X)_0 \xleftarrow{d_0} \mathrm{Path}(X)_1 \xleftarrow{d_0} \cdots \xleftarrow{d_0} \mathrm{Path}(X)_n$$

and the prefix filtration $p = \left(p_n^{(i)}\right)$ given by

$$\mathrm{Path}(X)_0 \xleftarrow{d_1} \mathrm{Path}(X)_1 \xleftarrow{d_2} \cdots \xleftarrow{d_{n-1}} \mathrm{Path}(X)_n$$

Notation. *Assume $X \in \mathbb{T}$ and $x, x' \colon X$. We shall write $\omega\colon x \rightsquigarrow x'$ as an abbreviation for a path $\omega\colon X^{\langle I \rangle}$ such that $\partial^-(\omega) = x$ and $\partial^+(\omega) = x'$. Let $f\colon X \to Y$ be a morphism. We shall abuse notation and write $f(\omega)\colon f(x) \rightsquigarrow f(x')$ for $f^{\langle I \rangle}(s)$.*

Notation. *Assume a finite face filtration* $d = (d^{(i)})_{0 \leqslant i < n}$. *We shall write* $d_{\flat(i)}$ *for the face operator*

$$d^{(i)} \circ d^{(i+1)} \circ \cdots \circ d^{(n-1)}$$

Definition 46.

1. *A cosimplicial interval has the Hurewicz property if for any* $\omega\colon \mathrm{Path}(X)_{m(n)}$ *and any finite face filtration the list*

$$[d_{\flat(0)}(\omega); d_{\flat(1)}(\omega); \cdots; d_{\flat(n-1)}(\omega); \omega]$$

   determines a path in $X^{\langle I \rangle}$.

2. *A Hurewicz topos is an arithmetic topos equipped with a Hurewicz interval (part of the data).*

Example 47. $(\mathcal{E}ff, \Delta\mathbf{2})$ is Hurewicz. Paths can be characterised in terms of equalities and existence predicates (c.f. Example 32). $\Delta\mathbf{2}$ is cosimplicial (c.f. Example 36). To see that it is Hurewicz, recall that the set underlying an exponential $(Y, \approx)^{(X, \approx)}$ in $\mathcal{E}ff$ is $\{\phi\colon X \times Y \to \mathcal{P}\mathbb{N}\}$, while existence is the set of quadruplets $\langle k, l, m, n\rangle$ witnessing that ϕ is a functional relation. Now a face operator $d\colon X^{I_m} \to X^{I_n}$ is given by precomposition

$$\begin{aligned} d\colon X^{I_m} &\longrightarrow X^{I_n} \\ \omega &\longmapsto \omega \circ \delta \end{aligned}$$

for some coface operator $\delta\colon I_n \rightarrowtail I_m$, so we have $E(\omega \circ \delta) \subseteq E(\omega)$.

Homotopy

Definition 48. *Let $f, g\colon X \to Y$ be morphisms. A* homotopy *$H\colon f \looparrowright g$ from f to g is given by a commuting diagram*

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, "H"] \arrow[dr, "g"] & \\ Y & Y^{\langle I \rangle} \arrow[l, "\partial^-"] \arrow[r, "\partial^+"'] & Y \end{tikzcd}$$

H is constant *on a subobject $X' \lhd X$ provided $H(x) = \iota(x)$ for any $x\colon X$ such that $x \in X'$.*

Remark 49. A homotopy $H\colon f \looparrowright g$ informs us that for any $x\colon X$ there is $H(x)\colon Y^{\langle I \rangle}$ such that $f(x) = \partial^-(\omega_x)$ and $g(x) = \partial^+(\omega_x)$.

Definition 50. *A* homotopy equivalence *is a morphism $u\colon X \to Y$ which has an up-to-homotopy inverse $v\colon Y \to X$.*

Remark 51. The morphism v is a homotopy equivalence as well, called *the inverse homotopy equivalence*.

Remark 52. Homotopy equivalences verify 3-for-2.

Definition 53. *A section h of the canonical morphism $\langle f^{\langle I\rangle}, \partial_X^-\rangle$ in*

$$\begin{tikzcd}
X^{\langle I\rangle} \arrow[rr, "\partial_X^-"] \arrow[dr, "\langle f^{\langle I\rangle}{,} \partial_X^-\rangle"'] \arrow[dd, "f^{\langle I\rangle}"'] & & X \arrow[dd, "f"] \\
& (\partial_Y^-)^* X \arrow[ul, bend right, "h"'] \arrow[ur] \arrow[dl] \arrow[dr, phantom, "\lrcorner", very near start] & \\
Y^{\langle I\rangle} \arrow[rr, "\partial_Y^-"'] & & Y
\end{tikzcd}$$

is called connection*. A morphism which admits a connection is called* Hurewicz fibration*. A Hurewicz fibration which is also a homotopy equivalence is called* trivial*.*

Notation. *We shall write $\mathcal{H}$ for the class of Hurewicz fibrations.*

Remark 54. A Hurewicz fibration $f\colon X \to Y$ is thus a morphism with a strong path lifiting property: for any path $v\colon y \rightsquigarrow y'$ in Y and any $x \in X$ such that $f(x) = y$ there is a path u in X such that $f \circ u = v$ along with an explicit construction of one such lift for any pair (x, v) such that $f(x) = \partial^-(v)$.

Proposition 55. *Projections from products are Hurewicz fibrations.*

Proposition 56. $\langle \partial^-, \partial^+ \rangle \colon X^{\langle I \rangle} \to X \times X$ *is a Hurewicz fibration for any* $X \in \mathbb{H}$.

Corollary 57. *The source map* $\partial^-_X \colon X^{\langle I \rangle} \to X$ *and the target map* $\partial^+_X \colon X^{\langle I \rangle} \to X$ *are Hurewicz fibrations for any* $X \in \mathbb{H}$.

Proposition 58. *Hurewicz fibrations are stable under pullback.*

Definition 59. *X is a deformation retract of Y if there is a map $e\colon X \to Y$ admitting a retraction $r\colon Y \to X$ such that there is a homotopy $H\colon e \circ r \looparrowright \mathrm{id}_Y$. We call the split epi r deformation retraction and the split mono e deformation insertion, respectively.*

Remark 60. A deformation insertion is a homotopy equivalence.

Proposition 61. *Trivial Hurewicz fibrations are stable under pullback.*

Proof. Assume $p\colon E \to B$ Hurewicz, witnessed by connection h_p. Assume $f\colon A \to B$. Now $p_0 \stackrel{\text{def.}}{=} f^*p$ is Hurewicz by the previous prop, so we only need to establish that it is a homotopy equivalence. Assume a homotopy inverse $u\colon B \to E$ of, p witnessed by homotopies

$$\begin{aligned} H\colon p \circ u &\looparrowright \mathrm{id}_B \\ K\colon u \circ p &\looparrowright \mathrm{id}_E \end{aligned}$$

Assume $a\colon A$. We have

$$\begin{tikzcd}
f^*E \arrow[r] \arrow[d, "p_0"'] & E \arrow[d, "p"'] \\
A \arrow[r, "f"'] & B \arrow[u, bend right, "u"']
\end{tikzcd}$$

$u(f(a))$

$H(f(a)) \colon p(u(f(a))) \rightsquigarrow f(a)$

But p is Hurewicz, so we have the lift $h_p(u(f(a)), H(f(a)))$. Let

$$e_a \overset{\text{def.}}{=} \partial^+(h_p(u(f(a)), H(f(a))))$$

As $p(e_a) = f(a)$, the term $\lambda a \colon A.\,(a, e_a)$ constructs a section $u_0 \colon A \to f^*E$ of p_0. We claim that u_0 is a deformation insertion. Assume $(a, e) \colon f^*E$. We have $u_0(p_0(a, e)) = u_0(a) = (a, e_a)$ and paths

$$\begin{aligned}
K(e) \colon u(p(e)) &\rightsquigarrow e \\
K(e_a) \colon u(p(e_a)) &\rightsquigarrow e_a
\end{aligned}$$

But $p(e) = f(a)$ by hypothesis and $p(e_a) = f(a)$ by construction so $u(p(e)) = u(p(e_a))$. Hence the term

$$\lambda(a, e) \colon f^*E.\, \mathbf{zip}(\iota_a, \mathbf{rev}(K(e_a)) \otimes K(e))$$

constructs a homotopy $H' \colon u_0 \circ p_0 \looparrowright \mathrm{id}_{f^* E}$. $\square$

Proposition 62. *Any object $X \in \mathbb{H}$ is Hurewicz fibrant.*

Proposition 63. *Hurewicz fibrations are stable under composition.*

Anodyne morphisms

Definition 64. *X is a strong deformation retract of Y if it is a deformation retract admitting a witnessing homotopy constant on X. We call the split epi r strong deformation retraction and the split mono e strong deformation insertion, respectively.*

Definition 65. *A morphism $f \in^{\pitchfork} \mathcal{H}$ is anodyne.*

Theorem 66. *A strong deformation insertion is anodyne.*

Proposition 67. *The constant path map $\iota_X \colon X \to X^{\langle I \rangle}$ is a strong deformation insertion.*

Remark 68. The diagonal factors through $X^{\langle I \rangle}$ as

$$\begin{tikzcd} & X^{\langle I \rangle} \arrow[dr, "\langle \partial^-_X, \partial^+_X \rangle"] & \\ X \arrow[ur, "\iota_X"] \arrow[rr, "\Delta_X"'] & & X \times X \end{tikzcd}$$

with ι_X anodyne and $\langle \partial^-_X, \partial^+_X \rangle$ Hurewicz.

Category of Fibrant Objects

Definition 69. (K.S.Brown, 1973) *A category $\mathbb{F}$ with finite limits equipped with a class of fibrations $\mathcal{F}$ and a class of weak equivalences $\mathcal{W}$ is a category of fibrant objects provided*

i. $\mathcal{I} \subset \mathcal{F} \cap \mathcal{W}$;

ii. $\mathcal{W}$ verifies 3-for-2;

iii. $\mathcal{F}$ and $\mathcal{F} \cap \mathcal{W}$ are stable under pullback;

iv. for any $X \in \mathbb{F}$ there is a an object $X^{\langle I \rangle}$ such that there is a factorisation

$$\begin{tikzcd} & X^{\langle I \rangle} \arrow[dr, "f \in \mathcal{F}"] & \\ X \arrow[ur, "\mathcal{W} \ni w"] \arrow[rr, "\Delta_X"'] & & X \times X \end{tikzcd}$$

Theorem 70. *Let I be a Hurewicz interval. $(\mathbb{H}, I)$ with $\mathcal{F} = \{\text{Hurewicz fibrations}\}$ and $\mathcal{W} = \{\text{homotopy equivalences}\}$ is a category of fibrant objects.*

A more general factorisation

Definition 71. *Let $f\colon X \to Y$ be a morphism in $\mathbb{H}$. The object M_f given by the pullback*

$$\begin{tikzcd} M_f \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y^{\langle I \rangle} \arrow[d, "\partial_Y^-"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is called f's mapping track.

Remark 72. $M_f = \{(x, \omega)\colon X \times Y^{\langle I \rangle} \mid f(x) = \partial_Y^-(\omega)\}$ is the object of paths that begin in the image of f.

Theorem 73. *A morphism $f\colon X \to Y$ factors through the mapping track as an anodyne morphism followed by a Hurewicz fibration.*

Open questions

- Are Hurewicz fibrations stable under $\prod_f$ for f Hurewicz? (Joyal's π-tribe)
- If so, is a "Hurewicz tribe" a model of HoTT? How about univalence?
- Is there a notion of cofibration? Model category?

That's all folks